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WEYL'S THEOREM FOR A GENERALIZED DERIVATION AND AN ELEMENTARY OPERATOR

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Abstract. For $a, b \in B(H)$, $B(H)$ the algebra of operators on a complex infinite dimensional Hilbert space H, the generalized derivation $\delta_{ab} \in B(B(H))$ and the elementary operator $\triangle_{ab} \in B(B(H))$ are defined by $\delta_{ab}(x) = ax - xb$ and $\triangle_{ab}(x) = axb - x$. Let $d_{ab} = \delta_{ab}$ or $\triangle_{ab}$. It is proved that if a, b^* are hyponormal, then $f(d_{ab})$ satisfies (generalized) Weyl's theorem for each function f analytic on a neighbourhood of $\sigma(d_{ab})$.

1. Introduction

Let $B(H)$ denote the algebra of operators (i.e., bounded linear transformations) on a complex infinite dimensional Hilbert space H. For $a, b \in B(H)$, let $\delta_{ab}\colon B(H) \to B(H)$ and $\triangle_{ab}\colon B(H) \to B(H)$ denotes the *generalized derivation* $\delta_{ab}(x) = ax - xb$ and the *elementary operator* $\triangle_{ab}(x) = axb - x$. Let $d_{ab} = \delta_{ab}$ or $\triangle_{ab}$. The following implications hold for a general bounded linear operator t on a normed linear space V, in particular for $t = d_{ab}$:

$$\begin{aligned} t^{-1}(0) \perp t(V) &\Longrightarrow t^{-1}(0) \cap clt(V) = \{0\} \\ &\Longrightarrow t^{-1}(0) \cap t(V) = \{0\} \Longleftrightarrow asc(t) \leq 1 \end{aligned}$$

[6, page 25]. Here $asc(t)$ denotes the *ascent* of t, $clt(V)$ denote the closure of the range of t and $t^{-1}(0) \perp t(V)$ denotes that the kernel of t is orthogonal to the range of t in the sense of G. Birkhoff. Recall that if M, N are linear subspaces of a normed linear space V, then $M \perp N$ in the sense of Birkhoff if $\|m\| \leq \|m+n\|$ for all $m \in M$ and $n \in N$. This concept of orthogonality is not symmetric, i.e., $M \perp N$ does not imply $N \perp M$, but the concept does agree with the usual concept of orthogonality in the case in which $V = H$. The range-kernel orthogonality of d_{ab} has been considered by a number of authors (see [1,6,10,15,22,23] for further references). A sufficient condition guaranteeing $d_{ab}^{-1}(0) \perp d_{ab}(B(H))$ is that $d_{ab}^{-1}(0) \subseteq d_{a^*b^*}^{-1}(0)$ [10, Theorem (i)]. The class of operators $a, b^* \in B(H)$ such that $d_{ab}^{-1}(0) \subseteq d_{a^*b^*}^{-1}(0)$ is large, and

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includes in particular the class of hyponormal a and b^* [9,20]. If $a, b^* \in B(H)$ are hyponormal, then $(d_{ab}-\lambda)^{-1}(0) \subseteq (d_{a^*b^*}-\overline{\lambda})^{-1}(0)$ and $asc(d_{ab}-\lambda) \leq 1$ for all complex numbers λ. This implies that d_{ab} has the single-valued extension property and hence satisfies Browder's theorem [8].

A detailed study of the spectral properties of the operator d_{ab} has been carried out in a series of papers by L. A. Fialkow (of which [14] is an earlier sample). Our aim here is a modest one. We show that if $a, b^* \in B(H)$ are hyponormal, and f is a function which is analytic on a neighbourhood of the spectrum of d_{ab}, then $f(d_{ab})$ satisfies Weyl's theorem. Indeed more is true: $f(d_{ab})$ satisfies the generalized Weyl's theorem. Problems of this type seem not to have previously been considered.

The plan of this paper is as follows. We use the remainder of this section to introduce some of our notation and terminology. (Any additional notation or terminology will be introduced as and when required.) Section 2 will be devoted to proving some complementary results, amongst them that d_{ab}, $a, b^* \in B(H)$ hyponormal, is isoloid and that the range of $d_{ab}-\lambda$ is closed for each isolated point λ of the spectrum of d_{ab}. We shall prove Weyl's theorems for d_{ab} in Section 3.

We shall denote the spectrum, the point spectrum and the set of isolated points of the spectrum of a Banach space operator $t \in B(V)$ by $\sigma(t), \sigma_p(t)$ and $iso\sigma(t)$, respectively. The range, the kernel and the restriction to an invariant subspace M of t will be denoted by $t(V)$ (or, $ran(t)$), $t^{-1}(0)$ (or, $\ker t$) and $t_{|M}$, respectively. The operator t is a *quasi-affinity* if it is injective and has dense range, and t is said to be *isoloid* if there is implication $\lambda \in iso\sigma(t) \Longrightarrow \lambda \in \sigma_p(t)$. Recall that the *ascent* $asc(t)$ of an operator t is the smallest non-negative integer n such that $t^{-n}(0) = t^{-(n+1)}(0)$.

Let V be a (complex) Banach space, and let $\mathcal{U}$ be an open subset of the complex plane $\mathbf{C}$. Let $\mathcal{O}(\mathcal{U}, V)$ denote the Frechet space of V-valued analytic functions from $\mathcal{U}$. The operator $t \in B(V)$ is said to satisfy *Bishop's condition* (β) if, for each open subset $\mathcal{U}$ of $\mathbf{C}$, the operator $t_{|\mathcal{U}}$ given by $(t_{|\mathcal{U}}f)(\lambda) := (t-\lambda)f(\lambda)$ is injective and has dense range in $\mathcal{O}(\mathcal{U}, V)$ for each $f \in \mathcal{O}(\mathcal{U}, V)$ and all $\lambda \in \mathcal{U}$. For a closed subset F of $\mathbf{C}$, let $V_t(F)$ denote the analytic spectral manifold

$$V_t(F) = \{v \in V : (t-\lambda)f(\lambda) = v \text{ has an analytic solution } f\colon \mathbf{C} \setminus F \to V\}.$$

The spaces $V_t(F)$ are t-invariant (generally, non-closed) manifolds of V. If, for every closed $F \subseteq \mathbf{C}$, $V_t(F)$ is closed, then t is said to satisfy *Dunford's property* (C). Condition (β) implies property (C), which in turn implies that the operator $t_{|\mathcal{U}}$ is injective for every open $\mathcal{U} \subseteq \mathbf{C}$. This last property is the *single-valued extension property*, shortened henceforth to SVEP. (Thus t has SVEP if, for every $v \in V$, $(t-\lambda)f(u) = v$ has a unique solution $f : \mathcal{U} \to V$ on $\mathcal{U} \subseteq \mathbf{C}$.) We shall denote the set of natural numbers by $\mathbf{N}$.

A Banach space operator $t \in B(V)$ is said to be *Fredholm* if $t(V)$ is closed, and both $t^{-1}(0)$ and $V \setminus clt(V)$ are finite dimensional. The *Fredholm index* $ind(t)$ of t is defined by $ind(t) = dim(t^{-1}(0)) - dim(V \setminus t(V))$. The operator t is *Weyl* if it is Fredholm of index 0, and it is *Browder* if it is Fredholm and both $asc(t)$ and

$dsc(t)$ are finite [11]. The (Fredholm) *essential spectrum* $\sigma_e(t)$, the *Weyl spectrum* $\sigma_w(t)$ and the *Browder spectrum* $\sigma_b(t)$ of t are defined by

$$\sigma_e(t) = \{\lambda \in \mathbf{C} : t - \lambda \text{ is not Fredholm}\},$$
$$\sigma_w(t) = \{\lambda \in \mathbf{C} : t - \lambda \text{ is not Weyl}\},$$
$$\sigma_b(t) = \{\lambda \in \mathbf{C} : t - \lambda \text{ is not Browder}\}.$$

Evidently, $\sigma_e(t) \subseteq \sigma_w(t) \subseteq \sigma_b(t) \subseteq \sigma_e(t) \cup acc\ \sigma(t)$. In general, the spectral mapping theorem holds for $\sigma_b(t)$ but fails for $\sigma_w(t)$ [12,13]. Let $\sigma_o(t)$ denote the set of *Riesz points* of t, and let $\sigma_{oo}(t) = \{\lambda \in iso\sigma(t) : 0 < dim(t-\lambda)^{-1}(0) < \infty\}$. Then $iso\sigma(t) \setminus \sigma_e(t) = iso\sigma(t) \setminus \sigma_w(t) = \sigma_o(t) \subseteq \sigma_{oo}(t)$. We say that t satisfies *Weyl's theorem* (resp., *Browder's theorem*) if

$$\sigma(t) \setminus \sigma_w(t) = \sigma_{oo}(t) \ \ (\text{resp., } \sigma(t) \setminus \sigma_w(t) = \sigma_o(t)).$$

2. Complementary results

We prove in this section that if $a, b^* \in B(H)$ are hyponormal, then d_{ab} is isoloid and $ran(d_{ab} - \lambda)$ is closed for each $\lambda \in \sigma_{oo}(d_{ab})$. But we start by working towards proving that d_{ab} has SVEP. Throughout the following, we write $t - \lambda$ for the operator $t - \lambda I$. The operators of "left multiplication by a" and "right multiplication by b" shall be denoted by L_a and R_b, respectively.

LEMMA 2.1. *Let $a, b \in B(H)$ be normal. If there exists a quasi- affinity $x \in \triangle_{ab}^{-1}(0)$, then b is invertible and $x \in \delta_{ab^{-1}}^{-1}(0)$.*

Proof. The operators a and b being normal, it follows from an application of the Putnam-Fuglede theorem for normal operators that $\triangle_{ab}^{-1}(0) = \triangle_{a^*b^*}^{-1}(0)$ [9, Corollary 3]. Let the quasiaffinity $x \in \triangle_{ab}^{-1}(0)$ have the polar decomposition $x = u|x|$ (where u is unitary). Since $\triangle_{ab}(x) = 0 = \triangle_{a^*b^*}(x)$, it follows from

$$b|x|^2 = bx^*(axb) = (bx^*a)xb = |x|^2 b$$

that $|x|^2$, and so also $|x|$, $\in \delta_{bb}^{-1}(0)$. Hence, since $\triangle_{ab}(x) = \triangle_{ab}(u|x|) = \triangle_{ab}(u)|x| = 0$, $u \in \triangle_{ab}^{-1}(0)$. Let $h \in H$. Then

$$\triangle_{ab}(u)h = 0 \Longrightarrow u^*aubh = h \Longrightarrow \|h\| = \|u^*aubh\| \leq \|a\|\|bh\|,$$

i.e., b is bounded below. It is clear from $\triangle_{a^*b^*}(x) = a^*xb^* - x = 0$ that b^* is injective. Hence b is invertible and $x \in \delta_{ab^{-1}}^{-1}(0)$. ∎

The following lemma is proved in [20] for the case in which $d = \delta$; for the case in which $d = \triangle$ a proof follows from the argument of the proof of [9, Lemma 4].

LEMMA 2.2. *If $a, b \in B(H)$ are normal, then $d_{ab}^{-2}(0) = d_{ab}^{-1}(0)$.*

The ascent of the operator d_{ab} (indeed, any operator) equals 0 if and only if d_{ab} is injective, and then $\{0\} = d_{ab}^{-1}(0) \subseteq d_{a^*b^*}^{-1}(0)$ trivially. The following proposition says that non-injective $d_{ab}-\lambda$ satisfying $(d_{ab}-\lambda)^{-1}(0)\subseteq(d_{a^*b^*}-\overline{\lambda})^{-1}(0)$ have ascent one.

PROPOSITION 2.3. *If $a,b\in B(H)$ and $\lambda\in\mathbf{C}$ are such that $(d_{ab}-\lambda)^{-1}(0)\subseteq(d_{a^*b^*}-\overline{\lambda})^{-1}(0)$, then $asc(d_{ab}-\lambda)\le 1$.*

Proof. We consider the cases $d_{ab}=\delta_{ab}$ and $d_{ab}=\triangle_{ab}$ separately.

Case $d_{ab}=\delta_{ab}$. Let $x\in(\delta_{ab}-\lambda)^{-1}(0)$. Then the hypothesis $(\delta_{ab}-\lambda)^{-1}(0)\subseteq(\delta_{a^*b^*}-\overline{\lambda})^{-1}(0)\Longrightarrow ax-x(b+\lambda)=0=a^*x-x(b^*+\overline{\lambda})\Longrightarrow\overline{ranx}$ reduces a and $ker^{\perp}x$ reduces $b+\lambda$. Since $x\in(\delta_{ab}-\lambda)^{-1}(0)\Longrightarrow ax$ and $x(b+\lambda)\in(\delta_{ab}-\lambda)^{-1}(0)$,

$$a^*ax=ax(b+\lambda)^*=aa^*x$$

and

$$x(b+\lambda)^*(b+\lambda)=a^*x(b+\lambda)=x(b+\lambda)(b+\lambda)^*.$$

Hence $a_1=a|_{\overline{ranx}}$ and $b_1=(b+\lambda)|_{ker^{\perp}x}$ are normal operators.

Suppose now that $y\in(\delta_{ab}-\lambda)^{-2}(0)$. Set $(\delta_{ab}-\lambda)y=x$, let $x_1:ker^{\perp}x\to\overline{ranx}$ be the quasi-affinity defined by setting $x_1h=xh$ for each $h\in H$ and let $y:ker^{\perp}x\oplus kerx\to\overline{ranx}\oplus\overline{ranx}^{\perp}$ have the matrix representation $y=[y_{ij}]_{i,j=1}^2$. Then $0=\delta_{ab}(x)=\delta_{a_1b_1}(x_1)\oplus 0=\delta^2_{a_1b_1}(y_{11})\oplus 0$. The operators a_1 and b_1 being normal, it follows from Lemma (2.2) that $\delta_{a_1b_1}(y_{11})=0$. Hence $x=\delta_{a_1b_1}(y_{11})\oplus 0=0\Longrightarrow(\delta_{ab}-\lambda)(y)=0\Longrightarrow asc(\delta_{ab}-\lambda)\le 1$.

Case $d_{ab}=\triangle_{ab}$. The proof is split into the cases $\lambda=-1$ and $\lambda\ne-1$.

If $\lambda=-1$, then $y\in(\triangle_{ab}-\lambda)^{-2}(0)\Longrightarrow ayb\in(\triangle_{ab}-\lambda)^{-1}(0)\Longrightarrow|a|^2y|b^*|^2=0$. If both $|a|$ and $|b^*|$ are injective, then $y=0$ (and we are done). If only one of $|a|$ and $|b^*|$ is injective, say $|a|$, then $y|b^*|^2=0$. Letting $|b^*|=0\oplus b_2$, b_2 invertible, and $y=[y_{ij}]_{i,j=1}^2$ it then follows that $y_{12}=y_{22}=0$. Hence $y|b^*|=0$, which implies that $|a|y|b^*|=0\Longrightarrow ayb=0$. Finally, if both $|a|$ and $|b^*|$ are not injective, then upon letting $|a|=0\oplus a_2$, $|b^*|=0\oplus b_2$ and $y=[y_{ij}]_{i,j=1}^2$ it follows that $y_{22}=0\Longrightarrow|a|y|b^*|=0\Longrightarrow ayb=0$. In either case $asc(\triangle_{ab}-\lambda)\le 1$.

If $\lambda\ne-1$, then $\triangle_{ab}-\lambda=(1+\lambda)\triangle_{cb}$ and $(\triangle_{ab}-\lambda)^{-1}(0)\subseteq(\triangle_{a^*b^*}-\overline{\lambda})^{-1}(0)\Longleftrightarrow\triangle_{cb}^{-1}(0)\subseteq\triangle_{c^*b^*}^{-1}(0)$, where we have set $\frac{1}{1+\lambda}a=c$. Let $x\in\triangle_{cb}^{-1}(0)$. Then $cxb-x=0=c^*xb^*-x\Longrightarrow\overline{ranx}$ reduces c, $ker^{\perp}x$ reduces b and $b|_{ker^{\perp}x}$ is invertible (see the proof of Lemma 2.1). Obviously, $x\in\triangle_{cb}^{-1}(0)\Longrightarrow cx$ and $xb\in\triangle_{cb}^{-1}(0)$. Since $\triangle_{cb}^{-1}(0)\subseteq\triangle_{c^*b^*}^{-1}(0)$,

$$\triangle_{c^*b^*}(cx)=0=c\triangle_{c^*b^*}(x)\ \text{ and }\ \triangle_{c^*b^*}(xb)=0=\triangle_{c^*b^*}(x)b,$$

which implies that $c_1=c|_{\overline{ranx}}$ and $b_1=b|_{ker^{\perp}x}$ are normal operators. Assume now that $y\in\triangle_{cb}^{-2}(0)$. Set $\triangle_{cb}(y)=x$, let $y:ker^{\perp}x\oplus kerx\to\overline{ranx}\oplus\overline{ranx}^{\perp}$ have the matrix representation $y=[y_{ij}]_{i,j=1}^2$, and define the quasi-affinity $x_1:ker^{\perp}x\to\overline{ranx}$ as above. Then

$$0=\triangle_{cb}(x)=\triangle_{c_1b_1}(x_1)\oplus 0=\triangle^2_{c_1b_1}(y_{11})\oplus 0.$$

The operators c_1 and b_1 being normal, it follows (from an application of Lemma (2.2)) that $\triangle_{c_1b_1}(y_{11})=0$. Hence $x=\triangle_{c_1b_1}(y_{11})\oplus 0=0\Longrightarrow(\triangle_{ab}-\lambda)(y)=0\Longrightarrow asc(\triangle_{ab}-\lambda)\le 1$. ∎

If $t \in B(H)$ is hyponormal, then so are the operators λt and $t+\lambda$ for every $\lambda \in \mathbf{C}$. Since the inclusion $d_{ab}^{-1}(0) \subseteq d_{a^*b^*}^{-1}(0)$ holds for hyponormal $a, b^* \in B(H)$ [9,20], it follows that

$$(\delta_{ab}-\lambda)^{-1}(0) = \delta_{a(b+\lambda)}^{-1}(0) \subseteq \delta_{a^*(b+\lambda)^*}^{-1}(0) = (\delta_{a^*b^*}-\overline{\lambda})^{-1}(0).$$

Again, if $\lambda \neq -1$, then

$$(\triangle_{ab}-\lambda)^{-1}(0) = (L_{\frac{1}{1+\lambda}a}R_b - 1)^{-1}(0) \subseteq (L_{\frac{1}{1+\overline{\lambda}}a^*}R_{b^*} - 1)^{-1}(0) = (\triangle_{a^*b^*}-\overline{\lambda})^{-1}(0);$$

and if $\lambda = -1$, then

$$(\triangle_{ab}-\lambda)^{-1}(0) = (L_aR_b)^{-1}(0) \subseteq (L_{a^*}R_{b^*})^{-1}(0) = (\triangle_{a^*b^*}-\overline{\lambda})^{-1}(0).$$

COROLLARY 2.4. *If $a, b^* \in B(H)$ are hyponormal, then $asc(d_{ab}-\lambda) \leq 1$ for all $\lambda \in \mathbf{C}$. In particular, d_{ab} has SVEP.*

Proof. Since $(d_{ab}-\lambda)^{-1}(0) \subseteq (d_{a^*b^*}-\overline{\lambda})^{-1}(0)$ for all $\lambda \in \mathbf{C}$, Proposition 2.3 applies. The finite ascent property of $(d_{ab}-\lambda)$ implies SVEP [17]. ∎

REMARKS 2.5. (i) The asymmetric hypotheses on a and b in Corollary 2.4 are not surprising; for the record the corollary fails if a and b are hyponormal (even, subnormal). Specifically, take u to be the (forward) unilateral shift and let $x = \begin{bmatrix} 0 & 0 \\ 1-uu^* & 0 \end{bmatrix}$ (on $H \oplus H$). Choose $a = b = u \oplus 0$ in the case in which $d = \delta$,and $a = u \oplus I$ and $b = (I+u) \oplus 0$ in the case in which $d = \triangle$. Then $x \in d_{ab}^{-1}(0)$, but $x \notin d_{a^*b^*}^{-1}(0)$.

(ii) More is true in Corollary 2.4 in the case in which $d = \delta$. The hypothesis $a, b^* \in B(H)$ are hyponormal implies that a, b^* satisfy Bishop's condition (β) [17]. Hence δ_{ab} satisfies condition (C) [17, Theorem 3.6.10, page 277] (which implies that δ_{ab} has SVEP). Denoting $B(H)$ by V and δ_{ab} by t, this implies that $E_t(F)$ is closed for all closed sets $F \subseteq \mathbf{C}$ if and only if $E_t(F) = V_t(F)$ [17, Proposition 1.4.13], where the algebraic spectral subspace $E_t(F)$ is the largest subspace of V on which all restrictions of $t-\lambda$, $\lambda \in \mathbf{C} \setminus F$, are surjective. (We note here that $E_t(F) = \cap_{\lambda \notin F, n \in \mathbf{N}}(t-\lambda)^n$ for all subsets F of $\mathbf{C}$, because of the finite ascent property of t.)

(iii) Does $\triangle_{ab}$, a and $b^* \in B(H)$ hyponormal, satisfy condition (C)?

For the remainder of this section we assume that $a, b^* \in B(H)$ are hyponormal.

THEOREM 2.6. *$(d_{ab}-\lambda)$ has closed range for each $\lambda \in iso\sigma(d_{ab})$.*

Proof. Before proceeding with the proof proper let us recall that if $t \in B(H)$ is hyponormal, then: (i) $t-\lambda$ is hyponormal; (ii) t quasi-nilpotent implies $t=0$; (iii) the isolated points of $\sigma(t)$ are poles of order one of the resolvent of t; and (iv) the eigenvalues of t are normal eigenvalues. Let $\lambda \in iso\sigma(d_{ab})$.

The case $d_{ab} = \triangle_{a,b}$. We divide the proof into the cases $\lambda = -1$ and $\lambda \neq -1$. Let $\Phi_{ab} = L_aR_b$. If $\lambda = -1$, then $0 \in iso\sigma(\Phi_{ab})$. Since $\sigma(\Phi_{ab}) = \cup\{\sigma(za) : z \in$

$\sigma(b)\}$ (this well known fact follows from [11, Theorem 3.2]), we must have that either $0 \in iso\sigma(b)$ or $0 \in iso\sigma(a)$. Suppose that $0 \in iso\sigma(b)$. (The other case is similarly dealt with.) Then 0 can not be a limit point of $\sigma(a)$. For if 0 is a limit point of $\sigma(a)$, then there exists a sequence $\{\alpha_n\} \in \sigma(a)$ such that $\alpha_n \to 0 \in \sigma(a)$. Choosing a non-zero $z \in \sigma(b)$ we then have a sequence $\{z\alpha_n\} \in \sigma(\Phi_{ab})$ such that $z\alpha_n \to 0$, which contradicts the fact that $0 \in iso\sigma(\Phi_{ab})$. (We remark here that such a choice of z is always possible, for if not then $\sigma(b) = \{0\}$ and b is the zero operator.) The conclusion that 0 can not be a limit point of $\sigma(a)$ implies that either $0 \notin \sigma(a)$ or $0 \in iso\sigma(a)$. If $0 \notin \sigma(a)$, then a is invertible and $ran(\Phi_{ab})$ is closed whenever $ran(\Phi_{Ib})$ is closed. Notice that $0 \in iso\sigma(b) \Longrightarrow 0 \in iso\sigma(b^*)$. Since b^* is hyponormal, $ker(b^*)$ reduces b and $b = 0 \oplus b_2$ with respect to the decomposition $H = ker(b^*) \oplus ker^{\perp}(b^*) = H_1 \oplus H_2$, say, of H. Clearly, the operator $b_2 = b|_{H_2}$ is invertible. Let $x : H_1 \oplus H_2 \to H_1 \oplus H_2$ have the matrix representation $x = [x_{ij}]_{i,j=1}^2$. Then $\Phi_{Ib}(x) = \begin{bmatrix} 0 & x_{12}b_2 \\ 0 & x_{22}b_2 \end{bmatrix}$. The operator b_2 being invertible, Φ_{Ib_2} is invertible, and hence $ran(\Phi_{Ib})$ (and so also $ran(\Phi_{ab})$) is closed. Now let $0 \in iso\sigma(a)$. Then $a = 0 \oplus a_2$ with respect to the decomposition $H = ker(a) \oplus ker^{\perp}(a) = H_1' \oplus H_2'$, say, of H, where the operator $a_2 = a|_{H_2'}$ is invertible. Let $x : H_1 \oplus H_2 \to H_1' \oplus H_2'$ have the matrix representation $x = [x_{ij}]_{i,j=1}^2$. Then $\Phi_{ab}(x) = \begin{bmatrix} 0 & 0 \\ 0 & a_2x_{22}b_2 \end{bmatrix}$. The operator $\Phi_{a_2b_2}$ being invertible, $ran(\Phi_{a_2b_2})$ (and so also $ran(\Phi_{ab})$) is closed. This leaves us with the case $\lambda \neq -1$, which we consider next.

If $\lambda \neq -1$, then $(\triangle_{ab}-\lambda)(x) = axb-(1+\lambda)x$, and it follows from [11, Theorem 3.2] that

$$\sigma(\triangle_{ab} - \lambda) = \bigcup\{\sigma(-(1+\lambda) + za) : z \in \sigma(b)\}.$$

If $\lambda \in iso\sigma(\triangle_{ab})$, then $0 \in iso\sigma(\triangle_{ab} - \lambda)$. There exists a finite set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ of distinct non-zero values of $z \in iso\sigma(b)$, and corresponding to these values of z a finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of distinct non-zero values $\alpha_i \in iso\sigma(a)$ such that $\alpha_i\beta_i = 1 + \lambda$ for all $1 \leq i \leq n$. Let

$$H_1 = \bigvee\nolimits_{i=1}^{n} ker(b-\beta_i)^*, \;\; H_1' = \bigvee\nolimits_{i=1}^{n} ker(a-\alpha_i), \;\; H_2 = H \ominus H_1 \text{ and } H_2' = H \ominus H_1'.$$

Then a and b have the direct sum decompositions $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$, where $a_1 = a|_{H_1'}$ and $b_1 = b|_{H_1}$ are normal operators with finite spectrum, b_1 is invertible, $a_2 = a|_{H_2'}$, $b_2 = b|_{H_2}$, and $\sigma(a_1) \cap \sigma(a_2) = \emptyset = \sigma(b_1) \cap \sigma(b_2)$. Let $x : H_1 \oplus H_2 \to H_1' \oplus H_2'$ have the matrix representation $x = [x_{ij}]_{i,j=1}^2$. Then

$$(\triangle_{ab} - \lambda)x = \begin{bmatrix} (\triangle_{a_1b_1} - \lambda)x_{11} & (\triangle_{a_1b_2} - \lambda)x_{12} \\ (\triangle_{a_2b_1} - \lambda)x_{21} & (\triangle_{a_2b_2} - \lambda)x_{22} \end{bmatrix},$$

where $0 \notin \sigma(\triangle_{a_ib_j} - \lambda)$ for all $1 \leq i,j \leq 2$ such that $i,j \neq 1$. To prove that $ran(\triangle_{ab} - \lambda)$ is closed it thus remains to prove that $ran(\triangle_{a_1b_1} - \lambda)$ is closed. This follows from [1, (2.4) Theorem], since b_1 invertible implies $(\triangle_{a_1b_1} - \lambda)x_{11} = (a_1x_{11} - x_{11}(1+\lambda)b_1^{-1})b_1 = \delta_{a_1((1+\lambda)b_1^{-1})}(x_{11}b_1)$, where the normal operators a_1 and $(1+\lambda)b_1^{-1}$ have finite spectrum.

The case $d_{ab} = \delta_{ab}$. Let $\lambda \in iso\sigma(\delta_{ab})$. Then $0 \in iso\sigma(\delta_{ab} - \lambda)$, where $\sigma(\delta_{ab} - \lambda) = \sigma(a) - \sigma(b + \lambda)$ [11]. Hence $\sigma(a) \cap \sigma(b + \lambda)$ consists of points which are isolated in both $\sigma(a)$ and $\sigma(b + \lambda)$. In particular, $\sigma(a) \cap \sigma(b + \lambda)$ does not contain any limit points of $\sigma(a) \cup \sigma(b + \lambda)$. There exists a finite set $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of distinct values α_i such that $S = \sigma(a) \cap \sigma(b + \lambda)$ and each α_i, $1 \leq i \leq n$, is an isolated point of both $\sigma(a)$ and $\sigma(b + \lambda)$. Let

$$H_1 = \bigvee\nolimits_{i=1}^{n} ker(b - \alpha_i)^*, \quad H_1' = \bigvee\nolimits_{i=1}^{n} ker(a - \alpha_i), \quad H_2 = H \ominus H_1 \text{ and } H_2' = H \ominus H_1'.$$

Then, upon defining the normal operators a_1 and b_1 as before and letting $x : H_1 \oplus H_2 \to H_1' \oplus H_2'$ have the matrix representation $x = [x_{ij}]_{i,j=1}^2$, it is seen that

$$(\delta_{ab} - \lambda)x = \begin{bmatrix} (\delta_{a_1 b_1} - \lambda)x_{11} & (\delta_{a_1 b_2} - \lambda)x_{12} \\ (\delta_{a_2 b_1} - \lambda)x_{21} & (\delta_{a_2 b_2} - \lambda)x_{22} \end{bmatrix},$$

where $a_2 = a|_{H_2'}$, $b_2 = b|_{H_2}$ and $\sigma(a_i) \cap \sigma(b_j + \lambda) = \emptyset$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$ (so that $0 \notin \sigma(\delta_{ab} - \lambda)$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$). That $ran(\delta_{ab} - \lambda)$ is closed now follows (see the proof above). ■

As earlier stated, hyponormal operators are isoloid. The following theorem says that d_{ab} retains this property in the case in which a, b^* are hyponormal.

THEOREM 2.7. *d_{ab} is isoloid.*

Proof. If $\lambda \in iso\sigma(d_{ab})$, then $0 \in iso\sigma(d_{ab} - \lambda)$. Let P denote the spectral projection of $d_{ab} - \lambda$ at 0. Then

$$0 \neq P(B(H)) = \{x \in B(H) : \lim_{n \to \infty} \|(d_{ab} - \lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

We prove that

$$P(B(H)) = \{x \in B(H) : (d_{ab} - \lambda)x = 0\}.$$

Let $d = \delta$. Then upon arguing as in the proof of Theorem (2.6) it is seen that there exist decompositions $H = H_1 \oplus H_2$ and $H = H_1' \oplus H_2'$ such that $x : H_1 \oplus H_2 \to H_1' \oplus H_2'$ has the representation $x = [x_{ij}]_{i,j=1}^2$ and

$$(\delta_{ab} - \lambda)x = [(\delta_{a_i b_j} - \lambda)x_{ij}]_{i,j=1}^2,$$

where $a_1 = a_{|H_1'}$ and $b_1 = b_{|H_1}$ are normal operators with finite spectrum, and where $0 \notin \sigma(\delta_{a_i b_j} - \lambda)$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. Since

$$\{\|(\delta_{a_i b_j} - \lambda)^n x_{ij}\|\}^{\frac{1}{n}} \leq \|(\delta_{ab} - \lambda)^n x\|^{\frac{1}{n}} \to 0$$

as $n \to \infty$ implies that $(\delta_{a_i b_j} - \lambda)$ is quasi-nilpotent for all $1 \leq i, j \leq 2$, it follows that $x_{ij} = 0$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. Consequently,

$$P(B(H)) = \{x = x_{11} \oplus 0 \in B(H) : \lim_{n \to \infty} \|(\delta_{a_1 b_1} - \lambda)^n x_{11}\|^{\frac{1}{n}} = 0\}.$$

The operators a_1 and $b_1 + \lambda$ in $\delta_{a_1(b_1 + \lambda)} = \delta_{a_1 b_1} - \lambda$ being normal,

$$\lim_{n \to \infty} \|(\delta_{a_1 b_1} - \lambda)^n x_{11}\|^{\frac{1}{n}} \Leftrightarrow (\delta_{a_1 b_1} - \lambda)x_{11} = 0$$

[20, Lemma 2], i.e., if and only if 0 is an eigenvalue of $\delta_{a_1 b_1} - \lambda$. Hence

$$P(B(H)) = \{x = x_{11} \oplus 0 \in B(H) : (\delta_{ab} - \lambda)x_{11} \oplus 0 = 0\}.$$

Now let $d = \triangle$, and let $0 \in iso\sigma(\triangle_{ab})$. If $\lambda \neq -1$, then (it follows from Theorem 2.6 that)

$$(\triangle_{ab} - \lambda)x = [(\triangle_{a_i b_j} - \lambda)x_{ij}]_{i,j=1}^2,$$

where a_1 and b_1 are normal invertible operators with finite spectrum, and where $0 \notin \sigma(\triangle_{a_i b_j} - \lambda)$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. Consequently,

$$P(B(H)) = \{x = x_{11} \oplus 0 \in B(H) : \lim_{n \to \infty} \|(\triangle_{a_1 b_1} - \lambda)^n x_{11}\|^{\frac{1}{n}} = 0\}.$$

The operator b_1 being invertible normal,

$$\|(\delta_{a_1 b_1^{-1}} - \lambda)^n x_{11}\|^{\frac{1}{n}} = \|(\triangle_{a_1 b_1} - \lambda)^n x_{11})b_1^{-n}\|^{\frac{1}{n}} \leq \|b_1^{-1}\| \|((\triangle_{a_1 b_1} - \lambda)^n x_{11}\|^{\frac{1}{n}} \to 0$$

as $n \to \infty$. This implies that

$$(\delta_{a_1 b_1^{-1}} - \lambda)x_{11} = 0 \Leftrightarrow (\triangle_{a_1 b_1} - \lambda)x_{11} = 0$$

and hence that

$$P(B(H)) = \{x = x_{11} \oplus 0 \in B(H) : (\triangle_{ab} - \lambda)(x_{11} \oplus 0) = 0\}$$

in the case in which $\lambda \neq -1$. Arguing similarly it is seen that if $\lambda = -1$, then

$$P(B(H)) = \{x = \left[\begin{smallmatrix} x_{11} & 0 \\ x_{21} & 0 \end{smallmatrix}\right] \text{ or } \left[\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & 0 \end{smallmatrix}\right] \in B(H) : (\triangle_{ab} - \lambda)x = \Phi_{Ib}(x) \text{ or } \Phi_{ab}(x) = 0\}. \quad \blacksquare$$

REMARK 2.8. It is clear from the proof of Theorem 2.7 that if $0 \in iso\sigma(d_{ab})$, then $P(B(H)) = d_{ab}^{-1}(0)$. Since $B(H) = P(B(H)) \oplus P^{-1}(0)$ and $P^{-1}(0) \subset d_{ab}(B(H))$, it follows from [16, Theorem 3.4] that 0 is a pole of order one of the resolvent of d_{ab} and $B(H) = d_{ab}^{-1}(0) \oplus d_{ab}(B(H))$.

The *descent* of the operator t, $dsc(t)$, is the smallest non-negative integer n such that $ran(t^n) = ran(t^{n+1})$. The operator t is said to be *Drazin invertible* if there is an operator s and an $n \in \mathbf{N}$ such that

$$t^n st = t^n, sts = s \text{ and } st = ts.$$

It is known that t is Drazin invertible if and only if both $asc(t)$ and $dsc(t)$ are finite (and this is equivalent to the existence of a decomposition $t = t_0 \oplus t_1$, where t_0 is nilpotent and t_1 is invertible) [18]. The following theorem relates the Drazin invertibility of d_{ab} to the finiteness of a subset of $\sigma(d_{ab})$. But before that we recall (once again) from [2, Theorem 3.3] that if $s, t \in B(H)$ are normal, then $cl(\delta_{st}(B(H))) \oplus \delta_{st}^{-1}(0) = \delta_{st}(B(H)) \oplus \delta_{st}^{-1}(0) = B(H)$ if and only if the set $\sigma(s) \cap \sigma(t)$ is isolated in $\sigma(\delta_{st})$. Since $\sigma(\delta_{ab}) = \sigma(a) - \sigma(b)$ (and $\sigma(\triangle_{ab}) = \cup\{\sigma(-1 + za) : z \in \sigma(b)\}$), $0 \in iso\sigma(\delta_{ab})$ (resp., $0 \in iso\sigma(\triangle_{ab})$)if and only if the set $\sigma(a) \cap \sigma(b)$ is isolated in $\sigma(\delta_{ab})$ (resp., the set $\{\alpha\beta : \alpha \in \sigma(a), \beta \in \sigma(b),$ and $\alpha\beta = 1\}$ is isolated in $\sigma(\triangle_{ab})$.)

THEOREM 2.9. *δ_{ab} (resp., $\triangle_{ab}$) is Drazin invertible if and only if the set $\{\sigma(a)\cap\sigma(b)\}$ is isolated in $\sigma(\delta_{ab})$(resp., the set $\{\alpha\in\sigma(a):\alpha^{-1}\in\sigma(b)\}$ is isolated in $\sigma(\triangle_{ab})$).*

Proof. We prove the case in which $d=\delta$; the other case is similarly proved. If δ_{ab} is Drazin invertible, then both $asc(\delta_{ab})$ and $dsc(\delta_{ab})$ are finite. Since $asc(\delta_{ab})\leq 1$ by Corollary (2.4), it follows from [21, Theorem V.6.2] that $asc(\delta_{ab})=dsc(\delta_{ab})\leq 1$ and $\delta_{ab}(B(H))\oplus\delta_{ab}^{-1}(0)=B(H)$. Hence $0\in iso\sigma(\delta_{ab})$, which implies that $\{\sigma(a)\cap\sigma(b)\}$ is isolated in $\sigma(\delta_{ab})$.

Conversely, $\{\sigma(a)\cap\sigma(b)\}$ isolated in $\sigma(\delta_{ab})\Longrightarrow 0\in iso\sigma(\delta_{ab})$. (Clearly, $0\notin\sigma(\delta_{ab})\Longrightarrow$ Drazin invertibility, trivially.) By Remark 2.8, 0 is a pole of order 1 of δ_{ab} and $\delta_{ab}(B(H))\oplus\delta_{ab}^{-1}(0)=B(H)$. Hence $asc(\delta_{ab})=dsc(\delta_{ab})\leq 1$ [17, Proposition 4.10.6] and δ_{ab} is Drazin invertible. ■

Note that the Drazin invertibility of d_{ab} implies the existence of a projection p and a bijection c on $B(H)$ such that $d_{ab}=pc=cp$ (see [17, Proposition 4.10.7]).

3. Weyl's Theorem

The implication *Weyl's theorem $\Longrightarrow$ Browder's theorem* holds, but the reverse implication is in general false. SVEP $\Longrightarrow$ Browder's theorem [8], but this implication fails if one replaces "Browder's theorem" by "Weyl's theorem" [7]. Let $V=B(H)$ and let (as before) $a,b^*\in B(H)$ be hyponormal. Then the SVEP of $d_{ab}\Longrightarrow$ Browder's theorem holds for d_{ab}. Recall from [7, Theorem 2.5] that if an operator t on a Banach space has SVEP, then t satisfies Weyl's theorem $\Longleftrightarrow ran(t-\lambda)$ is closed for every $\lambda\in\sigma_{oo}(t)$. Hence, in view of Theorem (2.6), d_{ab} satisfies Weyl's theorem. More is true.

THEOREM 3.1. *If f is analytic on a neighbourhood of $\sigma(d_{ab})$, then $f(d_{ab})$ satisfies Weyl's theorem.*

Proof. SVEP being stable under the functional calculus [17], d_{ab} has SVEP $\Longrightarrow$ $f(d_{ab})$ has SVEP for each f analytic in a neighbourhood of $\sigma(d_{ab})\Longrightarrow\sigma_b(f(d_{ab}))=\sigma_w(f(d_{ab}))$ [13]. Since the spectral mapping theorem holds for σ_b, we have

$$\sigma_w(f(d_{ab}))=\sigma_b(f(d_{ab}))=f(\sigma_b(d_{ab}))=f(\sigma_w(d_{ab})).$$

To complete the proof we have to show that $f(\sigma_w(d_{ab}))=\sigma(f(d_{ab}))\setminus\sigma_{oo}(f(d_{ab}))$: this follows from Theorem 2.7 and a limit argument applied to [19, Proposition 1]. ■

REMARK 3.2. Browder's theorem is transmitted to and from dual operators, but the same does not in general hold for Weyl' theorem [13]. It is known that if $T\in B(H)$ is hyponormal, then both T and T^* satisfy Weyl's theorem. A formal dual of the operator d_{ab} may be defined by $d_{a^*b^*}$. Does $d_{a^*b^*}$ satisfy Browder's theorem? Notice that $\sigma(d_{a^*b^*})=\overline{\sigma(d_{ab})}$ and $\lambda\in iso\sigma(d_{a^*b^*})\Longrightarrow\lambda\in iso\sigma(d_{ab})\Longrightarrow\lambda\in\sigma_{oo}(d_{ab})\Longrightarrow\overline{\lambda}\in\sigma_{oo}(d_{a^*b^*})$. Since Weyl's theorem holds for d_{ab},

$$\sigma(d_{a^*b^*})\setminus\sigma_{oo}(d_{a^*b^*})=\overline{\sigma(d_{ab})\setminus\sigma_{oo}(d_{ab})}=\overline{\sigma_w(d_{ab})}.$$

Does $\overline{\sigma_w(d_{ab})}=\sigma_w(d_{a^*b^*})$?

Generalized Weyl's Theorem. An operator $t \in B(V)$ is said to be *generalized Fredholm*, or *B-Fredholm*, if there is an $n \in \mathbf{N}$ for which the induced operator $t_n : t^n(V) \to t^n(V)$ is Fredholm in the usual sense, and *generalized Weyl*, or "B-Weyl", if in addition t_n has index zero. The generalized Weyl spectrum $\sigma_{Bw}(t)$ of t is defined to be the set $\{\lambda \in \mathbf{C} : (t-\lambda)$ is not generalized Weyl$\}$, and we say that t satisfies *generalized Weyl's theorem* (resp., *generalized Browder's theorem*) if $\sigma_{Bw}(t) = \sigma(t) \setminus E(t)$ (resp., $\sigma_{Bw}(t) = \sigma(t) \setminus \Pi(t)$), where $E(t) = \{\lambda \in iso\sigma(t) : \lambda$ is an eigenvalue of $t\}$ and $\Pi(t)$ is the set of *poles* of t. (See [3,4,5] for further information.) The implication t satisfies genarlized Weyl's theorem $\Longrightarrow$ t satisfies Weyl's theorem holds, but the reverse implication in general fails [5, Example 4.1]. Operators d_{ab}, recall a, b^* are hyponormal, satisfy generalized Weyl's theorem.

Theorem 3.3. *$\sigma_{Bw}(d_{ab}) = \sigma(d_{ab}) \setminus E(d_{ab})$. Furthermore, if f is analytic on a neighbourhood of $\sigma(d_{ab})$, then $f(d_{ab})$ satisfies generalized Weyl's theorem.*

Proof. Let $\lambda \in \sigma(d_{ab}) \setminus \sigma_{Bw}(d_{ab})$. Since d_{ab} has SVEP, it follows upon arguing as in the proof of [5, Theorem 3.12] and an application of Theorem (2.7) that $\lambda \in iso\sigma(d_{ab}) = E(d_{ab})$. Conversely, if $\lambda \in E(d_{ab})$, then $d_{ab} - \lambda$ is Fredholm of index 0 (by Theorems (2.6) and (2.7)). Hence d_{ab} satisfies generalized Weyl's theorem.

Now let f be as in the statement of the theorem, and let $\sigma_D(d_{ab}) = \{\lambda \in \mathbf{C} : (d_{ab} - \lambda)$ is not Drazin invertible $\}$ denote the Drazin spectrum of d_{ab}. Then $\sigma_D(f(d_{ab})) = f(\sigma_D(d_{ab}))$ [3, Corollary 2.4]. Also, since d_{ab} and $f(d_{ab})$ have SVEP, $\sigma_D(d_{ab}) = \sigma_{Bw}(d_{ab})$ and $\sigma_D(f(d_{ab})) = \sigma_{Bw}(f(d_{ab}))$ [5, Theorem 3.12]. Hence

$$f(\sigma_{Bw}(d_{ab})) = f(\sigma(d_{ab}) \setminus E(d_{ab})) = \sigma_{Bw}(f(d_{ab})).$$

The isoloid property of $\sigma(d_{ab})$, Theorem 2.7, now implies that

$$\sigma_{Bw}(f(d_{ab})) = \sigma(f(d_{ab})) \setminus E(f(d_{ab}))$$

[4, Lemma 2.9], and the proof is complete. ■

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