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A MULTIVALUED FIXED POINT THEOREM IN ULTRAMETRIC SPACES

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Abstract. The purpose of this paper is to prove that a class of generalized contractive multivalued mappings on a spherically complete ultrametric space has a fixed point.

Let (X,d) be a metric space. If the metric d satisfies strong triangle inequality: for all $x,y,z \in X$
$$d(x,y) \leqslant \max\{d(x,z), d(z,y)\},$$
it is called *ultrametric* on X [4]. Pair (X,d) is now an *ultrametric space.*

REMARK. If $X \neq 0$, then the so called discrete metric d defined on X by
$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$
is an ultrametric.

EXAMPLE. For $a \in \mathbb{R}$ let $[a]$ be the entire part of a. By
$$d(x,y) = \inf\{\, 2^{-n} : n \in \mathbb{Z},\ [2^n(x-e)] = [2^n(y-e)]\,\}$$
(here e is any irrational number) an ultrametric d on $\mathbb{Q}$ is defined which determines the usual topology on $\mathbb{Q}$ [4].

An ultrametric space (X,d) is said to be *spherically complete* if every shrinking collection of balls in X has a nonempty intersection.

In [3] authors proved a fixed point theorem for contractive function on spherically complete ultrametric space X. Let us recall: $T\colon X \to X$ is said to be contractive if for every $x,y \in X$, $x \neq y$, $d(Tx,Ty) < d(x,y)$. This result is generalized in [2] for multivalued mappings $T\colon X \to 2^X_c$ (2^X_c is the space of all nonempty compact subsets in X with Hausdorff metric H).

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On the other side, the result from [3] is generalized for a class of functions $T\colon X \to X$ such that for every $x, y \in X$, $x \neq y$

$$d(Tx, Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Now we are going to prove the related result for multivalued mappings.

THEOREM. *Let (X,d) be a spherically complete ultrametric space. If $T\colon X \to 2_c^X$ is such that for any $x, y \in X$, $x \neq y$,*

$$H(Tx, Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty)\}, \tag{1}$$

then T has a fixed point (i.e., there exists $x \in X$ such that $x \in Tx$).

Proof. Let $B_a = B[a; d(a,Ta)]$ denote the closed ball centered at a with radius $d(a,Ta) = \inf_{z \in Ta} d(a,z)$, and let $\mathcal{A}$ be the collection of these spheres for all $a \in X$. The relation

$$B_a \leqslant B_b \quad \text{iff} \quad B_b \subseteq B_a$$

is a partial order on $\mathcal{A}$. Let $\mathcal{A}_1$ be a totally ordered subfamily of $\mathcal{A}$. Since X is spherically complete, $\bigcup_{B_a \in \mathcal{A}_1} B_a = B \neq \emptyset$. Let $b \in B$ and $B_a \in \mathcal{A}_1$. Obviously, $b \in B_a$, so $d(b,a) \leqslant d(a,Ta)$.

Take $u \in T(a)$ such that $d(a,u) = d(a,Ta)$ (it is possible since Ta is a nonempty compact set). Then

$$\begin{aligned} d(b,Tb) &\leqslant \inf_{c \in Tb} d(b,c) \leqslant \max\{d(b,a), d(a,u), \inf_{c \in Tb} d(u,c)\} \\ &\leqslant \max\{d(a,Ta), H(Ta,Tb)\} < \max\{d(a,Ta), d(a,b), d(b,Tb)\} \\ &= \max\{d(a,Ta), d(b,Tb)\} \end{aligned}$$

which is possible only for $d(b,Tb) < d(a,Ta)$. Now, for any $x \in B_b$,

$$\begin{gathered} d(x,b) \leqslant d(b,Tb) < d(a,Ta), \\ d(x,a) \leqslant \max\{d(x,b), d(b,a)\} < d(a,Ta), \end{gathered}$$

so $x \in B_a$. We have just proved that $B_b \subseteq B_a$ for any $B_a \in \mathcal{A}_1$. Thus B_b is an upper bound in $\mathcal{A}$ for the family $\mathcal{A}_1$. By Zorn's lemma there is a maximal element in $\mathcal{A}$, say B_z. We shall prove that $z \in Tz$.

In opposite case, $z \notin Tz$, there exists $\bar{z} \in Tz$, $\bar{z} \neq z$, such that $d(z,\bar{z}) = d(z,Tz)$. Let us prove that $B_{\bar{z}} \subseteq B_z$.

$$\begin{aligned} d(\bar{z}, T\bar{z}) &\leqslant H(Tz, T\bar{z}) < \max\{d(z,\bar{z}), d(z,Tz), d(\bar{z},T\bar{z})\} \\ &= \max\{d(z,Tz), d(\bar{z},T\bar{z})\}, \end{aligned}$$

which is possible only for $d(\bar{z},T\bar{z}) < d(z,Tz)$. Now, for any $y \in B_{\bar{z}}$,

$$\begin{gathered} d(y,\bar{z}) \leqslant d(\bar{z},T\bar{z}) < d(z,Tz), \\ d(y,z) \leqslant \max\{d(y,\bar{z}), d(\bar{z},z)\} \leqslant d(z,Tz), \end{gathered}$$

which means that $y \in B_z$, so $B_{\bar{z}} \subseteq B_z$. But $d(z, \bar{z}) = d(z, Tz) > d(\bar{z}, T\bar{z})$, hence $z \notin B_{\bar{z}}$, so $B_{\bar{z}} \subsetneqq B_z$. This fact contradicts the maximality of B_z. So we have proved that T has a fixed point. ■

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