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ON THE THREE-SPACE-PROBLEM FOR dF SPACES AND THEIR DUALS

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Abstract. It is shown that dF spaces of K. Brauner behave more regularly than DF spaces in connection with the three-space-problem. In particular, this problem has a positive answer in the class of Fréchet spaces for the property of being the strong dual of a barrelled dF space. Thus, a partial positive answer to a question of D. Vogt is obtained.

1. Introduction

K. Brauner introduced in [4] the class of dF spaces which have some similar properties as more familiar DF spaces of A. Grothendieck, but are also significantly different. Thus, for example, this class is stable under passing to an arbitrary closed subspace, which is not the case for DF spaces. Also, each Fréchet space is the strong dual of a dF space, which is not true for DF spaces. In this paper we shall prove some more properties of dF spaces, particularly those connected with the three-space-problem for dF and dual spaces.

A Fréchet space is usually called a dual space if it is the strong dual of a barrelled DF space. D. Vogt posed the question in [18] whether the property of being a dual space is three-space-stable, i.e., whether if in a short exact sequence

$$0 \to F \xrightarrow{j} E \xrightarrow{q} G \to 0 \tag{1}$$

of Fréchet spaces, F and G are dual spaces, the same must be true for E. It was proved in [6; Examples 1 and 2] that the answer is negative; it is negative also in the class of Banach spaces [5]. In this paper we shall show that the answer to the Vogt's question is positive if the class of dual spaces is replaced with the class of strong duals of barrelled dF spaces.

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Passing to the three-space-problem for dF spaces themselves, we shall show that the condition of lifting of precompact subsets with closure for the quotient map is even more tightly connected with this problem than the condition of lifting of bounded subsets with closure was with the three-space-problem for DF, respectively, for dual spaces.

This paper can also be considered as a continuation of our paper [10].

2. Terminology and notation

All spaces mentioned in this article will be Hausdorff locally convex linear topological spaces. For the rest of the terminology we shall mostly follow H.H. Schaefer and M.P. Wolff [14]. If E is a locally convex space, index p in E'_p shall denote the topology of precompact convergence on the dual space E'. Following [12; 23.9], E will be called *polar semi-reflexive* (*p-semi-reflexive*) if $(E'_p)' = E$ and *polar reflexive* (*p-reflexive*) if $(E'_p)'_p = E$.

K. Brauner introduced in [4] the notion of a *dF space* as a p-reflexive space with a fundamental sequence of compact subsets. The class of dF spaces is not comparable with the class of DF spaces of A. Grothendieck. In particular, a barrelled dF space is always DF, but there exist barrelled DF spaces which are not dF (for example, a Banach space of infinite dimension or any dense hyperplane of a barrelled DF space). However, a Montel space is a dF space if and only if it is a DF space. Consequently, the common distribution spaces $\mathcal{D}'$, $\mathcal{E}'$ and $\mathcal{S}'$ are dF spaces.

The class of dF spaces has a nicer behaviour than the class of DF spaces. In particular, each closed subspace of a dF space is of the same kind [4; Prop. 1.9] and a locally convex space is a Fréchet space if and only if it is the strong dual of a dF space. In fact, if E is a Fréchet space, then E'_p is a dF space and we have $(E'_p)'_p = (E'_p)'_b = E$.

A Fréchet space E is said to be a *dual space*, if E is the strong dual X'_b of a barrelled DF space X. Similarly, a short exact sequence (1) of Fréchet spaces is said to be a dual sequence, if there exists a short topologically exact sequence

$$0 \to X \overset{i}{\to} Y \overset{p}{\to} Z \to 0$$

of barrelled DF spaces such that $X'_b = G$, $Y'_b = E$, $Z'_b = F$ and $j = p^t$, $q = i^t$.

3. Dual Fréchet and dF spaces

Proposition 1. *A Fréchet space E is the strong dual of a barrelled dF space if and only if it is a Montel space.*

Proof. If $E = X'_b$, where X is a barrelled dF space, then $X'_b = X'_p = X'_\tau$ (τ is the Mackey topology on X'). It follows that X is reflexive, wherefrom $E = X'_b$ is reflexive. Finally, let A be a bounded absolutely convex subset in $X'_b = X'_p$. Since X is barrelled we have that A is equicontinuous, that is A is weakly compact, i.e., X'_p-compact. Hence, $E = X'_p = X'_b$ is a Montel space.

Conversely, let E be a Fréchet-Montel space. Then $E'_b = E'_p$ is p-reflexive and it has a fundamental sequence of compact subsets, i.e., E'_p is a barrelled dF space. Since $(E'_p)'_p = E$ for each Fréchet space E, taking $X = E'_p$, the proof is complete. ∎

THEOREM 1. *The property "being the strong dual of a barrelled dF space" is three-space-stable in the class of Fréchet spaces.*

Proof. Let (1) be a short exact sequence in the class of Fréchet spaces, such that $F = Z'_b$, $G = X'_b$ where X and Z are barrelled dF spaces. According to Proposition 1, F and G are Fréchet-Montel spaces, and by [15; Prop. 4.4], E is also a Fréchet-Montel space. Consequently, there exists a barrelled dF space Y such that $E = Y'_b$, i.e., the intermediate space E in (1) must be a dual space, if F and G are such. ∎

REMARK 1. Since every barrelled dF space is also a DF space, and since there exist barrelled DF spaces which are not dF, this Theorem gives a partial positive answer to the Vogt's question.

COROLLARY 1. *If the spaces F and G in sequence* (1) *are the strong duals of barrelled dF spaces, then this sequence is a dual short exact sequence.*

Proof. Since Fréchet spaces F, E, G are Montel spaces, we have that q lifts bounded subsets and thus the dual sequence

$$0 \to G'_b \xrightarrow{q^t} E'_b \xrightarrow{j^t} F'_b \to 0 \tag{2}$$

is exact according to [13; 26.12]. Since $F = (F'_b)'_b$, $G = (G'_b)'_b$, $E = (E'_b)'_b$, it follows that the given sequence (1) is dual to the sequence (2). ∎

REMARK 2. It was shown in [7] that the concepts "being the strong dual of a DF space" and "being the strong dual of a barrelled DF space" are different. The same is true when "DF" is replaced by "dF". Indeed, it follows from Proposition 1 that a Fréchet space is Montel if and only if it is the strong dual of a barrelled dF space, while, as we have already mentioned, a locally convex space is a Fréchet space if and only if it is the strong dual of a dF space.

4. Three-space-problem for dF spaces

PROPOSITION 2. *Let*

$$0 \to F \to E \to E/F \to 0 \tag{3}$$

be a short exact sequence of dF spaces. Then

$$0 \to (E/F)'_p \to E'_p \to F'_p \to 0 \tag{4}$$

is a short exact sequence of Fréchet spaces.

Dually, if (3) is a short exact sequence of Fréchet spaces, then (4) is a short exact sequence of dF spaces.

Proof. Suppose that (3) is a short exact sequence of dF spaces and consider the short exact sequence

$$0 \to E'_p | F^\circ \to E'_p \to E'_p / F^\circ \to 0. \tag{5}$$

Let us prove that $E'_p|F^\circ = (E/F)'_p$ and $E'_p/F^\circ = F'_p$. Indeed, $E'_p|F^\circ$ and $(E/F)'_p$ are two comparable topologies ($E'_p|F^\circ \le (E/F)'_p$ is obvious) and F° is a Fréchet space in both of them. So, the Open-Mapping Theorem implies that these topologies coincide. Similarly, the topologies E'_p/F° and F'_p are two comparable topologies ($E'_p/F^\circ \ge F'_p$) and the space $E'/F^\circ \cong F'$ is complete in both of them, so by the same reasoning they coincide. This completes the proof that the sequence (4) is both algebraically and topologically exact.

Suppose now that (3) is a short exact sequence of Fréchet spaces and consider again sequence (5) which is now a sequence of dF spaces (since a quotient and a subspace of each dF space is again a dF space). It is always $E'_p|F^\circ \le (E/F)'_p$ and the reverse inequality is valid by the Banach-Dieudonné Theorem [14; IV.6.3] since E'_p is the finest locally convex topology on E' which coincides with the weak topology E'_σ on equicontinuous subsets of E'. Similarly, we always have $F'_p \le E'_p/F^\circ$. In order to prove the reverse inequality, notice first that $F'_\sigma = E'_\sigma/F^\circ$ and $E'_p|U^\circ = E'_\sigma|U^\circ$ for each absolutely convex neighbourhood U of 0 in E. It follows that $(E'_p/F^\circ)|A = (E'_\sigma/F^\circ)|A$ for each equicontinuous subset A of F. So, $F'_p|A = F'_\sigma|A = (E'_\sigma/F^\circ)|A = (E'_p/F^\circ)|A$. Applying the Banach-Dieudonné Theorem again we conclude that $F'_p \ge E'_p/F^\circ$. Since all the spaces $(E/F)'_p$, E'_p and F'_p are dF spaces, sequence (4) is a short exact sequence of such spaces. ∎

We are going to prove now that for dF spaces the positive answer to the three-space-problem is equivalent to the condition of lifting of precompact subsets with closure (for the quotient map).

THEOREM 2. *Let the spaces F and E/F in the short exact sequence (3) be of the type dF. Then the space E is of the same type if and only if the quotient map $q\colon E \to E/F$ lifts precompact subsets with closure, i.e. for each precompact subset N from E/F there exists a precompact subset M in E such that $N \subset \overline{q(M)}$.*

Proof. Notice first that the condition of lifting of precompact subsets with closure is equivalent to $E'_p|F^\circ = (E/F)'_p$. Suppose that this condition is satisfied and consider again the short exact sequence (5). Let us prove that $E'_p/F^\circ = F'_p$. The inequality $E'_p/F^\circ \ge F'_p$ is always satisfied and the reverse inequality can be proved similarly as the analogous inequality for DF spaces and strong topologies in [14; Prob. IV.24(c)].

It follows that sequence (4) is exact, too, and since $(E/F)'_p$ and F'_p are Fréchet spaces, E'_p is a Fréchet space, too (see [9] and [15]). But then, by the second part of Proposition 2, the sequence

$$0 \to ((E/F)'_p)'_p \to (E'_p)'_p \to (F'_p)'_p \to 0$$

is a short exact sequence of dF spaces, and since dF spaces are semi-reflexive (and so $((E/F)'_p)'_p = E/F$ and $(F'_p)'_p = F$), we have a short exact sequence

$$0 \to E/F \to (E'_p)'_p \to F \to 0.$$

From [15; Prop. 4.2] it follows that E is semi-reflexive and so $(E_p')' = E$. Since the topology $(E_p')_p'$ is not coarser than the topology of the space E, and these two topologies induce the same topologies on F and on E/F, they coincide on E by [8; Lemma 1]. Hence, the space E is of the type dF, too.

Conversely, if all the spaces in sequence (3) are of the type dF, then, as it was shown in the proof of the first part of Proposition 2, we have $E_p'|F^\circ = (E/F)_p'$. But this condition is, as we have already mentioned, equivalent to the condition of lifting of precompact subsets with closure. The proof is complete. ■

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