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INTEGRATED SEMIGROUPS OF UNBOUNDED LINEAR OPERATORS AND C_0-SEMIGROUPS ON SUBSPACES

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Abstract. We give conditions under which an infinitesimal generator of an integrated semigroups of unbounded linear operator becomes an infinitesimal generator of a perturbated semigroup of bounded linear operators. Also, we analyze when a linear operator in a Banach space is an infinitesimal generator of an integrated semigroups of unbounded linear operators.

0. Introduction

Integrated semigroups of unbounded linear operators in Banach spaces have been studied in [8], [9], [10]. This paper is a continuation of such investigations. Here we use also some results of [17] for exponentially bounded integrated semigroups.

We proved in [9] that any semigroup of unbounded linear operators under additional conditions is an exponentially bounded semigroup on a subspace with a possibly stronger norm. On the other hand, in [17] is showed that every exponentially bounded integrated semigroup is an integrated C_0-semigroup on a subspace (with a possibly stronger norm). We obtain this result for semigroups of unbounded operators. Some assertions from the perturbation theory for bounded operators are also used.

1. Preliminaries

Let E be a Banach space with the norm $\|\cdot\|$ and let $(S(t))_{t\geq 0}$ be a family of unbounded linear operators in E. We denote by $D(S(t))$ the domain od $S(t)$ and set

$$D = \{\, x \in D(S(s)S(t)) : S(0)x = 0,\ S(t)x \text{ is strongly continuous for } t \geq 0,$$
$$S(s)S(t)x = \int_0^s (S(r+t) - S(r))x\,dr = S(t)S(s)x, \text{ for } s,t \geq 0. \}$$

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If $D \neq \{0\}$, then $(S(t))_{t\geq 0}$ is said to be *1-integrated semigroup of unbounded linear operators in* E or *an integrated semigroup of unbounded linear operators.*

Differentiation spaces C^n, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are defined as follows:

(i) $C^0 = D$,

(ii) $C^n = \{\, x \in D : S(t)x$ is n-times continuously differentiable function of $t \geq 0\,\}$.

If $x \in D,\ S(t)x \in C^1$, then $S'(s)S(t)x = S(s+t)x - S(s)x,\ s,t \geq 0$. This implies

$$S'(s)S'(t)x = S'(s+t)x,\ x \in C^1,\ s,t \geq 0.$$

Let $\mathcal{N} = \{\, x \in D : S(t)x = 0, t \geq 0\}$. A semigroup $(S(t))_{t\geq 0}$ is called *non-degenerate* if $\mathcal{N} = \{0\}$. We shall observe only non-degenerate semigroups.

Let $\omega \in \mathbb{R}^+ = (0,\infty)$. Define

$$\|x\|_\omega := \sup_{t\geq 0} e^{-\omega t}\|S(t)x\|,\quad x \in \bigcap_{t\geq 0} D(S(t)),$$
$$E_\omega := \{x \in D : \|x\|_\omega < \infty\}$$

We assume in this paper an additional condition:

$$\text{For every } \omega > 0 \text{ exists } C_\omega > 0, \text{ such that } \|\cdot\|_\omega \geq C_\omega\|\cdot\|.$$

Proposition 1. ([8]) *a) If* $\omega_1 \leq \omega_2$ *and* $x \in D$, *then* $\|x\|_{\omega_2} \leq \|x\|_{\omega_1}$. *Hence, if* $\omega_1 \leq \omega_2$, *then* $E_{\omega_1} \subseteq E_{\omega_2}$.

b) If $x \in E_\omega$, *then* $S(t)x \in E_\omega$ *and* $\|S(t)x\|_\omega \leq \dfrac{2e^{\omega t}}{\omega}\|x\|_\omega$.

Let $\overline{E}_\omega$ denote the closure of the set E_ω under norm $\|\cdot\|$ and $S(t)|\overline{E}_\omega$ be the part of $S(t)$, i.e.

$$D(S(t)|\overline{E}_\omega) = \{\, x \in \overline{E}_\omega : x \in D(S(t)) \text{ and } S(t)x \in \overline{E}_\omega \,\}.$$

Theorem 1. ([8]) *Let* $(S(t))_{t\geq 0}$ *be an integrated semigroup of unbounded linear operators in* E.

a) Let $\omega > 0$ *be fixed. Suppose that for every* $t \geq 0$, $S(t)|\overline{E}_\omega$ *is a closed operator in* $\overline{E}_\omega$. *Then* $(E_\omega, \|\cdot\|_\omega)$ *is a Banach space.*

b) If $S(t)$ *is a closed operator in* E, *then* $S(t)|\overline{E}_\omega$ *is closed in* $\overline{E}_\omega$ *for* $t \geq 0$ *and* $\omega > 0$.

For fixed $\omega > 0$ and $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega$, define

$$R^\omega(\lambda)x := \lambda \int_0^\infty e^{-\lambda t} S(t)x\,dt,\quad x \in E_\omega.$$

Observe that as an operator in $(E_\omega, \|\cdot\|_\omega)$, $R^\omega(\lambda)$ is bounded but in general unbounded in $(E, \|\cdot\|)$.

THEOREM 2. ([8)] *The family* $(R^\omega(\lambda))_{\operatorname{Re}\lambda>\omega}$ *is the resolvent of a closed operator* A^ω *in* E_ω *(closed in the* $\|\cdot\|_\omega$ *norm topology),* $\omega>0$.

Let $D(A)=\bigcup_{\omega>0}D(A^\omega)$ and $x\in D(A)$. Then there exists $\omega\in\mathbb{R}^+$ such that $x\in D(A^\omega)$ and there exists $y\in E_\omega$ such that $x=R^\omega(\lambda)y$ for $\operatorname{Re}\lambda>\omega$. Put $Ax=\lambda-y$. Then, we call A *the infinitesimal generator* of the integrated semigroup $(S(t))_{t\geq 0}$.

THEOREM 3. ([10]) *a) For* $x\in D(A)$, $S(t)x$ *is a differentiable function of* t *for* $t\geq 0$ *and* $S'(t)x-x=S(t)Ax$, *or, equivalently,*

$$S(t)x-tx=\int_0^t S(s)Ax\,ds.$$

b) If $x\in D(A)$, *then there exists* $\omega'\in\mathbb{R}^+$ *such that* $R^\omega(\lambda)Ax=AR^\omega(\lambda)x$, $\operatorname{Re}\lambda>\omega$, $\omega\geq\omega'$.

Let $\omega\in\mathbb{R}^+$ and set

$$\begin{aligned} D(A_1^\omega)=\{x\in E_\omega:(i)\ &S(t)x \text{ is differentiable for } t\geq 0 \text{ with respect to } \|\cdot\|;\\ &(ii)\ \exists y\in E_\omega \text{ such that } S'(t)x-x=S(t)y\,\}\end{aligned}$$

For $x\in D(A_1^\omega)$ define $y:=A_1^\omega x$. Let $D(A_1)=\bigcup_{\omega>0}D(A_1^\omega)$ and for $x\in D(A_1^\omega)$ we define $A_1x:=A_1^\omega x$.

THEOREM 4. ([10]) *Let* A *be the infinitesimal generator of the integrated semigroup of unbounded linear operators* $(S(t))_{t\geq 0}$. *Then* $A=A_1$.

For all $\omega\in\mathbb{R}^+$, let $\mathcal{D}_\omega:=\overline{D(A^\omega)}^{\|\cdot\|_\omega}$.

THEOREM 5. ([9]) *Let* $\omega>0$ *be fixed. Then, for* $t\geq 0$, $S(t)\mathcal{D}_\omega\subset\mathcal{D}_\omega$ *and* $(S^\omega(t))_{t\geq 0}=(S(t)|\mathcal{D}_\omega)_{t\geq 0}$ *is an exponentially bounded integrated semigroup on* $(\mathcal{D}_\omega,\|\cdot\|_\omega)$ *with the infinitesimal generator equal to* $A^\omega|\mathcal{D}_\omega$.

2. Integrated semigroups and C_0-semigroups

Let $\omega>0$, and

$$\begin{aligned}\mathcal{D}_\omega^1=\{\,x\in\mathcal{D}_\omega: S(t)x &\text{ is strongly continuously differentiable function of}\\ &t\geq 0 \text{ with respect to the norm } \|\cdot\|_\omega.\}\end{aligned}$$

Then $(S^{'\omega}(t))_{t\geq 0}=(S'(t)|\mathcal{D}_\omega^1)_{t\geq 0}$ is strongly continuous but not exponentially bounded. For $x\in\mathcal{D}_\omega^1$ and $\nu\in\mathbb{R}^+$, let

$$\|x\|_{\omega,\nu}=\sup_{t\geq 0}e^{-\nu t}\|S^{'\omega}(t)x\|,\quad \tilde{\mathcal{D}}_{\omega,\nu}=\{x\in\mathcal{D}_\omega^1:\|x\|_{\omega,\nu}<\infty\}.$$

It is easy to prove that $\tilde{\mathcal{D}}_{\omega,\nu}$ is a Banach space (with respect to the norm $\|\cdot\|_{\omega,\nu}$) and

$$\|S'(t)x\|_{\omega,\nu}\leq e^{\nu t}\|x\|_{\omega,\nu},\ \ t\geq 0,\ x\in\tilde{\mathcal{D}}_{\omega,\nu}.$$

Moreover, if $\nu_1 \leq \nu_2$, then $\|\cdot\|_{\omega,\nu_2} \leq \|\cdot\|_{\omega,\nu_1}$ and $\tilde{\mathcal{D}}_{\omega,\nu_1} \subset \tilde{\mathcal{D}}_{\omega,\nu_2}$. But, on $\tilde{\mathcal{D}}_{\omega,\nu}$ we have lost the strong continuity of $(S'^{\omega}(t))_{t\geq 0}$. In order to enforce the strong continuity, we consider the following subspace

$$\mathcal{D}_{\omega,\nu} = \{\, x \in \tilde{\mathcal{D}}_{\omega,\nu} : \|S'^{\omega}(t)x - x\|_{\omega,\nu} \to 0, \ \text{ for } t \downarrow 0 \,\}.$$

It is clear that $\mathcal{D}_{\omega,\nu}$ is a closed subspace of $\tilde{\mathcal{D}}_{\omega,\nu}$ with the norm $\|\cdot\|_{\omega,\nu}$. Further, $S'^{\omega}(t)\mathcal{D}_{\omega,\nu} \subset \mathcal{D}_{\omega,\nu}$ and $S'^{\omega}(t)$ is strongly continuous on $\mathcal{D}_{\omega,\nu}$ (under the norm $\|\cdot\|_{\omega,\nu}$). The question is, whether after these restrictions, the space $\mathcal{D}_{\omega,\nu}$ contains $D(\tilde{A}^{\omega}) = D(A^{\omega}|\mathcal{D}_{\omega})$? Since $\|S^{\omega}(t)\|_{\omega} \leq \frac{2}{\omega}\, e^{\omega t}$, $t \geq 0$, we have the next theorem which is an appropriate modification of Theorem 4.1 ([17]).

Theorem 6. *Let $\nu > \omega$. Then:*

a) $(\mathcal{D}_{\omega,\nu}, \|\cdot\|_{\omega,\nu})$ is a Banach space. Moreover, $D(\tilde{A}^{\omega}) \subset \mathcal{D}_{\omega,\nu} \subset \mathcal{D}^1_{\omega}$, and

$$\|x\|_{\omega} \leq \|x\|_{\omega,\nu},\ x \in \mathcal{D}_{\omega,\nu},\ \|x\|_{\omega,\nu} \leq \tilde{M}_{\omega}\|x\|_{\omega,\tilde{A}^{\omega}}, x \in D(\tilde{A}^{\omega}).$$

Here $\|x\|_{\omega,\tilde{A}^{\omega}} = \|x\|_{\omega} + \|\tilde{A}^{\omega}x\|_{\omega}$ denotes the graf ω-norm of $\tilde{A}^{\omega}$ and $\tilde{M}^{\omega}$ is a positive constant.

b) $\mathcal{D}_{\omega,\nu}$ is invariant under $S'^{\omega}(t)$ and the restriction $(T^{\omega}(t))_{t\geq 0} = (S'^{\omega}(t)|\mathcal{D}_{\omega,\nu})_{t\geq 0}$ is a strongly continuous semigroup on $(\mathcal{D}_{\omega,\nu}, \|\cdot\|_{\omega,\nu})$ with the infinitesimal generator $\tilde{A}^{\omega,\nu}$ which is the part of $\tilde{A}^{\omega}$ in $\mathcal{D}_{\omega,\nu}$.

Thus, for an integrated semigroup of unbounded linear operators $(S(t))_{t\geq 0}$ which satisfies Theorem 1.a), there exists a family of subspaces with the stronger norms and the restriction $(S(t))_{t\geq 0}$ on these subspaces are integrated C_0-semigroup (Theorem 5. and Theorem 6.).

3. Perturbations. Characterization of generators of integrated semigroups of unbounded linear operators

We will use some results of perturbation theory [1], [7], [14], [17].

Let $(S(t))_{t\geq 0}$ be an integrated semigroup of unbounded linear operators in a Banach space E, with an infinitesimal generator A. The first question is, what are the conditions which we have to impose on an integrated semigroup of bounded linear operators such that its infinitesimal generator is "nearly" close to A?

The second question is, when a linear operator in E is an infinitesimal generator of an integrated semigroup of bounded linear operators?

In Theorem 8 ([10]), we have given an answer to these questions. But, now, we use Theorem 4.5 from [17] and give conditions under which a linear operator is the infinitesimal generator of an exponentially bounded integrated semigroup on the spaces $(\mathcal{D}_{\omega}, \|\cdot\|_{\omega})$, $\omega > 0$.

Theorem 7. (see Theorem 4.5 in [17]) *Let $\tilde{A}^{\omega}$ generate a non-degenerate exponentially bounded integrated semigroup on $(\mathcal{D}_{\omega}, \|\cdot\|_{\omega})$ and let $B^{\omega,\nu}$ be a bounded linear operator on $(\mathcal{D}_{\omega,\nu}, \|\cdot\|_{\omega,\nu})$, $\nu > \omega$. Then:*

a) $A_0^\omega = \tilde{A}^\omega + B^{\omega,\nu}$ *with* $D(A_0^\omega) = D(\tilde{A}^\omega)$ *generates a non-degenerate exponentially bounded integrated semigroup on* $(\mathcal{D}_\omega, \|\cdot\|_\omega)$.

b) The part $A_0^{\omega,\nu}$ *of* A_0^ω *in* $\mathcal{D}_{\omega,\nu}$ *generates a strongly continuous semigroup on* $(\mathcal{D}_{\omega,\nu}, \|\cdot\|_{\omega,\nu})$.

THEOREM 8. *Let* $(S(t)_{t\geq 0}$ *be an integrated semigroup of unbounded linear operators in a Banach space* E *with an infinitesimal generator* A *and* $(S^\omega(t))_{t\geq 0}$ *be the restriction of* $(S(t))_{t\geq 0}$ *on the subspace* $(\mathcal{D}_\omega, \|\cdot\|_\omega)$ *with the infinitesimal generators* $\tilde{A}^\omega, \omega > 0$. *If there exists a family of bounded linear operators* $(B^{\omega,\nu})_{\nu>\omega>0}$ *defined on* $(\mathcal{D}_{\omega,\nu}, \|\cdot\|_{\omega,\nu})$ *such that* $\omega_1 \leq \omega_2$ *implies* $B^{\omega_1,\nu_1} \subset B^{\omega_2\nu_2}$ *(for some* $v_1 > \omega_1$ *and for some* $v_2 > \omega_2$*), then there exists an integrated semigorup of unbounded linear operators defined on a space "no less than"* $\mathcal{D} = \bigcup_{\omega>0} \mathcal{D}_\omega$.

Proof. By Theorem 7, for every $\omega > 0$ the operator $A_0^\omega = \tilde{A}^\omega + B^{\omega,\nu}$ with $D(A_0^\omega) = D(\tilde{A}^\omega)$, generates an exponentially bounded integrated semigroup $(T^\omega(t))_{t\geq 0}$ on $(\mathcal{D}_\omega, \|\cdot\|_\omega)$. Then $\|T^\omega(t)\|_\omega \leq M' e^{\omega' t}$, $t \geq 0$.

We will prove that $\omega_1 \leq \omega_2$ implies $T^{\omega_1}(t) \subset T^{\omega_2}(t)$. Indeed, by Proposition 3.1 ([17])

$$R(\lambda, \tilde{A}^{\omega_1}+B^{\omega_1}) = \lambda \int_0^\infty e^{-\lambda t} T^{\omega_1}(t)\, dt, \quad R(\lambda, \tilde{A}^{\omega_2}+B^{\omega_2,\nu_2}) = \lambda \int_0^\infty e^{-\lambda t} T^{\omega_2}(t)\, dt.$$

Here $\nu_1 > \omega_1, \nu_2 > \omega_2$,

$$\|T^{\omega_1}(t)\|_{\omega_1} \leq M_1' e^{\omega_1' t} \text{ and } \|T^{\omega_2}(t)\|_{\omega_2} \leq M_2' e^{\omega_2' t},\ t \geq 0.$$

For $\lambda > \max\{\omega_1', \omega_2'\}$ the integrals exist in the norms $\|\cdot\|_{\omega_1}$ and $\|\cdot\|_{\omega_2}$ and therefore in the norm $\|\cdot\|$. We have $\tilde{A}^{\omega_1} + B^{\omega_1,\nu_1} \subset \tilde{A}^{\omega_2} + B^{\omega_2,\nu_2}$ and for $x \in \mathcal{D}_{\omega_1}$

$$R(\lambda, \tilde{A}^{\omega_1} + B^{\omega_1,\nu_1})x = R(\lambda, \tilde{A}^{\omega_2} + B^{\omega_2,\nu_2})x.$$

This implies $T^{\omega_1}(t)x = T^{\omega_2}(t)x$, $t \geq 0$. For $x \in \mathcal{D} = \bigcup_{\omega>0} \mathcal{D}_\omega$, set

$$T(t)x = T^\omega(t)x\ , t \geq 0, if\ x \in \mathcal{D}_\omega.$$

It easy to prove that family of operators $(T(t))_{t\geq 0}$ on $\mathcal{D}$ satisfies all conditions for the integrated semigroup. ■

EXAMPLE 1. Let $(S(t)_{t\geq 0}$ be the given integrated semigroup of unbounded linear operators in E. Fix $t_0 > 0$. Then the family $(S'^{\omega,\nu}(t_0))_{\nu>\omega>0}$ satisfies the assumptions of Theorem 8. Namely, for $\omega_1 \leq \omega_2$ we choose $\nu_1 > \omega_1$ and $\nu_2 \geq \nu_1$. Clearly, $S'^{\omega_1,\nu_1}(t_0) \subset S'^{\omega_2,\nu_2}(t_0)$. Moreover, the operators $(S'^{\omega_1,\nu_1}(t_0))_{\nu>\omega>0}$ are bounded on $(\mathcal{D}_\omega, \|\cdot\|_\omega)$. Then, we can take $B^{\omega,\nu} = \tilde{A}^\omega + S'^{\omega_1,\nu_1}(t_0)$.

Let $\mathcal{D}'_\omega = \overline{D(\tilde{A}_0^\omega)} = \overline{D(\tilde{A}^\omega)}$. Then $\mathcal{D}_\omega \subset \mathcal{D}'_\omega \subset \overline{E}_\omega$ because $\mathcal{D}_\omega \subset E_\omega$.

THEOREM 9. *Let* $(T(t))_{t\geq 0}$ *be an integrated semigroup on* $\mathcal{D} = \bigcup_{\omega>0} \mathcal{D}_\omega$ *by Theorem 8. Then there exists an extension* $(\tilde{T}(t))_{t\geq 0}$ *of* $(T(t))_{t\geq 0}$ *defined on* $\mathcal{D}'_\omega = \bigcup_{\omega>0} \mathcal{D}'_\omega$ *if one of two equivalent conditions holds:*

a) The operator $\mathbf{J}^\omega(t) = \int_0^t T(s)A_0^\omega\, ds$, $t \geq 0$, *defined on* $D(A_0^\omega) = D(\tilde{A})$, *is closable (in the norm* $\|\cdot\|$*).*

b) The operator $\mathbf{I}^\omega(t) = \frac{1}{2\pi i}\int_0^t \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} R^\omega(\lambda) A_0^\omega \frac{d\lambda}{\lambda}\, ds$, $t \geq 0$, *defined on* $D((A_0^\omega)^3)$, *is closable (in the norm* $\|\cdot\|$*).*

In the case that one of the equivalent conditions holds, we have that $\tilde{T}^\omega(t) = \tilde{T}(t)|\mathcal{D}'_\omega, t \geq 0$ *are closed operators on* $\mathcal{D}'_\omega$ *for* $\omega > 0$.

Proof. a)$\Rightarrow$ b) Let $\mathbf{J}^\omega(t)$ be closable on $D(A_0^\omega)$. Let $(x_n)_n$ be a sequence on $D(A_0^\omega)$ such that $x_n \to 0$ and $\mathbf{J}^\omega(t)x_n \to y$ as $n \to \infty$. Then $y = 0$. Moreover, we have

$$T^\omega(t)x_n - tx_n = \int_0^t T(t)A_0^\omega x_n\, ds.$$

If $x_n \to 0$ and $T^\omega(t)x_n \to z$ as $n \to \infty$, then $T^\omega(t)x_n - tx_n \to z$ and this implies $z = 0$. Hence $T^\omega(t)$ is closable.

Let $(x_n)_n$ be a sequence on $D\big((A_0^\omega)^3\big)$ such that $x_n \to 0$ and $\mathbf{I}^\omega(t)x_n \to u$ as $n \to \infty$. Then $\mathbf{I}^\omega(t)x_n = T^\omega(t)x_n - tx_n$. We have then that $T^\omega(t)$ is closable and it follows $u = 0$ and $\mathbf{I}^\omega(t)$ is closable. The converse is also true (we use the fact that $\overline{D(A_0^\omega)} = \overline{D((A_0^\omega)^3)}$).

If conditions in a) or b) hold, then the operator $T^\omega(t)$ is closable. Denote the smallest closure by $\overline{T}^\omega(t)$ and define

$$\tilde{T}(t)x = \overline{T}^\omega(t)x, \text{ if } x \in D(\overline{T}^\omega(t)) = \mathcal{D}'_\omega.$$

Then $\tilde{T}(t)$ is well defined and $(\tilde{T}(t))_{t\geq 0}$ is an extension of $(T(t))_{t\geq 0}$. For $x \in D(A_0) = \bigcup_{\omega>0} D(A_0^\omega)$, define

$$A_0 x = A_0^\omega x, \text{ if } x \in D(A_0^\omega).$$

We will prove that the infinitesimal generator A_1 is an extension of the infinitesimal generator A_0. It is sufficient to prove $A_0^\omega \subset A_1^\omega$.

Let $x \in D(A_0^\omega)$. Then

$$T^{'\omega}(t)x - x = T^\omega(t)A_0^\omega x.$$

Thus $T^\omega(t)x$ is differentiable and there exists $y = A_0^\omega x \in E_\omega$ such that $T^{'\omega}(t)x - x = T^\omega(t)y$. This implies $x \in D(A_1^\omega)$ and $A_1^\omega x = A_0^\omega x$. Hence, $A_0^\omega \subset A_1^\omega$. ∎

Let A be a closed linear operator in E. Then, by Theorem 4.2 [17], A generates a non-degenerate exponentially bonunded integrated semigroup if and only if the following conditions hold:

a) There exists some $\lambda \in \mathbb{R}$ such that $R(\lambda, A)$ exists as an everywhere defined bounded linear operator on E.

b) There exists a norm $\|\cdot\|_\omega$ on $D(A)$ such that

(i) $\|x\| \leq c_1\|x\|_\omega \leq c_2\|x\|_A$ for $x \in D(A)$,

(ii) the part A^ω of A in $E_\omega = \overline{D(A)}^{\|\cdot\|_\omega}$ generates a strongly continuous semigrup $(T^\omega(t))_{t\geq 0}$ on E_ω. This leads to the next theorem.

THEOREM 10. *Let E be a Banach space and let $(E_\omega, \|\cdot\|^\omega)_{\omega>0}$ be a nested family of nontrivial subspaces of E such that $\omega_1 \le \omega_2$ implies $E_{\omega_1} \subset E_{\omega_2}$. Let A be a linear operator with the domain and range in $E' = \bigcup_{\omega>0} E_\omega$. Let A^ω denote the part of A in E_ω. In addition, suppose:*

(i) For every $\omega > 0$ there exists $K_\omega > 0$ such that $\|x\|^\omega \ge K_\omega \|x\|$, $x \in E_\omega$.

(ii) If $\omega_1 \le \omega_2$ and $x \in E_{\omega_1}$, then $\|x\|^{\omega_2} \le \|x\|^{\omega_1}$.

(iii) For every $\omega > 0$ there exists $\lambda_\omega \in \mathbb{R}$ such that $R(\lambda_\omega, A^\omega)$ exists as an everywhere defined bounded linear operator on E_ω.

(iv) There exists a norm $\|\cdot\|^{\omega,\nu}$ on $D(A^\omega)$ such that

$$\|x\|^\omega \le c_1(\omega)\|x\|^{\omega,\nu} \le c_2(\omega)\|x\|^\omega_{A^\omega},$$

and the part of A^ω in $E_{\omega,\nu} = \overline{D(A^\omega)}^{\|\cdot\|_{\omega,\nu}}$ generates a strongly continuous semigroup $(T^{\omega,\nu}(t))_{t\ge 0}$ on $E_{\omega,\nu}$.

Then there exists an integrated semigroup $(S(t))_{t\ge 0}$ of (in general unbounded) linear operators defined "no less than" on $E' = \bigcup_{\omega>0} E_\omega$.

Proof. By Theorem 4.2 [17], for every $\omega > 0$, A^ω generates an exponentially bounded integrated semigroup $(S^\omega(t))_{t\ge 0}$. Then $A^{\omega_1} \subset A^{\omega_2}$ implies $S^{\omega_1}(t) \subset S^{\omega_2}(t)$ (see the proof of Theorem 8) and we can define an integrated semigorup $(S(t))_{t\ge 0}$ on $E' = \bigcup_{\omega>0} E_\omega$. ■

REMARKS. The norms $\|\cdot\|^\omega$ and $\|\cdot\|^{\omega,\nu}$ in Theorem 10 are different from the norms $\|\cdot\|_\omega$ and $\|\cdot\|_{\omega,\nu}$. Namely, the norms $\|\cdot\|_\omega$ and $\|\cdot\|_{\omega,\nu}$ are defined in the case where we have an integrated semigroups but in Theorem 10 we have not got such an assumption.

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