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DISTRIBUTIONS GENERATED BY BOUNDARY VALUES OF FUNCTIONS OF THE NEVANLINNA CLASS N

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Abstract. In this work necesarry and sufficient conditions are given for a regular distribution in D' to be distribution generated by the boundary function of some function from the Nevanlinna class N.

1. Introduction

1.1. Denotations which will be used in the paper

Let $\mathcal{U}$ denote the open unit disk in $\mathcal{C}$, i.e., $\mathcal{U} = \{z \in \mathbf{C} \mid |z| < 1\}$, $T = \partial\mathcal{U}$ and Π^+ denote the upper half-plane, i.e., $\Pi^+ = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$. For a given function f which is analytic on some region Ω we will write $f \in H(\Omega)$.

For a function f, $f\colon \Omega \to \mathbf{C}^n$, $\Omega \subseteq \mathbf{R}^n$, $x \in \Omega$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \in \mathbf{N} \cup \{0\}$, $D^\alpha f = D_x^\alpha f(x)$ denotes

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.$$

$L^p(\Omega)$ is the space of locally integrable functions on Ω, i.e., $f(x) \in L^p_{loc}(\Omega)$ if $f(x) \in L^p(\Omega')$, for every bounded subregion Ω' of Ω.

1.2. The Nevanlinna class N defined on $\mathcal{U}$ and on Π^+ and some properties of N

The Nevanlinna class, $N(\mathcal{U})$, consists of all $f \in H(\mathcal{U})$ whose characteristic function

$$T(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

is bounded for $0 \le r < 1$.

It is known [4] that a function $f \in H(\mathcal{U})$ belongs to the class $N(\mathcal{U})$ if and only if it is the quotient of two bounded analytic functions. It is also known [4] that for

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each function $f \in N(\mathcal{U})$ the nontangential limit $f^*(e^{i\theta})$ exists almost everywhere on T and $\log|f^*(e^{i\theta})|$ is integrable over T, unless $f \equiv 0$.

For a function $f \in H(\mathcal{U})$, $\log(1+|f|)$ is subharmonic, so the integrals

$$L(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log(1+|f(re^{i\theta})|)\,d\theta$$

increase with r. Thus the (possibly infinite) limit $\|f\| = \lim_{r\to 1^-} L(r,f)$ exists, and the inequalities

$$\log^+ x \le \log(1+x) \le \log 2 + \log^+ x, \quad (x > 0)$$

show that f belongs to $N(\mathcal{U})$ if and only if $\|f\| < \infty$

In the case of the upper half-plane Π^+, $N(\Pi^+)$ consists of all $f \in H(\Pi^+)$, for which

$$\sup_{0<y<\infty} \int_{-\infty}^{+\infty} \log(1+|f(x+iy)|)dx < \infty.$$

Note. From now on, we will write N instead of $N(\Pi^+)$.

1.3. Some notions of distributions

$C^\infty(\mathbf{R}^n)$ denotes the space of all complex valued infinitely differentiable functions on $\mathbf{R}^n$ and $C_0^\infty(\mathbf{R}^n)$ denotes the subspace of $C^\infty(\mathbf{R}^n)$ that consists of those functions of $C^\infty(\mathbf{R}^n)$ which have compact support. Support of a continuous function f, denoted by $\mathrm{supp}(f)$, is the closure of $\{x | f(x) \neq 0\}$ in $\mathbf{R}^n$.

$D = D(\mathbf{R}^n)$ denotes the space of $C_0^\infty(\mathbf{R}^n)$ functions in which convergence is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in D$ converges to $\varphi \in D$ in D as $\lambda \to \lambda_0$ if and only if there is a compact set $K \subset \mathbf{R}^n$ such that $\mathrm{supp}(\varphi_\lambda) \subseteq K$ for each λ, $\mathrm{supp}(\varphi) \subseteq K$ and for every n-tuple α of nonnegative integers the sequence $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $\{D_t^\alpha \varphi(t)\}$ uniformly on K as $\lambda \to \lambda_0$.

$D' = D'(R^n)$ is the space of all continuous, linear functionals on D, where continuity means that $\varphi_\lambda \to \varphi$ in D as $\lambda \to \lambda_0$, implies $\langle T, \varphi_\lambda\rangle \to \langle T, \varphi\rangle$, as $\lambda \to \lambda_0$, $T \in D'$. D' is called the space of distributions.

Note. $\langle T, \varphi\rangle$ denotes the value of the functional T, when it acts on the function φ.

Let $\varphi \in D$ and let $f(x) \in L^1_{loc}(\mathbf{R}^n)$. Then the functional T_f from D to C, defined by:

$$\langle T_f, \varphi\rangle = \int_{R^n} f(t)\varphi(t)\,dt, \quad \varphi \in D$$

is a distribution on D called regular distribution generated with f.

2. Main results

The idea for Theorem 1 and Theorem 2 comes from the following theorem, that is given in [7].

THEOREM. *Necessary and sufficient condition for a measurable function $\varphi(e^{i\theta})$, defined on T to coincide almost everywhere on T with boundary value $f^*(e^{i\theta})$ of some function $f(z)$ of the Nevanlinna class $N(\mathcal{U})$, is the existence of a sequence of polynomials $\{P_n(z)\}$ such that:*

(i) $\{P_n(e^{i\theta})\}$ converges to $\varphi(e^{i\theta})$ almost everywhere on T,

(ii) $\overline{\lim\limits_{n\to\infty}} \int_0^{2\pi} \log^+ |P_n(e^{i\theta})|\, d\theta < \infty.$

THEOREM 1. *Let T_{f^*} be the distribution in D' generated with the boundary value $f^*(x)$ of some function $f(z)$ from the space N. Then there exist a sequence of polynomials $\{P_n(z)\}$, $z \in \Pi^+$ and a respective sequence of distributions $\{T_n\}$, $T_n \in D'$ generated with the boundary values $P_n^*(x)$ of $P_n(z)$, satisfying ($T_n = T_{P_n^*}$):*

(i) $T_n \to T_{f^*}, \quad n \to \infty \quad in \quad D'$,

(ii) $\overline{\lim\limits_{n\to\infty}} \int_{-\infty}^{\infty} \log(1 + |P_n^*(x)|)|\varphi(x)|\, dx < \infty, \quad \forall \varphi \in D.$

Proof. Let the conditions of the Theorem be satisfied. Since $f \in N$, it follows that $f \in H(\Pi^+)$ and there exists a constant $C > 0$, such that

$$\int_{-\infty}^{\infty} \log(1 + |f(x+iy)|)\, dx \le C, \quad \text{for all} \quad x + iy \in \Pi^+. \tag{1}$$

Let $\{y_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} y_n = 0$.

We consider the sequence of functions $\{F_n(z)\}$, defined by $F_n(z) = f(z + iy_n)$. Then $F_n(z)$ are analytic functions on $\Pi^+ \cup \mathbf{R}$. Using the theorem of Mergelyan we get that for a compact subset K of $\Pi^+ \cup R$, whose complement is connected and for the function $F_n(z)$ there exists a polynomial $P_n(z)$, such that $|F_n(z) - P_n(z)| < \varepsilon_n$, for $z \in K$, where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Now we will prove (i) and (ii).

Let $\varphi \in D$ and let $K \subset \mathbf{R}$ be a compact set that contains $\operatorname{supp}(\varphi)$ and whose complement (in $\mathbf{C}$) is connected. (It is possible to be $K = \operatorname{supp}(\varphi)$).

(i) We have:

$$|\langle T_n, \varphi\rangle - \langle T_{f^*}, \varphi\rangle| = \left|\int_{-\infty}^{+\infty} P_n^*(x)\varphi(x)\, dx - \int_{-\infty}^{+\infty} f^*(x)\varphi(x)\, dx\right| =$$
$$= \left|\int_{-\infty}^{+\infty} [P_n^*(x) - f^*(x)]\varphi(x)\, dx\right| \le \int_K |P_n^*(x) - f^*(x)||\varphi(x)|\, dx \overset{\varphi \in D \subset S}{\le}$$
$$\le M(\int_K |P_n^*(x) - f^*(x)|\, dx \le M\varepsilon_n' m(K) \to 0 \quad \text{as} \quad n \to \infty$$

where $m(K)$ is the Lebesque measure of the set K, M is positive real number and $\varepsilon_n' = \varepsilon_n + |f^*(x) - F_n(x)|$. Clearly, $\varepsilon_n' \to 0$ as $n \to \infty$. From the above computations we conclude that $\langle T_n, \varphi\rangle \to \langle T_{f^*}, \varphi\rangle$ as $n \to \infty$, for every $\varphi \in D$.

(ii)

$$\begin{aligned}
\int_{-\infty}^{+\infty} & \log(1+|P_n^*(x)|)|\varphi(x)|\,dx \\
&= \int_{-\infty}^{+\infty} \log(1+|P_n^*(x)-F_n(x)+F_n(x)|)|\varphi(x)|\,dx \\
&\le \int_{-\infty}^{+\infty} \log(1+|P_n^*(x)-F_n(x)|+|F_n(x)|)|\varphi(x)|\,dx \\
&= \int_K \log(1+|F_n(x)|+|P_n^*(x)-F_n(x)|)|\varphi(x)|\,dx \\
&\le \int_K [\log(1+|F_n(x)|)+|P_n^*(x)-F_n(x)|]|\varphi(x)|\,dx \\
&= \int_K \log(1+|F_n(x)|)|\varphi(x)|dx + \int_K |P_n^*(x)-F_n(x)||\varphi(x)|\,dx \\
&\le M\int_K \log(1+|F_n(x)|)dx + M\int_K |P_n^*(x)-F_n(x)|\,dx \\
&\le M\int_K \log(1+|f(x+iy_n)|)\,dx + M\varepsilon_n m(K) \overset{(1)}{\le} \\
&\le MC + M\varepsilon_n m(K) \to M, \quad as \quad n\to\infty.
\end{aligned}$$

In the proof of (ii) we used the inequality $|a+b|\le |a|+|b|$, monotonicity of the function $\log x$ and the inequality $\log(1+a+b)\le \log(1+a)+b$, for $a,b>0$. ∎

THEOREM. *Let φ_0 be a locally integrable function on $\mathbf{R}$ and T_{φ_0} be the distribution in D' generated by φ_0. Let there exists a sequence of polynomials $P_n(z)$, $z\in\Pi^+$ such that the following conditions are satisfied:*

(i) The sequence of distributions, generated by the boundary values $P_n^*(x)$ of $P_n(z)$ converges to T_{φ_0} in D' as $n\to\infty$.

(ii) $\overline{\lim\limits_{n\to\infty}} \int_{-\infty}^{+\infty} \log(1+|P_n(x+iy)|)|\varphi(x)|\,dx < \infty$, *for all* $x+iy\in\Pi^+$, $\varphi\in D$.

Then there exists a function $f\in H(\Pi^+)$, such that

$$\int_K \log(1+|f(x+iy)|)\,dx < C < \infty, \quad \forall (x+iy)\in\Pi^+$$

for every compact subset K of $\mathbf{R}$ and

$$\lim_{y\to 0^+}\int_{-\infty}^{+\infty} f(x+iy)\varphi(x)\,dx = \langle T_{\varphi_0},\varphi\rangle, \quad \varphi\in D.$$

Proof. Let the conditions of the Theorem be satisfied. In [6] it is proven that the condition (i), i.e.

$$\lim_{n\to\infty}\int_R P_n^*(x)\varphi(x)\,dx = \int_R \varphi_0(x)\varphi(x)\,dx, \quad \varphi\in D,$$

implies

$$\text{there exists a function } f \in H(\Pi^+), \text{ such that the sequence of polynomials } \{P_n(z)\} \text{ converges to } f(z) \text{ uniformly on compact subsets of } \Pi^+ \text{ as } n \to \infty. \tag{2}$$

First we will prove that this analytic function f also satisfies

$$\int_K \log(1 + |f(x+iy)|)\, dx < C < \infty, \quad \forall (x+iy) \in \Pi^+$$

for every compact subset K of R.

In order to do that, we will use the second condition (ii), i.e.,

$$\overline{\lim_{n\to\infty}} \int_K \log(1 + |P_n(x+iy)|)|\varphi(x)|\, dx < C < \infty,\ \forall (x+iy) \in \Pi^+,\ \varphi \in D. \tag{3}$$

Let K be a compact subset of $\mathbf{R}$. Then there exists $\varphi(x) \in C_0^\infty(\mathbf{R})$, $\varphi(x) = 1$, $\forall x \in K$. Substituting $\varphi(x)$, chosen in this way, in (3), we get

$$\overline{\lim_{n\to\infty}} \int_K \log(1 + |P_n(x+iy)|)\, dx < C < \infty,\ \forall (x+iy) \in \Pi^+. \tag{4}$$

Now,

$$\begin{aligned}\int_K \log(1 + |f(x+iy)|)\, dx &= \int_K \lim_{n\to\infty} \log(1 + |P_n(x+iy)|)\, dx \\ &\le \overline{\lim_{n\to\infty}} \int_K \log(1 + |P_n(x+iy)|)\, dx \overset{(4)}{<} C < \infty,\end{aligned}$$

i.e. $\int_K \log(1 + |f(x+iy)|)\, dx < C < \infty$, for every compact subset K of $\mathbf{R}$ and for every $x + iy \in \Pi^+$.

It remains to prove that

$$\lim_{y\to 0^+} \int_{-\infty}^{+\infty} f(x+iy)\varphi(x)\, dx = \langle T_{\varphi_0}, \varphi \rangle, \quad \varphi \in D. \tag{5}$$

Let $\varphi \in D$ and $\operatorname{supp}(\varphi) = K \subset R$. Then

$$\begin{aligned}&\lim_{y\to 0^+} \int_{\mathbf{R}} f(x+iy)\varphi(x)\, dx \overset{(2)}{=} \lim_{y\to 0^+} \int_K \lim_{n\to\infty} P_n(x+iy)\varphi(x)\, dx \overset{u.c.}{=} \\ &= \lim_{y\to 0^+} \lim_{n\to\infty} \int_K P_n(x+iy)\varphi(x)\, dx = \lim_{n\to\infty} \lim_{y\to 0^+} \int_K P_n(x+iy)\varphi(x)\, dx = \\ &\qquad = \lim_{n\to\infty} \int_K P_n^*(x)\varphi(x)\, dx = \int_{\mathbf{R}} \varphi_0(x)\varphi(x)\, dx = \langle T_{\varphi_0}, \varphi \rangle, \quad \forall \varphi \in D.\end{aligned}$$

In the proof above, we used that

$$\lim_{y\to 0^+} \lim_{n\to\infty} \int_K P_n(x+iy)\varphi(x)\, dx = \lim_{n\to\infty} \lim_{y\to 0^+} \int_K P_n(x+iy)\varphi(x)\, dx. \tag{6}$$

We will show that (6) holds.

Let us consider the sequence $\{g_n(y)\}$, where

$$g_n(y) = \int_K P_n(x+iy)\varphi(x)\, dx, \quad x + iy \in K_1,$$

K_1 is any compact set in Π^+ whose elements $z \in K_1$ satisfy $\mathrm{Re}\, z \in K$. Since $\{P_n(x+iy)\}$ converges to $f(x+iy)$, uniformly on K_1 as $n \to \infty$, we have

$$\lim_{n\to\infty} g_n(y) = \lim_{n\to\infty} \int_K P_n(x+iy)\varphi(x)\,dx = \int_K f(x+iy)\varphi(x)\,dx = g(y),$$

i.e., the sequence $\{g_n(y)\}$ converges to $g(y)$, as $n \to \infty$. We will prove that the convergence is uniform.

$$\begin{aligned} 0 \le \sup_y |g_n(y) - g(y)| &= \sup_y \left| \int_K P_n(x+iy)\varphi(x)\,dx - \int_K f(x+iy)\varphi(x)\,dx \right| \\ &= \sup_y \left| \int_K [P_n(x+iy) - f(x+iy)]\varphi(x)\,dx \right| \\ &\le \sup_y \int_K |P_n(x+iy) - f(x+iy)||\varphi(x)|\,dx \\ &\overset{\varphi\in D\subset S}{\le} M \sup_y \int_K |P_n(x+iy) - f(x+iy)|\,dx. \end{aligned}$$

Since

$$\lim_{n\to\infty} \int_K |P_n(x+iy) - f(x+iy)|\,dx = 0,$$

we get that $\lim_{n\to\infty} \sup |g_n(y) - g(y)| = 0$.

So we have proved that $\{g_n(y)\}$ converges to $g(y)$ uniformly on K_1, as $n \to \infty$, which implies (6). This concludes the proof of (5) and of Theorem 2. ■

Comment. This work is a continuation of [6], where two similar theorems were proved in the spaces H^p, $1 \le p < \infty$.

Similar theorems can be given in the Smirnov space.

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