МАТЕМАТИЧКИ ВЕСНИК
54 (2002), 139–144

UDK 517.984.3
оригинални научни рад
research paper

DISPLACEMENT STRUCTURE OF GENERALIZED INVERSE $A_{T,S}^{(1,2)}$

Mei Qin and Yimin Wei

Abstract. It is well known that matrices with a UV-displacement structure possess generalized inverse with a VU-displacement structure. Estimation for the displacement rank of $A_{T,S}^{(1,2)}U - VA_{T,S}^{(1,2)}$ are presented, where $A_{T,S}^{(1,2)}$ is the $(1,2)$-inverse of A with prescribed range T and null space S. We extend the results due to G. Heinig and F. Hellinger, Wei and Ng, Cai and Wei for the Moore-Penorse inverse, group inverse and weighted Moore-Penrose inverse, respectively.

1. Introduction and Preliminaries

The aim of the present paper is to study the generalized inverse of a structured matrix. To begin with, we recall some facts concerning the regular inversion of structured matrices, which are motivated by [5, 6]. If the rank of a matrix's displacement is small, fast algorithms for the matrix are available.

A matrix is called *matrix with displacement structure* [2, 3, 8] if and only if the rank of the matrix $AU - VA$ or $A - VAU$ is small compared with the order of the matrix A. The rank of $AU - VA$ is said to be *Sylvester UV-displacement rank* and the rank of $A - VAU$ is called the *Stein UV-displacement rank* of A, since A is the solution of a *Sylvester* or *Stein* equation, respectively.

As is well known, fast inversion algorithms for matrix A can be constructed if A is a matrix with displacement structure. We are interested in the generalized inverse with as small as possible displacement rank. We present the estimate for the rank of $A_{T,S}^{(1,2)}U - VA_{T,S}^{(1,2)}$, where $A_{T,S}^{(1,2)}$ is the $(1,2)$-inverse of A, in Section 2. Then we give an explicit estimate for *general displacement*, which is defined in the paper, Section 3. Our results cover the previous results [2, 4, 14] for the Moore-Penrose inverse, weighted Moore-Penrose, inverse and group inverse, respectively.

AMS Subject Classification: 15 A 09,65 F 20

Keywords and phrases: Displacement, $(1,2)$-inverse, structured matrix.

Communicated at the 5th International Symposium on Mathematical Analysis and its Applications, Niška banja, Yugoslavia, October, 2–6, 2002.

This project is supported by National Natural Science Foundation of China under grant 19901006 and 10171021 and Doctoral Point Foundation of China.

A matrix X is said to be a $(1,2)$-inverse of A if the following condition is fulfilled:

$$XAX = X, \qquad AXA = A.$$

Let $A \in C_r^{m\times n}$, T and S are subspaces of C^n and C^m, respectively, the dimensions of T and S are r and $m-r$, then A has a unique $(1,2)$-inverse X satisfying $R(X) = T$, $N(X) = S$ if and only if $R(A) \bigoplus S = C^m$, $Ker(A) \bigoplus T = C^n$. We denote X by $A_{T,S}^{(1,2)}$.

If $G \in C^{n\times m}$, and G satisfies $R(G) = T$, $N(G) = S$, then we get [9]

$$A_{T,S}^{(1,2)} = (GA)_g G = G(AG)_g, \quad AA_{T,S}^{(1,2)} = AG(AG)_g, \quad A_{T,S}^{(1,2)}A = (GA)_g GA,$$

where $(AG)_g$ and $(GA)_g$ are group inverses of AG and GA.

From Jordan canonical form theory, we get that for any complex $m\times n$ matrix A with $rank(A) = r$ with prescribed range T and null space S, there exist nonsingular matrices R and N [13] such that

$$A = R^{-1}\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} N, \quad G = N^{-1}\begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} R \tag{1}$$

where R is an $m \times m$ nonsingular matrix and N is $n \times n$. Note that A_{11}, G_{11} are nonsingular matrices. Now we can write $A_{T,S}^{(1,2)}$ of A in the form [13],

$$A_{T,S}^{(1,2)} = N^{-1}\begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} R \tag{2}$$

Let

$$Q = A_{T,S}^{(1,2)}A, \quad P = I - Q, \quad Q_* = AA_{T,S}^{(1,2)}, \quad P_* = I - Q_*. \tag{3}$$

Obviously, Q and P are oblique projections, where $Im(A)$ denotes the range of A and $Ker(A)$ is the null space of A. It is easy to check that

$$\begin{aligned} &Im(Q) = Im(A_{T,S}^{(1,2)}) = Ker(P) = Im(G), \qquad Im(Q_*) = Ker(P_*) = Im(A), \\ &Im(P_*) = Ker(A_{T,S}^{(1,2)}) = Ker(Q_*) = Ker(G), \quad Im(P) = Ker(Q) = Ker(A). \end{aligned} \tag{4}$$

For recent results on the generalized inverse $A_{T,S}^{(1,2)}$, we refer to [10, 11, 12, 15].

2. Sylvester displacement rank

Throughout the paper, $U \in C^{n\times n}$ and $V \in C^{m\times m}$ are some fixed matrices. The operator

$$d(U,V) = AU - VA$$

is called the *UV-displacement* of A. To distinguish this displacement concept from the more general case in Section 3 we call it *the Sylvester UV-displacement*, since A is the solution of a certain Sylvester equation. We can easily find that for a nonsingular matrix A with *UV-displacement* structure the inverse matrix A^{-1} possesses a *VU-displacement* structure, and the relation

$$rank(A^{-1}V - UA^{-1}) = rank(AU - VA)$$

holds.

We want to get an estimate for the *VU-displacement* rank of the generalized inverse of a matrix with displacement structure. First we show that the displacement structure of $A_{T,S}^{(1,2)}$ is the following representation.

PROPOSITION 2.1. *Let $A \in C^{m\times n}$, $U \in C^{n\times n}$ and $V \in C^{m\times m}$, and $A_{T,S}^{(1,2)}$ is the $(1,2)$-inverse of A. Then*

$$A_{T,S}^{(1,2)}V - UA_{T,S}^{(1,2)} = A_{T,S}^{(1,2)}VP_* - PUA_{T,S}^{(1,2)} - A_{T,S}^{(1,2)}(AU - VA)A_{T,S}^{(1,2)}, \qquad (5)$$

where Q, P, Q_* and P_* are defined in (4).

Proof. The relation is immediately proved by the following equation

$$A_{T,S}^{(1,2)}(AU - VA)A_{T,S}^{(1,2)} = (I-P)UA_{T,S}^{(1,2)} - A_{T,S}^{(1,2)}V(I-P_*). \quad \blacksquare \qquad (6)$$

From (4) we obtain the following Corollary.

COROLLARY 2.1. *The VU-displacement rank of $A_{T,S}^{(1,2)}$ satisfies the following estimate:*

$$rank(A_{T,S}^{(1,2)}V - UA_{T,S}^{(1,2)}) \leq rank(AU - VA) + rank(Q_*VP_*) + rank(PUQ). \quad (7)$$

Proof. We will prove $dim(PUA_{T,S}^{(1,2)}) = rank(PUQ)$, and $rank(A_{T,S}^{(1,2)}VP) = rank(QVP)$.

$$rank(PUA_{T,S}^{(1,2)} = dim[PUIm(A_{T,S}^{(1,2)})] = dim[Im(PUQ)] = rank(PUQ), \qquad (8)$$

the second equality can be proved similarly.

Taking these into account, we obtain the above estimate. ■

Now we aim to obtain the estimate for the second and third terms on the right-hand side.

PROPOSITION 2.2. *With the notation above, we get the estimate*

$$rank(Q_*VP_*) + rank(PUQ) \leq rank(UG - GV). \qquad (9)$$

Proof. We set $F = UG - GV$ and make partitions

$$NUN^{-1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad RVR^{-1} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}. \qquad (10)$$

From (1) we write the matrix F in the following form

$$\begin{aligned} NFR^{-1} &= NUN^{-1}NGR^{-1} - NGR^{-1}RVR^{-1} \\ &= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \\ &= \begin{bmatrix} U_{11}G_{11} - G_{11}V_{11} & -G_{11}V_{12} \\ U_{21}G_{11} & 0 \end{bmatrix}. \end{aligned}$$

From [7] we know that

$$rank\begin{bmatrix} U_{11}G_{11}-G_{11}V_{11} & -G_{11}V_{12} \\ U_{21}G_{11} & 0 \end{bmatrix} \geq rank(-G_{11}V_{12})+rank(U_{21}G_{11})$$
$$= rank(V_{12})+rank(U_{21}) = rank\begin{bmatrix} 0 & V_{12} \\ 0 & 0 \end{bmatrix} + rank\begin{bmatrix} 0 & 0 \\ U_{21} & 0 \end{bmatrix}$$
$$= rank(Q_*VP_*)+rank(PUQ).$$

where G_{11} is nonsingular. Then the proof is over. ■

From Propositions 2.1 and 2.2 we derive the first main result.

Theorem 2.1. *Let $A \in C^{m\times n}$ and $A_{T,S}^{(1,2)}$ be its $(1,2)$-inverse. Then*

$$rank(A_{T,S}^{(1,2)}V - UA_{T,S}^{(1,2)}) \leq rank(AU-VA)+rank(UG-GV).$$

Proposition 2.3. *If $AU=UG, VA=GV$, then we get the estimate:*

$$rank(A_{T,S}^{(1,2)}V - UA_{T,S}^{(1,2)}) \leq 2\ rank(AU-VA). \tag{11}$$

3. Displacement structure for generalized displacement

In order to generalize Theorem 2.1 we introduce a generalized displacement concept [2]. Let $a=[a_{ij}]_0^1$ denote a nonsingular 2×2 matrix. We associate a with the polynomial in two variables

$$a(\lambda,\mu)=\sum_{i,j=0}^{1} a_{ij}\lambda^i\mu^j$$

and the linear fractional function

$$f_a(\lambda)=\frac{a_{10}+a_{11}\lambda}{a_{00}+a_{01}\lambda}. \tag{12}$$

For any fixed $U\in C^{n\times n}$ and $V\in C^{m\times m}$, the generalized (a,U,V) displacement of $A\in C^{m\times n}$ generated by $a(\lambda,\mu)$ is defined by

$$a(V,U)A=\sum_{i,j=0}^{1} a_{ij}V^iAU^j.$$

If $a=d\equiv\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we just get *Sylvester displacement* that we have discussed. If $a=d\equiv\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we get *Stein displacement.*

Lemma 3.1. [2] *Let $a=[a_{ij}]_0^1$, $b=[b_{ij}]_0^1$, $c=[c_{ij}]_0^1$, $d=[d_{ij}]_0^1$ be nonsingular 2×2 matrices such that*

$$a=b^Tdc. \tag{13}$$

Then

$$(b_{00}+b_{01}\lambda)^{-1}a(\lambda,\mu)(c_{00}+c_{01}\mu)^{-1}=d(f_b(\lambda),f_c(\mu)) \tag{14}$$

for all λ,μ with $b_{00}+b_{01}\lambda\neq 0$ and $c_{00}+c_{01}\mu\neq 0$.

LEMMA 3.2. [2] *If* $d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, *then there exist* 2×2 *matrices* b, c *such that (13) holds and* $b_{00} + b_{01}V$ *and* $c_{00} + c_{01}U$ *are invertible.*

Taking Lemma 3.1 and Lemma 3.2 together, we obtain the following

PROPOSITION 3.1. [2] *If* b *and* c *be matrices satisfying the conditions in Lemma 3.2, then for* $A \in C^{m\times n}$,

$$a(V,U)A = (b_{00} + b_{01}V)[Af_c(U) - f_b(V)A](c_{00} + c_{01}U).$$

The following is very important to generalize Theorem 2.1 for general (a, U, V) displacement.

PROPOSITION 3.2. *(a) If* $\psi = [\psi_{ij}]_0^1$ *is nonsingular and* $\psi_{00}+\psi_{01}V$ *is invertible, then* $rank(Q_*VP_*) = rank(Q_*\tilde{V}P_*)$, *where* $\tilde{V} \equiv f_\psi(V)$ *is defined in (12).*

(b) If $\phi = [\phi_{ij}]_0^1$ *is nonsingular and* $\phi_{00} + \phi_{01}U$ *is invertible, then*

$$rank(PUQ) = rank(P\tilde{U}Q),$$

where $\tilde{U} \equiv f_\phi(U)$ *is defined in (12).*

Proof. We define

$$\mathcal{S} = Ker(G) \cap Ker(GV), \quad \mathcal{S}_1 = Ker(G) \ominus \; CalS.$$

We show that Q_*VP_* is one-to-one on $\mathcal{S}_1$. If $Q_*VP_*x = 0$ and $x \in \mathcal{S}_1$, then $VP_*x \in Ker(Q_*) = Ker(G)$. That means $GVP_*x = GVx = 0$. Noting that $x \in Ker(G)$, we conclude $x \in \mathcal{S}$. Thus $x = 0$.

Furthermore, Q_*VP_* vanishes on $\mathcal{S}$. Since $Q_*VP_*x = Q_*Vx = (AG)_gAGVx = 0$ for all $x \in \mathcal{S}$ we have

$$rank(Q_*VP_*) = dim(\mathcal{S}_1). \tag{15}$$

Analogously we define $\tilde{\mathcal{S}} = Ker(G) \cap Ker(G\tilde{V})$, $\tilde{\mathcal{S}_1} = Ker(G) \ominus \tilde{\mathcal{S}}$, and we get

$$rank(Q_*\tilde{V}P_*) = dim(\tilde{\mathcal{S}_1}). \tag{16}$$

Now we show that the invertible matrix $\overline{\psi}_{00} + \overline{\psi}_{01}V$ bijectively maps $\mathcal{S}$ onto $\tilde{\mathcal{S}}$. Suppose that $x \in \mathcal{S}$. Then $x, Vx \in Ker(G)$. Hence $y \equiv (\overline{\psi}_{10} + \overline{\psi}_{11}V)x$ and $z \equiv (\overline{\psi}_{00} + \overline{\psi}_{01}V)x$ are all contained in $Ker(G)$. Thus $y = \tilde{V}z$ and we conclude that $z, \tilde{V}z \in Ker(G)$, which implies $z \in \tilde{\mathcal{S}}$. Convesely, with the same arguments we get $(\overline{\psi}_{00} + \overline{\psi}_{01}V)^{-1}z \in \mathcal{S}$ for $z \in \tilde{\mathcal{S}}$.

This implies

$$dim(\mathcal{S}_1) = dim[Ker(G)] - dim(\mathcal{S}) = dim[Ker(G)] - dim(\tilde{\mathcal{S}}) = dim(\tilde{\mathcal{S}_1}).$$

According to (15) and (16), we get assertion (a).

Assertion (b) is proved analogously. ■

Now we can generalize Theorem 2.1 for general (a, U, V) displacement.

THEOREM 3.1. *If* a, b *are* 2×2 *nonsingular matrices, then*

$$rank[a(U,V)A^{(1,2)}_{T,S})] \leq rank[a^T(V,U)A] + rank[b(U,V)G]. \tag{17}$$

Proof. According to Lemma 3.2 there exist 2×2 matrices w, x, y, z such that $w_{00} + w_{01}U$, $x_{00} + x_{01}V$, $y_{00} + y_{01}U$, $z_{00} + z_{01}V$ are invertible and $a = w^T dz$, $b = x^T dy$. Hence,

$$\begin{aligned} &rank[a(U,V)A^{(1,2)}_{T,S}] - rank[a^T(V,U)A] \\ &\quad = rank[f_w(U)A^{(1,2)}_{T,S} - A^{(1,2)}_{T,S} f_z(V)] - rank[f_z(V)A - Af_w(U)] \\ &\leq rank[Pf_w(U)Q] + rank[Q_* f_z(V)P_*] = rank[Pf_y(U)Q] + rank[Q_* f_x(V)P_*] \\ &\qquad \leq rank[f_y(U)G - Gf_x(V)] = rank[b(U,V)G]. \quad \blacksquare \end{aligned}$$

4. Concluding remarks

In this paper we study the displacement structure of $(1,2)$-inverse of a singular matrix. It is natural to ask if we can extend our results to linear operators in Hilbert space. This will be the future research.

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(received 12.12.2002)

Mei Qin, Institute of Mathematics, Fudan University, Shanghai, 200433, P.R.of China
Yimin Wei, Department of Mathematics, Fudan University, Shanghai, 200433, P.R.of China
E-mail: ymwei@fudan.edu.cn