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UNBOUNDED SOLUTIONS TO SOME SYSTEMS OF CONSERVATION LAWS—SPLIT DELTA SHOCK WAVES

Marko Nedeljkov

Abstract. The solution concept is based on splitting of delta measures along regular curves in $\mathbb{R}^2$. Now, their product with piecewise smooth functions with discontinuities along such curves makes sense.

The differentiation is defined by their mapping into the usual Radon measure space (naturally embedded into the space of Schwartz distributions).

1. Introduction

Let f_i, g_i be real valued C^1-functions, $i = 1, 2$. Suppose that the Riemann problem for the hyperbolic system

$$u_t + (f_1(u)v + f_2(u))_x = 0 \tag{1}$$

$$v_t + (g_1(u)v + g_2(u))_x = 0, \tag{2}$$

$$u(x,0) = \begin{cases} u_0, & x < 0 \\ u_1, & x > 0 \end{cases} \qquad v(x,0) = \begin{cases} v_0, & x < 0 \\ v_1, & x > 0 \end{cases} \tag{3}$$

has no admissible elementary solution (see [6]).

A very good survey paper on general problems of non-existence of classical solutions is [3], for example.

Our aim is to find a solution in the form of so called "delta shock wave" solution with speed c_1, where $u(x,t) = G(x - c_1 t)$, $v(x,t) = H(x - ct) + s_1(t)\delta(x - ct)$, and $s \in C^1([0,\infty))$.

Let us remind the reader that a wave is overcompressive if

$$\lambda_2(U_0) > \lambda_1(U_0) \geq c_1 \geq \lambda_2(U_1) > \lambda_1(U_1), \tag{4}$$

where λ_1 and λ_2 are the eigenvalues of system (1–2), and $U_i = (u_i, v_i)$, $i = 1, 2$. In this paper, admissible delta shock wave means that it is overcompressive.

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The references to some special cases of such systems one can find in [7].

Delta shock waves could occur when the total variation of initial data is not small (see for example [9]), or in some other special cases (see for example [2], [10] or [1]).

Our primary task will is to find sufficient conditions for the existence of a solution to the Riemann problem (1–3) depending on different initial data.

Colombeau generalized functions are used in [7] for solving the Riemann problems (1–3). The main idea of [7] was the splitting of a generalized function associated with the delta distribution contained in the solution along a shock propagation curve into two parts: the left- and the right-hand side.

In this paper we use the similar idea, but within a "more classical" setting. This will be explained in the second section.

We refer to [4] for some other possibilities in solving of system (1–3).

2. A measure space

Let $\mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2 : t > 0\}$ and $\overline{\mathbb{R}^2_+} = \{(x,t) \in \mathbb{R}^2 : t \geq 0\}$.

Suppose that $\overline{\mathbb{R}}^2_+$ is divided with a finite number of connected closed sets with mutual intersections of Lebesgue measure zero, $Z_1, \dots, Z_n$, where $Z_i = \Omega_i \cup \partial\Omega_i$, $\Omega_i \neq \emptyset$ is an open set, $i = 1, \dots, n$. Suppose $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, and let Γ_i, $i = 1, \dots, m$, be piecewise smooth curves dividing the sets Ω_i, $i = 1, \dots, n$.

Let $C(Z_i)$ be the space of bounded continuous functions on Z_i with values in $\mathbb{R}$ equipped with the L^∞-norm. Its dual is $\mathcal{M}(Z_i)$, the space of measures, $i = 1, \dots, n$.

Let $C_\Gamma = \prod_{i=1}^n C(Z_i)$ and equip it with the product topology. Its dual is $\mathcal{M}_\Gamma = \prod_{i=1}^n \mathcal{M}(Z_i)$ with the dual pairing

$$(D, G) = \sum_{i=1}^n (D_i, G_i),\ D = (D_1, \dots, D_n) \in \mathcal{M}_\Gamma,\ (G_1, \dots, G_n) \in C_\Gamma.$$

Clearly, if $D \in \mathcal{M}_\Gamma$ and $G \in C_\Gamma$, then the multiplication defined by

$$D \cdot G = (D_1 \cdot G_1, \dots, D_n \cdot G_n) \in \mathcal{M}_\Gamma \tag{5}$$

makes sense. The mapping $C(\mathbb{R}) \to C_\Gamma$ is defined by

$$G^* f = (f(G_1), \dots, f(G_n)),$$

where $G \in C_\Gamma$ and $f \in C(\mathbb{R})$.

Let $\mathcal{M}(\overline{\mathbb{R}^2_+})$ be the dual space for $C(\overline{\mathbb{R}}^2_+)$, equipped with the L^∞-norm. Let us define the mapping

$$m\colon \mathcal{M}_\Gamma \to \mathcal{M}(\overline{\mathbb{R}^2_+}),\ m(D) = D_1 + \dots + D_n, \tag{6}$$

$$(m(D), g) = \sum_{i=1}^n (D_i, g|_{Z_i}),\ g \in C(\mathbb{R}). \tag{7}$$

2.1. The solution concept

The solution concept which will be used in this paper can be described by the following steps.

The first step: Multiplications and compositions in the space $\mathcal{M}_\Gamma$ (such operations are closed in it) before the differentiation.

The second step: The mapping of $\mathcal{M}_\Gamma$ by m into $\mathcal{M}(\overline{\mathbb{R}^2_+})$ first, and then the mapping of this space by the usual embedding into the space of distributions.

The third step: The differentiation in the space of distributions.

If the result equals zero, then we have a solution.

REMARK 1. The above concept is an analogue of the usual one in the conservation laws theory:

(1) Make all the operations in the space of locally integrable functions before the differentiation.

(2) Map the space of locally integrable functions into the space of distributions by the usual embedding.

(3) Make the differentiation in the space of distributions.

If the result equals zero, the function is a solution.

3. The main assertion

We shall say that G is a step function with value (y_0, y_1) if

$$G(y) = \begin{cases} y_0, & y < 0 \\ y_1, & y > 0, \end{cases}$$

and denote its jump by $[G] = y_1 - y_0$.

Here are the new definitions of the delta shock wave and the delta locus.

DEFINITION 1. The delta shock wave solution to Riemann problem (1–3) is a solution in the form

$$\begin{aligned} u(x,t) &= G(x-ct) \\ v(x,t) &= H(x-ct) + s_0(t)\delta^-(x-ct) + s_1(t)\delta^+(x-ct), \end{aligned} \tag{8}$$

where $s_i(0) = 0$, $s_i \in C^1(\mathbb{R}_+)$, $i = 1, 2$, G and H are the step functions with values (u_0, u_1) and (v_0, v_1), respectively, $\delta^-(x-ct)$ is the delta measure on the set $\overline{\mathbb{R}}^2_+ \cap \{x \le ct\}$ supported by the line $x = ct$, while $\delta^+(x-ct)$ is the delta measure supported by the same line, but on the set $\overline{\mathbb{R}}^2_+ \cap \{x \ge ct\}$.

Solutions of such a type are observed in [5] for the first time.

DEFINITION 2. Let $(u_0, v_0) \in \mathbb{R}^2$. A point (u_1, v_1) is said to be in the delta locus of the point (u_0, v_0) if there exists a solution in the sense of the above solution concept to (1–3). A point is in the admissible delta locus if the delta shock solution is overcompressive.

A sufficient condition for a point (u_1, v_1) to be in the delta locus of a point (u_0, v_0) for system (1–3) is given in the following theorem. It is similar to the one obtained by using Colombeau generalized functions in [7].

THEOREM 1. *A point (u_1, v_1) is in the delta locus of a point (u_0, v_0) for the Riemann problem (1–3) if the following holds:*

(a) $g_1(u_0) \neq g_1(u_1)$.

(b) $f_1(u_0) \dfrac{k_1(g_1(u_1) - c)}{g_1(u_1) - g_1(u_0)} = f_1(u_1) \dfrac{k_1(g_1(u_0) - c)}{g_1(u_1) - g_1(u_0)}$,

where $k_1 = c[G] - [f_1(G)H + f_2]$, and c is a speed of the delta shock wave.

Proof. The substitution of functions in (8) into (1–3) and the use of the Rankine-Hugoniot conditions gives the following equation

$$
\begin{aligned}
&(-c[G]+[f_1(G)H+f_2(G)])\delta(x-ct)+(f_1(s_0(t)u_0)\delta^-(x-ct)+f_1(s_1(t)u_1)\delta^+(x-ct))_x \\
&= (-c[G] + [f_1(G)H + f_2(G)])\delta(x-ct) + (f_1(s_0(t)u_0) + f_1(s_1(t)u_1))\delta'(x-ct) = 0.
\end{aligned}
$$

Suppose that $u_0 \neq u_1$. From the above equation, one obtains the value of the speed c and the coupling equations for s_0 and s_1:

$$
c = \frac{[f_1(G)H + f_2(G)]}{[G]} \quad \text{and} \quad s_0(t)f_1(u_0) + s_1(t)f_1(u_1) = 0. \tag{9}
$$

Doing the same for the second equation, one obtains

$$
\begin{aligned}
&- c[H] + (s_0(t) + s_1(t))'\delta(x-ct) - c(s_0(t) + s_1)\delta'(x-ct) \\
&\quad + [g_1(G)H + g_2(G)]\delta(x-ct) + (s_0(t)g_1(u_0) + s_1(t)g_1(u_1))\delta'(x-ct) = 0.
\end{aligned} \tag{10}
$$

Since c is already determined,

$$
(s_0(t) + s_1(t))' = c[H] - [g_1(G)H + g_2(G)], \text{ i.e. } s_0(t) + s_1(t) = k_1 t,
$$

and k_1 is called Rankine-Hugoniot deficit ([4]). Now, one obtains the following system of equations for s_0 and s_1:

$$
\begin{aligned}
(g_1(u_0) - c)s_0(t) &+ (g_1(u_1) - c)s_1(t) = 0 \\
s_0(t) &+ s_1(t) = k_1 t.
\end{aligned}
$$

If $g_1(u_0) = g_1(u_1)$, then $k_1 = 0$, i.e. there is no delta shock wave solution. Otherwise,

$$
s_0(t) = \frac{k_1(g_1(u_1) - c)}{g_1(u_1) - g_1(u_0)} \qquad s_1(t) = \frac{k_1(c - g_1(u_0))}{g_1(u_1) - g_1(u_0)} \tag{11}
$$

are determined. Using these values and the second equation in (9), one gets the assertion of the theorem.

Now, let $u_0 = u_1$. Then, from the above equations, one can see that $k_1 = 0$ and there is no delta shock wave solution to (1–3). ∎

REMARK 2. One can easily see that the given definitions and the algorithm can be used for more general systems of conservation laws, linear in one variable, but non necessary in evolution form:

$$(f_1(u)v + f_2(u))_t + (f_3(u)v + f_4(u))_x = 0$$
$$(g_1(u)v + g_2(u))_t + (g_3(u)v + g_4(u))_x = 0.$$

REFERENCES

[**1**] Danilov, V.G., Shelkovich, V.M., *Propagation and interaction of delta-shock waves of a hyperbolic system of conservation laws*, Preprint.

[**2**] Hayes, B.T., Le Flock, P.G., *Measure solutions to a strictly hyperbolic system of conservation laws*, Nonlinearity, **9** (1996), 1547–1563.

[**3**] Keyfitz, B.L., *Conservation laws, delta shocks and singular shocks*, In: M. Grosser embed-dinget al: *Nonlinear Theory of Generalized Functions*, Research Notes in Math., Champman Hall/CRC, 1999.

[**4**] Keyfitz, B.L., Kranzer, H.C., *Spaces of weighted measures for conservation laws with singular shock solutions*, J. Diff. Eq., **118**, 2 (1995), 420–451.

[**5**] Korchinski, D.J., *Solution of a Riemann Problem for a* 2×2 *System of Conservation Laws Possesing No Classical Weak Solution*, Ph.D. Thesis, Adelphi University, Garden City, New York, 1977.

[**6**] Lax, P.D., *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, Philadelphia, 1973.

[**7**] Nedeljkov, M., *Delta and singular delta locus for one dimensional systems of conservation laws*, to appear in Math. Mod. Appl. Sci.

[**8**] Nedeljkov, M., Oberguggenberger, M., *Paper in preparation.*

[**9**] Tan, D., Zhang, T. and Zheng, Y., *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, J. Diff. Eq., **112** (1994), 1–32.

[**10**] Yang, H., *Riemann problems for a class of coupled hyperbolic systems of conservation laws*, J. Diff. Eq., **159** (1999), 447–484.

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Department for Mathematics and Informatics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Yugoslavia

E-mail: marko@im.ns.ac.yu