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GENERALIZED FRACTIONAL CALCULUS WITH APPLICATIONS IN MECHANICS

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Abstract. In the studies concerned with the stability of the viscoelastic rod of fractional type it is shown that many problems are described by stability of coupled systems of differential equations with fractional derivatives. Here, we are dealing with analysis of such systems in the space of distributions. The main result of this work is stated in Theorem 5.

1. Introduction

Fractional calculus is the field of mathematical analysis which deals with investigation and application of integrals and derivatives of real order. Fractional calculus may be considered as an old and yet novel topic. It is an old topic since its developing started from G.W. Leibniz and L. Euler. Afterwards there is a list of mathematicians who have provided important contribution to the theory. It is a novel topic since it has become an object of specialized conferences and treatises only about 25 years ago. First conference was "The First Conference on Fractional Calculus and its Applications" in 1974, first monograph was written by K.B. Oldham and Spanier who after a joint collaboration starting in 1968 published a book devoted to fractional calculus in 1974.

Nowadays, the list of texts and proceedings devoted solely or partly to fractional calculus and its applications includes about a dozen of titles, among which the encyclopedic treatise by Samko, Kilbas and Marichev [5] is the most prominent.

In recent years the interest for fractional calculus has been stimulated by its wide survey of applications which includes viscoelasticity, fractional capacitor theory, electrical circuits, electroanalytical chemistry, biology, physics, etc.

Viscoelasticity seems to be the field with the most extensive application of fractional operators. Using fractional derivatives for modeling viscoelastic materials is quite natural. Our aim in this note is to formulate and analyse a mathematical model for a viscoelastic rod described by a constitutive equation containing fractional derivatives that impacts against a rigid wall.

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2. The model

We consider light viscoelastic rod with a body B of mass m which is atached to its end. At the moment t_0 the rod is about to impact a rigid wall. Rod is moving at the speed v_0 and its length in undeformed state is L. At the moment $t > t_0$ the rod impacts the wall, its length gets reduced to $l(t)$, and at the contact point contact force (stress) f appears.

Strain of the rod $x(t)$ is given as

$$x(t) = \frac{L - l(t)}{L}, \qquad t \geq 0.$$

The theory of constitutive equations gives relation between f and x through the generalized Zener model $\tau_f f^{(\alpha)} + f = \tau_x x^{(\alpha)} + x$, where τ_f and τ_x are constants called time of relaxation of stress and strain, and $f^{(\alpha)}$ and $x^{(\alpha)}$ denote the Riemann-Liouville fractional derivative of stress and strain. The Riemann-Liouville fractional derivative for $0 < \alpha < 1$ is defined as

$$u^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^1 (t-\tau)^{-\alpha} u(\tau)\, d\tau, \qquad t \geq 0. \tag{1}$$

Differential equation of motion or the second Newton's law for the body B is expressed by $mx^{(2)} = -f$, where m is the mass of body B, and $x^{(2)}$ is its acceleration.

Initial conditions are $x(t_0) = 0, \quad x^{(1)}(t_0) = v_0$. Without loss of generality we take $m = 1$, $v_0 = 1$, $t_0 = 0$. Also we take $\tau_f = a$ and $\tau_x = b$ and we get the system

$$\begin{aligned} x^{(2)}(t) &= -f(t) \\ af^{(\alpha)}(t) + f(t) &= bx^{(\alpha)}(t) + x(t) \\ x(0) &= 0 \\ x^{(1)}(0) &= 1 \end{aligned} \tag{2}$$

We denote by T^* the moment of separation. This is the time instant when contact force between rod and wall is equal to zero, $f(T^*) = 0$.

The second law of thermodynamics requires that $b > a > 0$ (see [1]).

Our main goal is to estimate $|x'(T^*)|$. We claim that $b > a$ implies $|x'(T^*)| \leq 1$, for otherwise the rod would represent a perpetuum mobile of the first kind.

3. Mathematical preliminaries: Spaces $\mathcal{D}'_+$ and $\mathcal{S}'_+$. Operator $f_\alpha *$

Denote by $\mathcal{D}$ and $\mathcal{S}$ Schwartz test function spaces of compactly supported and rapidly decreasing smooth functions on the real line $\mathbb{R}$. Their duals are well known spaces of distributions $\mathcal{D}'$ and tempered distributions $\mathcal{S}'$. The space of Schwartz's distributions vanishing on interval $(-\infty, 0)$ is denoted by $\mathcal{D}'_+$, i.e. $\mathcal{D}'_+ = \{\, f \in \mathcal{D}'(\mathbb{R}) : \operatorname{supp} f \subset [0, \infty) \,\}$.

If $f \in \mathcal{D}'_+$ and $g \in \mathcal{D}'_+$, then convolution[1] $f * g$ exists in $\mathcal{D}'_+$. Recall that the operation of convolution is associative and commutative in $\mathcal{D}'_+$, so that $\mathcal{D}'_+$ is a convolution algebra with Dirac delta function as the unit element, since $\delta * f = f$.

[1] For localy integrable functions on $\mathbb{R}$ vanishing on $(-\infty, 0)$, $f * g(t) = \int_0^t g(t-\tau) f(\tau)\, d\tau$, $t \in \mathbb{R}$. For $f \in \mathcal{S}'_+$, the convolution is $\langle f \times g, \varphi(x+y) \rangle$, $\varphi \in \mathcal{S}_+$.

Space $\mathcal{S}'_+$ is the intersection of $\mathcal{D}'_+$ and $\mathcal{S}'$. It is a convolution algebra, a subalgebra of $\mathcal{D}'_+$.

We consider the equation in $\mathcal{D}'_+$

$$a * u = f, \tag{3}$$

where a and f are specified elements of $\mathcal{D}'_+$ and u is a unknown generalized function in $\mathcal{D}'_+$. The solution of (3) for $f = \delta$ is called the fundamental solution of convolution operator $a*$.

The Laplace transform (or Fourier-Laplace transform) of an $f \in \mathcal{S}'_+$ is defined as

$$\mathcal{L}f(s) = \left\langle f(t), \eta(t)e^{-st} \right\rangle, \qquad s \in D_+, \tag{4}$$

where η is a smooth function on $\mathbb{R}$ equal to one on $(-a, \infty)$ and zero on $(-\infty, -a)$ and $D_+ = \{z : \Re z > 0\}$. Note, $\mathcal{L}f$ is an analytic function on D_+ not depending on the cutoff function η.

The space $\mathcal{S}'_+$ is isomorphic, via Laplace transform, to the corresponding space of holomorphic functions in D_+ [6]. This is the claim of the following theorem.

THEOREM 1. [6] *Let $h(p)$ be holomorphic in half-plane $\Re p > \sigma_0$. Then there exists a distribution $T \in \mathcal{D}'_+$ the Laplace transform of which is $h(p)$ if and only if there exist numbers $\sigma_1 \in \mathbb{R}$, $\sigma_0 \le \sigma_1$, C and $k \ge 0$, such that*

$$|h(p)| \le C(1 + |p|)^k, \qquad p \in \mathbb{C},\ \Re p > \sigma_1.$$

Let $a \in \mathcal{S}'_+$ be given. Then by Theorem 1, equation (3) is uniquely solvable in $\mathcal{S}'_+$ if and only if there exist $p, q \in \mathbb{R}$, $C > 0$ such that

$$\frac{1}{|\mathcal{L}a(z)|} \le C\frac{(1 + |z|)^p}{x^q}, \qquad z = x + iy \in D_+. \tag{5}$$

Next we define a generalized function f_α, for arbitrary $\alpha \in \mathbb{R}$,

$$f_\alpha(x) = \begin{cases} \theta(x)\frac{x^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0 \\ f^{(N)}_{\alpha+N}, & \alpha \le 0, \alpha + N > 0, N \in \mathbb{N} \end{cases} \tag{6}$$

where $(\cdot)^{(N)}$ is the N-th derivative in the sense of distributions and $\theta(x)$ is the Heaviside's function. Then $f_\alpha \in \mathcal{D}'_+$ and the group property

$$f_\alpha * f_\beta = f_{\alpha+\beta} \qquad \alpha, \beta \in \mathbb{R} \tag{7}$$

is valid.

Let us consider the convolution operator $f_\alpha *$ in algebra $\mathcal{D}'_+$. Since $f_0 = \theta' = \delta$, (7) implies that a fundamental solution to equation $f_\alpha * u = \delta$ exists and is equal to $f_{-\alpha}$. Let $n \in \mathbb{Z}$. For $n < 0$, $f_n = \delta^{(n)}$, so that $f_n * u = \delta^{(n)} * u = u^{(n)}$, i.e. the operator $f_n *$ is the operator of n-folding differentiation. Finally, for $n > 0$, we have $(f_n * u)^{(n)} = f_{-n} * (f_n * u) = u$, which implies that f_n is the operator of n-folding integration.

For $\alpha < 0$ the operator $f_\alpha *$ is the operator of fractional differentiation, and the operator of fractional integration for $\alpha > 0$. This is the *Riemann-Liouville* operator.

REMARK 1. For $\alpha \in (0,1)$ we have

$$f_{-\alpha} * u = f'_{1-\alpha} * u = (f_{1-\alpha} * u)' = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u(\tau)\, d\tau.$$

This is (1).

Let $\alpha \in \mathbb{R}$. Then the Laplace transform of f_α is given by

$$\mathcal{L}[f_\alpha(t)](s) = \frac{1}{s^\alpha}, \qquad \Re s > 0. \tag{8}$$

3.1. Special functions of Mittag-Leffler type

Functions of Mittag Lefler's type have very important role in the theory of differential equations where fractional derivative appears. Here we will recall from [4] some facts concerning this class of functions. One-parameter generalization of exponential function e^z is given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \qquad z \in \mathbb{C},\ \alpha > 0, \tag{9}$$

and was introduced by G.M. Mittag-Leffler.

More interesting for us is $e_\alpha(t,\lambda) = E_\alpha(-\lambda t^\alpha)$, $t \geq 0, \alpha > 0$ with Laplace transform

$$\mathcal{L}[e_\alpha(t,\lambda)](s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda}, \qquad \Re s > |\lambda|^{\frac{1}{\alpha}} \tag{10}$$

4. Analysis of system (2)

Let $u \in \mathcal{S}'_+(\mathbb{R})$, $a > 0$, $\alpha \in (0,1)$. We define the operator $T_{a,\alpha}$

$$T_{a,\alpha} u = a u^\alpha + u = a f_{-\alpha} * u + u \qquad u \in \mathcal{S}'_+.$$

$T_{a,\alpha}$ is a linear and continuous operator from $\mathcal{S}'_+$ to $\mathcal{S}'_+$. We have

$$\mathcal{L}[T_{a,\alpha} f](s) = (a s^\alpha + 1)\mathcal{L}[f](s), \qquad \Re s > 0.$$

The following proposition is valid.

PROPOSITION 1. *Let $a > 0$ and $v \in \mathcal{S}'_+$. Then*

$$T_{a,\alpha} u = v \tag{11}$$

has a unique solution $u \in \mathcal{S}'_+$.

Proof. Applying the Laplace transform to (11), we get $(1 + as^\alpha)\mathcal{L}u(s) = \mathcal{L}v(s)$, $\Re s > 0$. Let $h(s) = \dfrac{\mathcal{L}v(s)}{1 + as^\alpha}$, $\Re s > 0$. Since $\Re s > 0$ and $\alpha \in (0,1)$ it follows that $\Re s^\alpha > 0$. Further

$$|1 + as^\alpha| = \Re^2(1 + as^\alpha) + \Im^2(1 + as^\alpha) > \Re^2(1 + as^\alpha) > 1$$

and this implies that h is holomorphic for $\Re s > 0$. Also,

$$|h(s)| = \frac{|\mathcal{L}v(s)|}{|1 + as^{\alpha}|} < |\mathcal{L}v(s)| \leq C, \qquad \Re s > 0, \tag{12}$$

Thus by (5) it follows that there exists $u^* \in \mathcal{S}_+^{'}(\mathbb{R})$ such that $h(s) = \mathcal{L}u^*(s)$.

Let us show that u^* is the unique solution to (11). Indeed, we know that $\mathcal{L}f = 0$ if and only if $f = 0$. So $\mathcal{L}(T_{a,\alpha}u^* - v) = 0$ implies $T_{a,\alpha}u^* = v$ and u^* is unique since if we had two solutions u_1 and u_2, then from $T_{a,\alpha}u_1 = T_{a,\alpha}u_2$ we would have $T_{a,\alpha}(u_1 - u_2) = 0$, i.e. $(1 + as^{\alpha})\mathcal{L}[u_1 - u_2](s) = 0$ which yields $\mathcal{L}[u_1 - u_2](s) = 0$ and $u_1 - u_2 = 0$. ■

Now, we formulate

THEOREM 2. *Let $a, b \in \mathbb{R}$, $a > 0$, $b > 0$, $\alpha \in \mathbb{R}$. Let f and x be solution to $x''(t) = -f(t)$, $af^{(\alpha)}(t) + f(t) = bx^{(\alpha)}(t) + x(t)$ in $\mathcal{S}_+^{'}$ and let f be a continuous function on the interval $[0, \infty)$ and $f(0) = 0$. Then*

(i) $x \in C^2([0, \infty))$ and x is the unique solution to given equations.

(ii) With the given assumptions on f, the solutions f and x of system (2) satisfies the following: $b > a$ if and only if $f(t) > x(t)$ in a neighborhood of zero.

Proof. (i) Let us define $f = 0$ on $(-\infty, 0)$. Then f belongs to $\mathcal{S}_+^{'}$. (Moreover it is continuous on $\mathbb{R}$).) We can rewrite the second equation of the system

$$T_{a,\alpha}f = T_{b,\alpha}x \tag{13}$$

Due to Proposition 1 equation (13) has a unique solution $x \in \mathcal{S}_+^{'}$. Since it satisfies $x''(t) = -f(t)$, x is a function with continuous first and second derivative on $[0, \infty)$. We use the general assertion of Schwartz: if u is a distribution defined by a locally integrable function and if its distributional derivative is a locally integrable function, then the function u has a classical derivative equal to that locally integrable function.

(ii) If we apply Laplace transform to (13) we get $\mathcal{L}f(s) = \dfrac{\mathcal{L}[T_{a,\alpha}x](s)}{1 + as^{\alpha}}$, $\Re s > 0$, i.e.

$$\mathcal{L}f(s) = \mathcal{L}x(s) + (\frac{b}{a} - 1)\frac{s^{\alpha}}{s^{\alpha} + \frac{1}{a}}\mathcal{L}x(s), \qquad \Re s > 0. \tag{14}$$

Now we apply inverse Laplace transform to (14). We obtain

$$f(t) = x(t) + (\frac{b}{a} - 1)\left(\delta + e'_{\alpha}\left(t, \frac{1}{a}\right)\right) * x(t). \tag{15}$$

Integration by parts with initial conditions gives

$$\left(\delta + e'_{\alpha}\left(t, \frac{1}{a}\right)\right) * x(t) = e_{\alpha}\left(t, \frac{1}{a}\right) * x'(t)$$

wherefrom it folows

$$f(t) = x(t) + (\frac{b}{a} - 1)e_{\alpha}\left(t, \frac{1}{a}\right) * x'(t). \tag{16}$$

The function $e_\alpha(\lambda,t)$ is completely monotone[2] for $\lambda>0$ and $0<\alpha<1$ (see [2]). Therefore $e_\alpha\left(t,\frac{1}{a}\right)>0$. Since $x'(0)=1$ there is $\varepsilon>0$ such that $x'(t)>0$ for $t\in[0,\varepsilon)$. Hence from (16) we have that $f(t)>x(t)$ if and only if $(\frac{b}{a}-1)>0$ which we wanted to prove. ■

5. Existence of solution

We solve system (2) by using two methods. Firstly, we use the Laplace transform method and, secondly, we formulate an iterative procedure for obtaining the solution.

5.1. Laplace transform method

THEOREM 3. *Let $a,b\in\mathbb{R}$, $a>0$, $b>0$, $\alpha\in(0,1)$. System (2) has a unique solution in $\mathcal{S}'_+\cap C^2(\mathbb{R})$ given by*

$$x(t)=\sum_{i=1}^{2}\frac{(1+as_i^\alpha)e^{s_it}}{a(\alpha+2)s_i^{\alpha+1}+2s_i+b\alpha s_i^{\alpha-1}}+ \\ +\frac{\sin\alpha\pi}{\pi}\int_0^\infty\frac{r^\alpha\,(b-a)\,e^{-rt}\,dr}{(1+r^2)^2+2\,(1+r^2)\,r^\alpha(b+r^2a)\cos\alpha\pi+(r^\alpha(b+r^2a))^2},\tag{17}$$

$$f(t)=\sum_{i=1}^{2}\frac{(1+bs_i^\alpha)e^{s_it}}{a(\alpha+2)s_i^{\alpha+1}+2s_i+b\alpha s_i^{\alpha-1}}+ \\ +\frac{\sin\alpha\pi}{\pi}\int_0^\infty\frac{r^\alpha\,(a-b)\,e^{-rt}\,dr}{(1+r^2)^2+2\,(1+r^2)\,r^\alpha(b+r^2a)\cos\alpha\pi+(r^\alpha(b+r^2a))^2},\tag{18}$$

$t\geq0$, where s_1 and s_2 ($s_1=\overline{s_2}$) are poles of the first order of the function $\mathcal{L}x(t)$ and $\mathcal{L}f(t)$, and $\Re s_i<0$.

REMARK 2. Initial conditions $x(0)=0$ and $f(0)=0$ imply

$$\sum_{i=1}^{2}\frac{(1+as_i^\alpha)}{a(\alpha+2)s_i^{\alpha+1}+2s_i+b\alpha s_i^{\alpha-1}} \\ =-\frac{\sin\alpha\pi}{\pi}\int_0^\infty\frac{r^\alpha\,(b-a)\;dr}{(1+r^2)^2+2\,(1+r^2)\,r^\alpha(b+r^2a)\cos\alpha\pi+(r^\alpha(b+r^2a))^2},$$

$$\sum_{i=1}^{2}\frac{(1+bs_i^\alpha)}{a(\alpha+2)s_i^{\alpha+1}+2s_i+b\alpha s_i^{\alpha-1}} \\ =-\frac{\sin\alpha\pi}{\pi}\int_0^\infty\frac{r^\alpha\,(a-b)\;dr}{(1+r^2)^2+2\,(1+r^2)\,r^\alpha(b+r^2a)\cos\alpha\pi+(r^\alpha(b+r^2a))^2}.$$

REMARK 3. From equations (17) and (18) we conclude that x and f are continuous. Also we can differentiate (17) and (18) two times and get continuous functions, i.e. f and x belong to $C^2([0,\infty))$, and the claim of Theorem 2 is valid.

[2]A function $f(x)$ is *completely monotone* for $t>0$ if $(-1)^nf^{(n)}(x)\geq0$ for all $n=0,1,2,\ldots$ and all $t>0$.

Proof of Theorem 3. The first equation in (2) gives

$$x = t - t * f, \qquad t \geq 0 \tag{19}$$

Substituting (19) into (13) we get $T_{a,\alpha} f = T_{b,\alpha}(t - t * f)$. Applying the Laplace transform to (11) and (19) we get

$$(as^{\alpha} + 1)\,\mathcal{L}f(s) = (bs^{\alpha} + 1)\mathcal{L}[t - t * f] \qquad \Re s > 0,$$

$$(as^{\alpha} + 1)\,\mathcal{L}f(s) = (bs^{\alpha} + 1)(\frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}f(s)) \qquad \Re s > 0,$$

$$\mathcal{L}f(s) = \frac{bs^{\alpha} + 1}{as^{\alpha+2} + s^2 + bs^{\alpha} + 1} \qquad \Re s > 0, \tag{20}$$

and

$$\mathcal{L}x(s) = \frac{1}{s^2}(1 - \mathcal{L}f(s)) \qquad \Re s > 0, \tag{21}$$

$$\mathcal{L}x(s) = \frac{as^{\alpha} + 1}{as^{\alpha+2} + s^2 + bs^{\alpha} + 1} \qquad \Re s > 0. \tag{22}$$

Now we apply the inverse Laplace transform to (20) and (22)

$$h(t) = \frac{1}{2\pi i}\int_{\gamma} e^{st}\mathcal{L}h(s)\,ds \qquad t \geq 0,$$

where $\gamma = \{s : \Re s = \sigma, \sigma > \sigma_0 = 0\}$.

After the use Cauchy formula on an appropriate conture we obtain

$$x(t) = \sum_{i=1}^{2} \frac{e^{s_i t}(1 + as_i^{\alpha})}{a(\alpha+2)s_i^{\alpha+1} + 2s_i + b\alpha s_i^{\alpha-1}} + \frac{\sin\alpha\pi}{\pi}\times$$
$$\times \int_0^{\infty} \frac{r^{\alpha}\,(b-a)\;dr}{(1+r^2)^2 + 2\,(1+r^2)\,r^{\alpha}(b + r^2 a)\cos\alpha\pi + (r^{\alpha}(b + r^2 a))^2},$$

$t \geq 0$. Similarly we get

$$f(t) = \sum_{i=1}^{2} \frac{e^{s_i t}(1 + bs_i^{\alpha})}{a(\alpha+2)s_i^{\alpha+1} + 2s_i + b\alpha s_i^{\alpha-1}} + \frac{\sin\alpha\pi}{\pi}\times$$
$$\times \int_0^{\infty} \frac{r^{\alpha}\,(a-b)\,e^{-rt}\,dr}{(1+r^2)^2 + 2\,(1+r^2)\,r^{\alpha}(b + r^2 a)\cos\alpha\pi + (r^{\alpha}(b + r^2 a))^2},$$

$t \geq 0$.

In a forthcoming paper we will show that zeros of $\Delta(s) = as^{\alpha+2} + s^2 + bs^{\alpha} + 1 = (s^2 + \frac{b}{a})(1 + as^{\alpha}) - (\frac{b}{a} - 1)$ exist and lie in the left complex halfplane. ∎

5.2. Iterative procedure

Recall that we have formulas (21) and (15),

$$\mathcal{L}x(s) = \frac{1}{s^2}(1 - \mathcal{L}f(s)), \qquad \Re s > 0, \tag{23}$$

$$\mathcal{L}f(s) = \frac{bs^{\alpha} + 1}{as^{\alpha} + 1}\mathcal{L}x(s), \qquad \Re s > 0. \tag{24}$$

Let us denote $B = \frac{bs^\alpha+1}{as^\alpha+1}$. If we substitute B and (23) into (24) we get

$$\mathcal{L}f(s) = \frac{B}{s^2}(1 - \mathcal{L}f(s)).$$

Now we calculate $\mathcal{L}x(s)$ and obtain

$$\mathcal{L}x(s) = \frac{1}{s^2}\sum_{k=0}^{n}\left(-\frac{B}{s^2}\right)^k + (-1)^{n+1}\left(\frac{B}{s^2}\right)^n \mathcal{L}f(s).$$

One can show that for $|s| > \frac{b}{a}$, $\Re s > 0$, $\left|\left(\frac{B}{s^2}\right)^n \mathcal{L}f(s)\right|$ tends to zero as $n \to \infty$. More precisely, for $|s| > \frac{b}{a}, \Re s > 0$ we have

$$\left|\left(\frac{B}{s^2}\right)^n \mathcal{L}f(s)\right| \le \left(\frac{b}{a}\right)^n \frac{|\mathcal{L}f(s)|}{s^{2n}}$$

Thus $\mathcal{L}x$ is an analitic function, $\mathcal{L}x(s) = \dfrac{1}{s^2}\sum\limits_{k=0}^{\infty}\left(-\dfrac{B}{s^2}\right)^k$, $|s| > \dfrac{b}{a}$, $\Re s > 0$. Since $\mathcal{L}x$ belongs to D_+ and has the given form for $|s| > \frac{b}{a}$, $\Re s > 0$, its inverse Laplace transform is given by

$$x(t) = t * \sum_{k=0}^{\infty}\left(\frac{b}{a}t + \left(\frac{b}{a} - 1\right)e'_\alpha(t, \frac{1}{a}) * t\right)^k \tag{25}$$

where $u^k = \underbrace{u * u * \cdots * u}_{k \text{ times}}$. Formula (25) represents one more form of the solution for system (2). The function f is of the form

$$f(t) = x * \left(\frac{b}{a}t + \left(\frac{b}{a} - 1\right)e'_\alpha(t, \frac{1}{a}) * t\right), \qquad t \ge 0.$$

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