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THE INVARIANT SUBSPACE LATTICE OF AN ALGEBRAIC OPERATOR

Marija Orovchanec and Biljana Nachevska

Abstract. The main objecr in this work is to analyze the invariant subspace lattice of an algebraic operator.

Let X be a Banach space. By $B(X)$ we mean the algebra of all bounded linear operators on X. A subspace $\mathcal{M}$ is *invariant* under an operator A if $Ax \in \mathcal{M}$ for every $x \in \mathcal{M}$. The collection of all subspaces of X invariant under A is denoted by $\operatorname{Lat} A$. The lattice $\operatorname{Lat} A$ is the direct sum of sublattices $\operatorname{Lat} A_1$ and $\operatorname{Lat} A_2$ if each $\mathcal{M} \in \operatorname{Lat} A$ is uniquely representable in the form $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_i \in \operatorname{Lat} A_i$, $i = 1, 2$. Notation: $\operatorname{Lat} A = \operatorname{Lat} A_1 \oplus \operatorname{Lat} A_2$.

An operator $A \in B(X)$ is *algebraic* if there exists a polynomial p other than 0 such that $p(A) = 0$. Let us consider the factorization $p(z) = \prod_{j=1}^{k} (z - \lambda_j)^{n_j}$ with the λ_j's mutually distinct. Then the spectrum of A is $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Every operator on a finite-dimensional space is algebraic. The algebraic operators on infinte-dimensional spaces can be characterized in terms of their invariant subspaces. An operator is algebraic if and only if the union of its finite-dimensional invariant subspaces is X.

The main motivation and the basis for the work in this paper are the results obtained by Brickman and Filmore in [1].

PROPOSITION 1. *Let A_1 and A_2 be algebraic operators with minimal polynomials p_1 and p_2 on the Banach spaces X_1 and X_2, respectively. Then*

$$\operatorname{Lat}(A_1 \oplus A_2) = \operatorname{Lat} A_1 \oplus \operatorname{Lat} A_2 \iff (p_1, p_2) = 1.$$

Proof. In general case, for every operator $A_i \in B(X_i)$, $i = 1, 2$, $\operatorname{Lat} A_1 \oplus \operatorname{Lat} A_2 \subset \operatorname{Lat}(A_1 \oplus A_2)$ holds. For the inverse inclusion, let $(p_1, p_2) = 1$. We must

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show that $\mathcal{M} \in \mathrm{Lat}(A_1 \oplus A_2)$ implies that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_i \in \mathrm{Lat}\, A_i$, $i = 1, 2$. Given $\mathcal{M} \in \mathrm{Lat}(A_1 \oplus A_2)$ let $\mathcal{M}_1 \oplus \{0\} = (1 \oplus 0)\mathcal{M}$ and $\{0\} \oplus \mathcal{M}_2 = (0 \oplus 1)\mathcal{M}$. Obviously $\mathcal{M} \subset \mathcal{M}_1 \oplus \mathcal{M}_2$. To prove that $\mathcal{M}_1 \oplus \mathcal{M}_2 \subset \mathcal{M}$, let r_1 and r_2 be polynomials such taht $r_1p_1 + r_2p_2 = 1$, and let $q_2 = r_2p_2$. We have $q_2(A_1) = 1 - r_1(A_1)p_1(A_1) = 1$ so $q_2(A_1 \oplus A_2) = q_2(A_1) \oplus q_2(A_2) = 1 \oplus 0$. Then $\mathcal{M}_1 \oplus \{0\} = (1 \oplus 0)\mathcal{M} = q_2(A_1 \oplus A_2)\mathcal{M} \subset \mathcal{M}$. Similarly $\{0\} \oplus \mathcal{M}_2 \subset \mathcal{M}$. Thus $\mathcal{M}_1 \oplus \mathcal{M}_2 \subset \mathcal{M}$, and it follows that $\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{M}$. Clearly $\mathcal{M}_i \in \mathrm{Lat}\, A_i$ for $i = 1, 2$.

Conversely, suppose that p_1 and p_2 have a common prime factor q, i.e., $p_1 = qr_1$ and $p_2 = qr_2$. Deviding q by its largest coefficient, we can assume that the leading coefficient of q is 1. If $q(A_i)x_i \neq 0$ for each $x_1 \in X_1$ and $x_2 \in X_2$, then r_1 and r_2 are the minimal polynomials of A_1 and A_2. That means that there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that $q(A_i)x_i = 0$ for $i = 1, 2$.

Let $\mathcal{M} = \{\, r(A_1)x_1 \oplus r(A_2)x_2 : \deg r < \deg q$, the leading coefficient of r is $1 \,\}$. It is easy to verify that $\mathcal{M}$ is a linear manifold in $X_1 \oplus X_2$. If (y_n) is a sequence in $\mathcal{M}$ and $y_n \to y$, then $y \in \mathcal{M}$. For, suppose that, for each n, r_n is a non-zero polynomial of degree less than $\deg q$ such that $y_n = r_n(A_1)x_1 \oplus r_n(A_2)x_2$. Each coefficient of r_n has absolute value at most 1 and at least one coefficient has absolute value equal to 1. Then a subsequence of (r_n) converges coefficient-wise to a polynomial r of degree less than $\deg q$; r is not 0, since at least one of its coefficients has modulus 1. Re-label so that (r_n) is such a subsequence. Then $(r_n(A_1)x_1 \oplus r_n(A_2)x_2)$ converges to $r(A_1)x_1 \oplus r(A_2)x_2$. Thus $\mathcal{M}$ is closed, and so $\mathcal{M}$ is a subspace of $X_1 \oplus X_2$.

Now, let $x \in \mathcal{M}$, $x = r(A_1)x_1 \oplus r(A_2)x_2$, $\deg r < \deg q$. Then $(A_1 \oplus A_2)x = r_1(A_1)x_1 \oplus r_1(A_2)x_2$, where $r_1(z) = zr(z)$. The leading coefficient of r_1 is 1 and $\deg r_1 = \deg r + 1 \leqslant \deg q$. Since $\deg(r_1 - q) < \deg r_1 \leqslant \deg q$, we have $(A_1 \oplus A_2)x = r_1(A_1)x_1 \oplus r_1(A_2)x_2 = (r_1 - q)(A_1)x_1 \oplus (r_1 - q)(A_2)x_2 \in \mathcal{M}$. Thus $\mathcal{M} \in \mathrm{Lat}(A_1 \oplus A_2)$. If $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_i \in \mathrm{Lat}\, A_i$, $i = 1, 2$, we shall have $r(A_1)x_1 \oplus 0 \in \mathcal{M}$, $r(A_2)x_2 = 0$, and therefore $r = 0$ (because q is prime). Thus $\mathcal{M}_1 = \{0\}$ and similarly $\mathcal{M}_2 = \{0\}$. Hence $\mathcal{M} = \{0\}$, a contradiction. ∎

Proposition 2. *Let A be an algebraic operator on X with primary summands A_i. Then* $\mathrm{Lat}\, A = \bigoplus_i \mathrm{Lat}\, A_i$.

Proof. Using the induction and previous proposition gives the result. ∎

Proposition 3. *Let A be a prime algebraic operator, i.e., $(A - \lambda)^n = 0$. If $\mathcal{M} \in \mathrm{Lat}\, A$, then $(A - \lambda)\mathcal{M} \subset \mathcal{M}_-$, where $\mathcal{M}_- = \bigvee\{\, \mathcal{M}' : \mathcal{M}' \in \mathrm{Lat}\, A, \mathcal{M}' \subset \mathcal{M}, \mathcal{M}' \neq \mathcal{M} \,\}$.*

Proof. Suppose $\mathcal{M}_-$ is a proper subset of $\mathcal{M}$. Let $\hat{A}$ be the quotient operator on $X \setminus \mathcal{M}_-$. Then $\hat{\mathcal{M}} = \mathcal{M} \setminus \mathcal{M}_-$ is a minimal non-zero element of $\mathrm{Lat}\, \hat{A}$. But $(\hat{A} - \lambda)\hat{\mathcal{M}} \subset \hat{\mathcal{M}}$ and $(\hat{A} - \lambda)\hat{\mathcal{M}} \in \mathrm{Lat}\, \hat{A}$. Since $\hat{A} - \lambda$ is nilpotent, $(\hat{A} - \lambda)\hat{\mathcal{M}} \neq \hat{\mathcal{M}}$. Hence $\hat{A} - \lambda$ annihilates $\hat{\mathcal{M}}$, i.e., $(A - \lambda)\mathcal{M} \subset \mathcal{M}_-$. ∎

Proposition 4. *Let A be an algebraic operator and* $\mathrm{Lat}\, A \subset \mathrm{Lat}\, B$. *Then B is algebraic.*

Proof. The proof is similar to the proof of Theorem 4.8 [2]. Let A be an algebraic operator with the minimal polynomial p_A, $p_A(z) = \prod_{j=1}^{k}(z-\lambda_j)^{n_j}$, and $\operatorname{Lat} A \subset \operatorname{Lat} B$. Let $x \in X$ and $\mathcal{M}_x = \bigvee\{A^n x\}$. We have $\dim \mathcal{M}_x \leqslant \deg p_A$ and $\mathcal{M}_x \in \operatorname{Lat} A \subset \operatorname{Lat} B$. Thus there exists q_x, polynomial in B, such that $q_x(B)x = 0$. Let F_k denote the set of all vectors x such that $q_x(B)x = 0$, $\deg q_x \leqslant k$. Then $X = \bigcup_{k=1}^{\infty} F_k$. Let (x_n) is a sequence in F_k, $x_n \to x$. We shall prove that $x \in F_k$. For each n, $q_{x_n}(B)x_n = 0$, $\deg q_{x_n} \leqslant k$. We can assume that each coefficient of q_{x_n} has absolute value at most 1 and at least one coefficient has absolute value equal to 1. Then a subsequence of (q_{x_n}) converges coefficient-wise to a polynomial q; $\deg q \leqslant k$; q is not 0. Since

$$\begin{aligned}\|q(B)x\| &= \|q(B)x - q_{x_n}(B)x\| + \|q_{x_n}(B)x_n - q_{x_n}(B)x\| \\ &\leqslant \|q(B) - q_{x_n}(B)\|\,\|x\| + \|q_{x_n}(B)\|\,\|x_n - x\| \to 0,\end{aligned}$$

$q(B)x = 0$. Thus each F_k is closed. By the Baire category theorem, there exists k_0 such that the interior of F_{k_0} is not empty. For $x_0 \in \operatorname{int} F_{k_0}$, there exists $r > 0$ such that $B(x_0, r) = \{x \in X : \|x - x_0\| < r\} \subset F_{k_0}$. If $y \in B(0,r)$, then $y = x - x_0$ for some $x \in B(x_0, r)$. Then $q_x(B)q_{x_0}(B)y = 0$ and $\deg q_x q_{x_0} \leqslant 2k_0$ and so $B(0,r) \subset F_{2k_0}$. Since $q_x(B)q_{x_0}(B)(\alpha y) = \alpha q_x(B)q_{x_0}(B)(y)$, it follows that $X = F_{2k_0}$.

Let $x \in X$ and n_x is the degree of the lowest-degree non-zero polynomial of degree n such that $q(B)x = 0$, where $n = \max\{n_x\}$. We claim that $q(B)y = 0$ for all y. Given y, let $\mathcal{M} = \bigvee_{j=0}^{\infty}\{B^j x, B^j y\}$. Then $\mathcal{M} \in \operatorname{Lat} B$ and $\dim \mathcal{M} \leqslant 2n$. Let r be the minimal polynomial of $B|\mathcal{M}$, then q devides r. Moreover, $\deg q = \deg r$. Thus q is a multiple of r, and $q(B|\mathcal{M}) = 0$. Then $q(B)y = 0$. ∎

In the subsequent work we prove that every commutative set of algebraic operators is triangularizable. First we give some definitions.

Definitions. A collection of bounded linear operaors on a complex Banach space is *triangularizable* if there is a chain of subspaces which is maximal as a subspace chain and which consists of common invariant subspaces for the operators in the collection.

A collection of properties is said to be *inherited by quotients* if for every collection of quotients of a set satisfying the properties also satisfies the same properties.

The Triangularization Lemma. *Let P be a collection of properties inherited by quotients. If every set of operators on a space of dimension greater than one, which satisfies P, has a non-trivial invariant subspace, then every such set is trinagularizable.*

Proposition 5. *Every commutative set of algebraic operators is triangularizable.*

Proof. Let $\mathcal{A}$ is a commutative set of algebraic operators. If every operator in $\mathcal{A}$ is a multiple of the identity, the result is trivial, so assume that $A \in \mathcal{A}$ is not a multiple of the identity. If p is a minimal polynomial of A, let $p(z)=\prod_{j=1}^{k}(z-\lambda_j)^{n_j}$. Then $\sigma(A)=\{\lambda_1,\lambda_2,\dots,\lambda_k\}$. Each $\lambda_j\in\Pi_0(A)$. For at least one $\lambda\in\sigma(A)$, the $\mathrm{cl}(A-\lambda)X\neq X$ (otherwise $\mathrm{cl}\bigl(\prod_{j=1}^{k}(A-\lambda_j)X\bigr)=X$). $\mathrm{Ker}(A-\lambda)$ is a nontrivial invariant subspace for A. Let $f\in\mathrm{Ker}(A-\lambda)$ and $B\in\mathcal{A}$. Then $(A-\lambda)Bf=B(A-\lambda)f=0$, i.e., $Bf\in\mathrm{Ker}(A-\lambda)$. It follows that the kernel of $A-\lambda$ is invariant under $\mathcal{A}$. The Triangularization Lemma completes the proof. ∎

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St. Cyril and Methodius University, Faculty of Natural Sciences and Mathematics, Institute of Mathematics, P.O.Box 162, 1000 Skopje, R. Macedonia

E-mail: marijaor@iunona.pmf.ukim.edu.mk