МАТЕМАТИЧКИ ВЕСНИК
54 (2002), 219–226

UDK 514.763.4
оригинални научни рад
research paper

INFINITESIMAL DEFORMATIONS OF CURVATURE TENSORS AT NON-SYMMETRIC AFFINE CONNECTION SPACE

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Abstract. At the present work we consider infinitesimal deformations of geometric objects, especially of curvature tensors, at a space L_N of non-symmetric affine connection.

0. Introduction

We consider a space L_N of a non-symmetric affine connection L^i_{jk} with the torsion tensor $T^i_{jk} = L^i_{jk} - L^i_{kj}$ at local coordinates x^i $(i = 1, \dots, N)$. Basic information on infinitesimal deformations one can find at [4–11].

Let x^i, $i = 1, \dots, N$ be local coordinates at L_N and let us define infinitesimal deformations of space L_N in the next manner.

Definition 0.1. A transformation $f\colon L_N \to L_N \colon x = (x^1, \dots, x^N) \equiv (x^i) \mapsto \bar{x} = (\bar{x}^1, \dots, \bar{x}^N) \equiv (\bar{x}^i)$, where

$$\bar{x} = x + z(x)\varepsilon, \tag{0.1}$$

or in local coordinates

$$\bar{x}^i = x^i + z^i(x^j)\varepsilon, \quad i, j = 1, \dots, N, \tag{0.1'}$$

where ε is an infinitesimal, is called *infinitesimal deformation of a space L_N*, determined by the vector $z = (z^i)$, which is called *the field of infinitesimal deformation* (0.1′).

Under deformation of a space according to (0.1′) some geometrical objects are deformed. The fact that geometric object is given at point x at local coordinates x^i, is denoted by $\mathcal{A}(i, x)$. Under deformation (0.1′) we get geometric object $\bar{\mathcal{A}}$,

AMS Subject Classification: 53 C 25, 53 A 45, 53 B 05

Keywords and phrases: Infinitesimal deformation, non-symmetric affine connection, Lie derivative, geometric object, curvature tensor.

Communicated at the 5th International Symposium on Mathematical Analysis and its Applications, Niška banja, Yugoslavia, October, 2–6, 2002.

which can be considered at the same point and at the same coordinate system (see [6,7,10,11]).

Remark 0.1. In this study of infinitesimal deformations according to (0.1′), quantities of an order higher then the first with respect to ε are neglected.

Definition 0.2. The magnitude $\mathcal{D}\mathcal{A}$, the difference between deformed object $\bar{\mathcal{A}}$ and initial object $\mathcal{A}$ at the same coordinate system and at the same point with respect to (0.1′), i.e.

$$\mathcal{D}\mathcal{A} = \bar{\mathcal{A}}(i,x) - \mathcal{A}(i,x), \tag{0.2}$$

is called *Lie difference (Lie differential)*, and the magnitude

$$\mathcal{L}_z\mathcal{A} = \lim_{\varepsilon\to 0}\frac{\mathcal{D}\mathcal{A}}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\bar{\mathcal{A}}(i,x) - \mathcal{A}(i,x)}{\varepsilon} \tag{0.2′}$$

is *Lie derivative* of geometric object $\mathcal{A}(i,x)$ with respect to the vector field $z = (z^i(x^j))$.

Using the relation (0.2), for deformed object $\bar{\mathcal{A}}(i,x)$ we have

$$\bar{\mathcal{A}}(i,x) = \mathcal{A}(i,x) + \mathcal{D}\mathcal{A}, \tag{0.2″}$$

and thus we can express $\bar{\mathcal{A}}$, finding previously $\mathcal{D}\mathcal{A}$.

From (0.2′) we get

$$\frac{\mathcal{D}A}{\varepsilon} = \mathcal{L}_z\mathcal{A} + \varepsilon_1, \varepsilon_1 \to 0, \quad \text{if} \quad \varepsilon \to 0,$$

and $\mathcal{D}A = \varepsilon\mathcal{L}_z\mathcal{A} + \varepsilon\varepsilon_1$, i.e.

$$\mathcal{D}A = \varepsilon\mathcal{L}_z\mathcal{A}, \tag{0.3}$$

where $\varepsilon\varepsilon_1$, as an infinitesimal value of the higher order with respect to ε, is omitted.

As it is known (see [6,7]), for an arbitrary tensor $t^{i_1,\dots,i_u}_{j_1,\dots,j_v}$ we have

$$\begin{aligned}\mathcal{D}t^{i_1\dots i_u}_{j_1\dots j_v} &= [t^{i_1\dots i_u}_{j_1\dots j_v,p}z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{,p}\binom{p}{i_\alpha}t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^p_{,j_\beta}\binom{j_\beta}{p}t^{i_1\dots i_u}_{j_1\dots j_v}]\varepsilon \\ &= \mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v}\varepsilon,\end{aligned} \tag{0.4}$$

where we denoted, for instance, $z^i_{,p} = \partial z^i/\partial x^p$, and

$$\binom{p}{i_\alpha}t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_{\alpha-1}p i_{\alpha+1}\dots i_u}_{j_1\dots j_v}, \quad \binom{j_\beta}{p}t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_{\beta-1}p j_{\beta+1}\dots j_v}.$$

For the *connection coefficients* we have

$$\mathcal{D}L^i_{jk} = (L^i_{jk,p}z^p + z^i_{,jk} - z^i_{,p}L^p_{jk} + z^p_{,j}L^i_{pk} + z^p_{,k}L^i_{jp})\varepsilon = \mathcal{L}_z L^i_{jk}\varepsilon. \tag{0.5}$$

Because of non-symmetry of the connection, at L_N we can consider two types of covariant derivatives for a vector and four types for general tensor. So, denoting by $\underset{\theta}{|}(\theta = 1,\dots,4)$ the derivative of the type θ, we have ([1]–[3]):

$$t^{i_1\dots i_u}_{j_1\dots j_v\underset{\substack{1\\2\\3\\4}}{|}m} = t^{i_1\dots i_u}_{j_1\dots j_v,m} + \sum_{\alpha=1}^{u} L^{i_\alpha}_{\substack{pm\\mp\\pm\\mp}} \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} - \sum_{\beta=1}^{v} L^{p}_{\substack{j_\beta m\\mj_\beta\\mj_\beta\\j_\beta m}} \binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}. \qquad (0.6a-d)$$

By virtue of Ricci-type identities twelve curvature tensors in L_N are obtained [1,2,3]. Among these tensors only five of them are independent:

$$\underset{1}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm}, \qquad (0.7)$$

$$\underset{2}{R}{}^i_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp}, \qquad (0.8)$$

$$\underset{3}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}T^i_{pj}, \qquad (0.9)$$

$$\underset{4}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}T^i_{pj}, \qquad (0.10)$$

$$\underset{5}{R}{}^i_{jmn} = \frac{1}{2}(L^i_{jm,n} + L^i_{mj,n} - L^i_{jn,m} - L^i_{nj,m} + L^p_{jm}L^i_{pn} + L^p_{mj}L^i_{np} - L^p_{jn}L^i_{mp} - L^p_{nj}L^i_{pm}). \qquad (0.11)$$

If we define symmetric connection

$$\underset{0}{L}{}^i_{jk} = \frac{1}{2}(L^i_{jk} + L^i_{kj}), \qquad (0.12)$$

its curvature tensor

$$\underset{0}{R}{}^i_{jmn} = \underset{0}{L}{}^i_{jm,n} - \underset{0}{L}{}^i_{jn,m} + \underset{0}{L}{}^p_{jm}\underset{0}{L}{}^i_{pn} - \underset{0}{L}{}^p_{jn}\underset{0}{L}{}^i_{pm} \qquad (0.13)$$

is an ordinary Riemann-Christoffel's curvature tensor of symmetric connection. Denoting by semicolon (;) covariant derivative with respect to symmetric connection then according to [3], we have

$$\underset{1}{R}{}^i_{jmn} = \underset{o}{R}{}^i_{jmn} + \frac{1}{2}T^i_{jm;n} - \frac{1}{2}T^i_{jn;m} + \frac{1}{4}T^p_{jm}T^i_{pn} - \frac{1}{4}T^p_{jn}T^i_{pm}, \qquad (0.14)$$

$$\underset{2}{R}{}^i_{jmn} = \underset{0}{R}{}^i_{jmn} - \frac{1}{2}T^i_{jm;n} + \frac{1}{2}T^i_{jn;m} + \frac{1}{4}T^p_{jm}T^i_{pn} - \frac{1}{4}T^p_{jn}T^i_{pm}, \qquad (0.15)$$

$$\underset{3}{R}{}^i_{jmn} = \underset{0}{R}{}^i_{jmn} + \frac{1}{2}T^i_{jm;n} + \frac{1}{2}T^i_{jn;m} - \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm} - \frac{1}{2}T^p_{mn}T^i_{pj}, \qquad (0.16)$$

$$\underset{4}{R}{}^i_{jmn} = \underset{0}{R}{}^i_{jmn} + \frac{1}{2}T^i_{jm;n} + \frac{1}{2}T^i_{jn;m} - \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm} + \frac{1}{2}T^p_{mn}T^i_{pj} \qquad (0.17)$$

$$\underset{5}{R}{}^i_{jmn} = \underset{0}{R}{}^i_{jmn} + \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm}. \qquad (0.18)$$

At (0.14–18) all the addends at the right side are tensors.

At the present work we will consider infinitesimal deformations of cited five curvature tensors.

1. Infinitesimal deformations of curvature tensor $\underset{1}{\boldsymbol{R}}$

1.1. According to the (0.7) for deformed first curvature tensor at L_N, i.e. for $\underset{1}{\bar{R}}$, we have

$$\underset{1}{\bar{R}}{}^i_{jmn}(x) = \bar{L}^i_{jm,n} - \bar{L}^i_{jn,m} + \bar{L}^p_{jm}\bar{L}^i_{pn} - \bar{L}^p_{jn}\bar{L}^i_{pm}, \tag{1.1}$$

and with respect to $\bar{L}^i_{jm}(x) = L^i_{jm}(x) + \mathcal{D}L^i_{jm}$, we have

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= (L^i_{jm} + \mathcal{D}L^i_{jm})_{,n} - (L^i_{jn} + \mathcal{D}L^i_{jn})_{,m}\\ &\quad + (L^p_{jm} + \mathcal{D}L^p_{jm})(L^i_{pn} + \mathcal{D}L^i_{pn}) - (L^p_{jn} + \mathcal{D}L^p_{jn})(L^i_{pm} + \mathcal{D}L^i_{pm}).\end{aligned}$$

Developing this and omitting the members of the form $\mathcal{D}L^{\cdots}_{\cdots} \cdot \mathcal{D}L^{\cdots}_{\cdots}$, as they include $(\varepsilon)^2$, we have

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= L^i_{jm,n} + (\mathcal{D}L^i_{jm})_{,n} - L^i_{jn,m} - (\mathcal{D}L^i_{jn})_{,m} + L^p_{jm}L^i_{pn}\\ &\quad + L^p_{jm}\mathcal{D}L^i_{pn} + (\mathcal{D}L^p_{jm})L^i_{pn} - L^p_{jn}L^i_{pm} - L^p_{jn}\mathcal{D}L^i_{pm} - (\mathcal{D}L^p_{jn})L^i_{pm}.\end{aligned} \tag{1.2}$$

As $\mathcal{D}L^i_{jm}$ is a tensor, we can consider covariant derivative:

$$(\mathcal{D}L^i_{jm})_{\underset{1}{|}n} = (\mathcal{D}L^i_{jm})_{,n} + L^i_{pn}\mathcal{D}L^p_{jm} - L^p_{jn}\mathcal{D}L^i_{pm} - L^p_{mn}\mathcal{D}L^i_{jp}$$

where

$$(\mathcal{D}L^i_{jm})_{,n} + L^i_{pn}\mathcal{D}L^p_{jm} = (\mathcal{D}L^i_{jm})_{\underset{1}{|}n} + L^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}L^i_{jp} \tag{1.3}$$

and in the same manner

$$(\mathcal{D}L^i_{jn})_{,m} + L^i_{pm}\mathcal{D}L^p_{jn} = (\mathcal{D}L^i_{jn})_{\underset{1}{|}m} + L^p_{jm}\mathcal{D}L^i_{pn} + L^p_{nm}\mathcal{D}L^i_{jp} \tag{1.3$'$}$$

If we have in mind (0.7), $(1.3, 3')$, the equation (1.2) becomes

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + (\mathcal{D}L^i_{jm})_{\underset{1}{|}n} + L^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}L^i_{jp}\\ &\quad - (\mathcal{D}L^i_{jn})_{\underset{1}{|}m} - L^p_{im}\mathcal{D}L^i_{pn} - L^p_{nm}\mathcal{D}L^i_{jp} + L^p_{jm}\mathcal{D}L^i_{pn} - L^p_{jn}\mathcal{D}L^i_{pm},\end{aligned}$$

i.e.

$$\underset{1}{\bar{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + (\mathcal{D}L^i_{jm})_{\underset{1}{|}n} - (\mathcal{D}L^i_{jn})_{\underset{1}{|}m} + T^p_{mn}\mathcal{D}L^i_{jp}. \tag{1.4}$$

From here

$$\mathcal{D}\underset{1}{R}{}^i_{jmn} = \underset{1}{\bar{R}}{}^i_{jmn} - \underset{1}{R}{}^i_{jmn} = (\mathcal{D}L^i_{jm})_{\underset{1}{|}n} - (\mathcal{D}L^i_{jn})_{\underset{1}{|}m} + T^p_{mn}\mathcal{D}L^i_{jp}, \tag{1.4$'$}$$

and (dividing with ε):

$$\mathcal{L}_z\underset{1}{R}{}^i_{jmn} = (\mathcal{L}_z L^i_{jm})_{\underset{1}{|}n} - (\mathcal{L}_z L^i_{jn})_{\underset{1}{|}m} + T^p_{mn}\mathcal{L}_z L^i_{jp}. \tag{1.4$''$}$$

1.2. We can also start from (0.14), and then we have

$$\begin{aligned}\bar{\underset{1}{R}}{}^i{}_{jmn} &= \bar{\underset{0}{R}}{}^i{}_{jmn} + \frac{1}{2}\bar{T}^i_{jm;n} - \frac{1}{2}\bar{T}^i_{jn;m} + \frac{1}{4}\bar{T}^p_{jm}\bar{T}^i_{pn} - \frac{1}{4}\bar{T}^p_{jn}\bar{T}^i_{pm}\\ &= \underset{0}{R}{}^i{}_{jmn} + \mathcal{D}\underset{0}{R}{}^i{}_{jmn} + \frac{1}{2}(T^i_{jm} + \mathcal{D}T^i_{jm})_{;n} - \frac{1}{2}(T^i_{jn} + \mathcal{D}T^i_{jn})_{;m}\\ &\quad + \frac{1}{4}(T^p_{jm} + \mathcal{D}T^p_{jm})(T^i_{pn} + \mathcal{D}T^i_{pn}) - \frac{1}{4}(T^p_{jn} + \mathcal{D}T^p_{jn})(T^i_{pm} + \mathcal{D}T^i_{pm}).\end{aligned}$$

Omitting the members containing $\mathcal{D}T\cdot\mathcal{D}T$, we get

$$\begin{aligned}\bar{\underset{1}{R}}{}^i{}_{jmn} &= \underset{0}{R}{}^i{}_{jmn} + \mathcal{D}\underset{0}{R}{}^i{}_{jmn} + \frac{1}{2}T^i_{jm;n} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} - \frac{1}{2}T^i_{jn;m} - \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m}\\ &\quad + \frac{1}{4}(T^p_{jm}T^i_{pn} + T^p_{jm}\mathcal{D}T^i_{pn} + \mathcal{D}T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm} - T^p_{jn}\mathcal{D}T^i_{pm} - \mathcal{D}T^p_{jn}T^i_{pm}).\end{aligned}$$

So, with respect to (0.14), one obtains

$$\bar{\underset{1}{R}}{}^i{}_{jmn} = \underset{1}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} - \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m} + \frac{1}{4}\mathcal{D}(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}), \tag{1.5}$$

wherefrom

$$\mathcal{D}\underset{1}{R}{}^i_{jmn} = \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} - \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m} + \frac{1}{4}\mathcal{D}(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}) \tag{1.5$'$}$$

where ; denotes covariant derivative with respect to $\underset{0}{L}$ (0.12), and (dividing with ε):

$$\mathcal{L}_z\underset{1}{R}{}^i_{jmn} = \mathcal{L}_z\underset{0}{R}{}^i_{jmn} + \frac{1}{2}(\mathcal{L}_zT^i_{jm})_{;n} - \frac{1}{2}(\mathcal{L}_zT^i_{jn})_{;m} + \frac{1}{4}\mathcal{L}_z(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}). \tag{1.5$''$}$$

2. Infinitesimal deformations of curvature tensor $\underset{2}{\boldsymbol{R}}$

2.1. In the similar manner as for $\underset{1}{R}$, using (0.8), we get

$$\bar{\underset{2}{R}}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + (\mathcal{D}L^i_{mj})_{\underset{2}{|}n} - (\mathcal{D}L^i_{nj})_{\underset{2}{|}m} + T^p_{nm}\mathcal{D}L^i_{pj}. \tag{2.1}$$

and corresponding expressions for $\mathcal{D}\underset{2}{R}{}^i_{jmn}$ and $\mathcal{L}_z\underset{2}{R}{}^i_{jmn}$.

2.1. Starting from (0.15), we obtain

$$\bar{\underset{2}{R}}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} - \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} + \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m} + \frac{1}{4}\mathcal{D}(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}). \tag{2.2}$$

3. Infinitesimal deformations of curvature tensor $\underset{3}{\boldsymbol{R}}$

3.1. Starting from (0.9), we obtain

$$\begin{aligned}\bar{\underset{3}{R}}{}^i_{jmn} &= L^i_{jm,n} + (\mathcal{D}L^i_{jm})_{,n} - L^i_{nj,m} - (\mathcal{D}L^i_{nj})_{,m} + L^p_{jm}L^i_{np} + L^p_{jm}\mathcal{D}L^i_{np} + \mathcal{D}L^p_{jm}L^i_{np}\\ &\quad - L^p_{nj}L^i_{pm} - L^p_{nj}\mathcal{D}L^i_{pm} - \mathcal{D}L^p_{nj}L^i_{pm} + L^p_{nm}T^i_{pj} + L^p_{nm}\mathcal{D}T^i_{pj} + \mathcal{D}L^p_{nm}T^i_{pj}.\end{aligned} \tag{3.1}$$

In relation to

$$(\mathcal{D}L^i_{jm})_{\underset{4}{|}n} = (\mathcal{D}L^i_{jm})_{,n} + L^i_{np}\mathcal{D}L^p_{jm} - L^p_{jn}\mathcal{D}L^i_{pm} - L^p_{mn}\mathcal{D}L^i_{jp},$$
$$(\mathcal{D}L^i_{nj})_{\underset{3}{|}m} = (\mathcal{D}L^i_{nj})_{,m} + L^i_{pm}\mathcal{D}L^p_{nj} - L^p_{mn}\mathcal{D}L^i_{pj} - L^p_{mj}\mathcal{D}L^i_{np},$$

one obtains respectively

$$(\mathcal{D}L^i_{jm})_{,n} + L^i_{np}\mathcal{D}L^p_{jm} = (\mathcal{D}L^i_{jm})_{\underset{4}{|}n} + L^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}L^i_{jp},$$
$$(\mathcal{D}L^i_{nj})_{,m} + L^i_{pm}\mathcal{D}L^p_{nj} = (\mathcal{D}L^i_{nj})_{\underset{3}{|}m} + L^p_{mn}\mathcal{D}L^i_{pj} + L^p_{mj}\mathcal{D}L^i_{np}.$$

Using these equations and (0.9), the equation (3.1) becomes

$$\begin{aligned}\underset{3}{\bar{R}}{}^i_{jmn} &= \underset{3}{R}{}^i_{jmn} + (\mathcal{D}L^i_{jm})_{\underset{4}{|}n} + L^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}L^i_{jp} - (\mathcal{D}L^i_{nj})_{\underset{3}{|}m}\\ &- L^p_{mn}\mathcal{D}L^i_{pj} - L^p_{mj}\mathcal{D}L^i_{np} + L^p_{jm}\mathcal{D}L^i_{np} - L^p_{nj}\mathcal{D}L^i_{pm} + L^p_{nm}\mathcal{D}T^i_{pj} + T^i_{pj}\mathcal{D}L^p_{nm},\end{aligned}$$

that is

$$\begin{aligned}\underset{3}{\bar{R}}{}^i_{jmn} &= \underset{3}{R}{}^i_{jmn} + (\mathcal{D}L^i_{jm})_{\underset{4}{|}n} - (\mathcal{D}L^i_{nj})_{\underset{3}{|}m}\\ &+ T^p_{jn}\mathcal{D}L^i_{pm} + T^p_{jm}\mathcal{D}L^i_{np} + L^p_{mn}\mathcal{D}T^i_{jp} + \mathcal{D}(T^i_{pj}L^p_{nm}).\end{aligned}\tag{3.2}$$

3.2. According to (0.16), we get

$$\begin{aligned}\underset{3}{\bar{R}}{}^i_{jmn} &= \underset{0}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{2}T^i_{jm;n} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} + \frac{1}{2}T^i_{jn;m} + \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m}\\ &- \frac{1}{4}T^p_{jm}T^i_{pn} - \frac{1}{4}T^p_{jm}\mathcal{D}T^i_{pn} - \frac{1}{4}\mathcal{D}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm} + \frac{1}{4}T^p_{jn}\mathcal{D}T^i_{pm}\\ &+ \frac{1}{4}\mathcal{D}T^p_{jn}T^i_{pm} - \frac{1}{2}T^p_{mn}T^i_{pj} - \frac{1}{2}T^p_{mn}\mathcal{D}T^i_{pj} - \frac{1}{2}\mathcal{D}T^p_{mn}T^i_{pj}\end{aligned}$$

i.e.

$$\begin{aligned}\underset{3}{\bar{R}}{}^i_{jmn} &= \underset{3}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} + \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m}\\ &+ \frac{1}{4}\mathcal{D}(T^i_{pm}T^p_{jn} - T^i_{pn}T^p_{jm} - 2T^i_{pj}T^p_{mn}).\end{aligned}\tag{3.3}$$

4. Infinitesimal deformations of curvature tensor $\underset{4}{\boldsymbol{R}}$

4.1. Starting from (0.10) and using the procedure which is similar to that used for $\underset{3}{R}$, we obtain

$$\begin{aligned}\underset{4}{\bar{R}}{}^i_{jmn} &= \underset{4}{R}{}^i_{jmn} + (\mathcal{D}L^i_{jm})_{\underset{4}{|}n} - (\mathcal{D}L^i_{nj})_{\underset{3}{|}m}\\ &+ T^p_{jm}\mathcal{D}L^i_{np} + T^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}T^i_{jp} + \mathcal{D}(T^i_{pj}L^p_{mn})\end{aligned}\tag{4.1}$$

4.2. Using (0.17), we obtain for $\underset{4}{R}$

$$\underset{4}{\bar{R}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} + \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m} + \frac{1}{4}\mathcal{D}(T^i_{pm}T^p_{jn} - T^i_{pn}T^p_{jm} + 2T^i_{pj}T^p_{mn}). \quad (4.2)$$

5. Infinitesimal deformations of the tensor $\underset{5}{R}$

5.1 According to (0.11) we have

$$\begin{aligned} 2\underset{5}{\bar{R}}{}^i_{jmn} = {} & L^i_{jm,n} + (\mathcal{D}L^i_{jm}), n + L^i_{mj,n} + (\mathcal{D}L^i_{mj}), n - L^i_{jn,m} - (\mathcal{D}L^i_{jn}), m \\ & - L^i_{nj,m} - (\mathcal{D}L^i_{nj}), m + L^p_{jm}L^i_{pn} + L^p_{jm}\mathcal{D}L^i_{pn} + (\mathcal{D}L^p_{jm})L^i_{pn} + L^p_{mj}L^i_{np} \\ & + L^p_{mj}\mathcal{D}L^i_{np} + (\mathcal{D}L^p_{mj})L^i_{np} - L^p_{jn}L^i_{mp} - L^p_{jn}\mathcal{D}L^i_{mp} \\ & - (\mathcal{D}L^p_{jn})L^i_{mp} - L^p_{nj}L^i_{pm} - L^p_{nj}\mathcal{D}L^i_{pm} - (\mathcal{D}L^p_{nj})L^i_{pm}. \end{aligned} \quad (5.1)$$

By differentiating the Lie differences one obtains

$$\begin{aligned} (\mathcal{D}L^i_{jm})_{\underset{3}{|}n} &= (\mathcal{D}L^i_{jm})_{,n} + L^i_{pn}\mathcal{D}L^p_{jm} - L^p_{nj}\mathcal{D}L^i_{pm} - L^p_{nm}\mathcal{D}L^i_{jp}, \\ (\mathcal{D}L^i_{mj})_{\underset{4}{|}n} &= (\mathcal{D}L^i_{mj})_{,n} + L^i_{np}\mathcal{D}L^p_{mj} - L^p_{mn}\mathcal{D}L^i_{pj} - L^p_{jn}\mathcal{D}L^i_{mp}, \\ (\mathcal{D}L^i_{nj})_{\underset{3}{|}m} &= (\mathcal{D}L^i_{nj})_{,m} + L^i_{pm}\mathcal{D}L^p_{nj} - L^p_{mn}\mathcal{D}L^i_{pj} - L^p_{mj}\mathcal{D}L^i_{np}, \\ (\mathcal{D}L^i_{jn})_{\underset{4}{|}} &= (\mathcal{D}L^i_{jn})_{,m} + L^i_{mp}\mathcal{D}L^p_{jn} - L^p_{jm}\mathcal{D}L^i_{pn} - L^p_{nm}\mathcal{D}L^i_{jp}. \end{aligned}$$

If we substitute at (5.1) three members from the right sides of previous four equations, we get

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \frac{1}{2}[(\mathcal{D}L^i_{jm})_{\underset{3}{|}n} - (\mathcal{D}L^i_{nj})_{\underset{3}{|}m} + (\mathcal{D}L^i_{mj})_{\underset{4}{|}n} - (\mathcal{D}L^i_{jn})_{\underset{4}{|}m}]. \quad (5.2)$$

5.2. If we start from (0.18), we have

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{0}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{4}[T^p_{jm}T^i_{pn} + T^p_{jm}\mathcal{D}T^i_{pn} + (\mathcal{D}T^p_{jm})T^i_{pn} + T^p_{jn}T^i_{pm} + T^p_{jn}\mathcal{D}T^i_{pm} + (\mathcal{D}T^p_{jn})T^i_{pm}].$$

According to (0.18) the previous equation becomes:

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{4}[T^p_{jm}\mathcal{D}T^i_{pn} + (\mathcal{D}T^p_{jm})T^i_{pn} + T^p_{jn}\mathcal{D}T^i_{pm} + (\mathcal{D}T^p_{jn})T^i_{pm}],$$

i.e.

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{4}\mathcal{D}(T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm}). \quad (5.3)$$

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(received 23.12.2002)

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