МАТЕМАТИЧКИ ВЕСНИК
55 (2003), 31–36

UDK 517.521
оригинални научни рад
research paper

GENERALIZED BINOMIAL LAW AND REGULARLY VARYING MOMENTS

Slavko Simić

Abstract. In this paper we demonstrate a method for estimating asymptotic behavior of the regularly varying moments $E(K_\rho(X_n))$, $(n\to\infty)$ in the case of generalized Binomial Law. Here $K_\rho(x)$ is from the class of regularly varying functions in the sense of Karamata. We prove that

$$E(K_\rho(X_n))\sim K_\rho(E(X_n)),\ \rho>0,\quad E(X_n)\to\infty\quad(n\to\infty),$$

i.e., that the asymptotics of the first moment determines the behavior of all other moments.

1. Introduction

1.1. We shall consider a polynomial $P_n(c):=\sum_{k\le n}p_{nk}c^k$ with non-positive zeros and a random variable X_n defined as follows:

$$P\{X_n=k\}=\frac{p_{nk}c^k}{P_n(c)},\qquad k\le n;\quad k,n\in N\cup\{0\}.$$

We call this a generalized Binomial Law with parameter $c>0$, since for

$$P_n(c)=(1+c)^n,\quad c/(1+c):=p;\quad 1/(1+c):=q,$$

we obtain the well-known Binomial Law.

Define also, in the usual way, the first moment $E(X_n)$ and variance $D^2(X_n)$:

$$E(X_n):=\frac{1}{P_n(c)}\sum_{k\le n}kp_{nk}c^k;\qquad D^2(X_n):=\frac{1}{P_n(c)}\sum_{k\le n}(k-E(X_n))^2p_{nk}c^k.$$

The aim of this paper is to determine the asymptotic behavior of the moments generalized in the following way.

Let $K_\rho(x):=x^\rho\ell(x),\ x>0;\ K_\rho(0):=0$ be a regularly varying function of index $\rho\in R$ in the sense of Karamata. Then

$$E(K_\rho(X_n)):=\frac{1}{P_n(c)}\sum_{k\le n}k^\rho\ell(k)p_{nk}c^k,\quad \rho\in R.$$

AMS Subject Classification: 26 A 12

We shall prove the following

Theorem A. *For the generalized Binomial Law, defined above, we have*

$$E(K_\rho(X_n)) \sim K_\rho(E(X_n)), \quad E(X_n) \to \infty \quad (n \to \infty).$$

for each $\rho \in R^+$.

Therefore, for this class of distributions, it is particularly simple to determine the asymptotic behavior of its moments.

1.2. Karamata's class K_ρ plays here an important role. We say that $c \in K_\rho$ if it can be represented in the form $c(x) := x^\rho \ell(x)$, $x > 0$, $\rho \in R$, where ρ is the index of regular variation and $\ell(x) \in K_0$ is a slowly varying function, i.e., positive, measurable and satisfying $\ell(tx) \sim \ell(x)$, $\forall t > 0$ $(x \to \infty)$. Some examples of $\ell(x)$ are:

$$1,\ \log^a x,\ \log^b(\log x),\ \exp\left(\frac{\log x}{\log\log x}\right),\ \exp(\log^c x), \quad a, b \in R;\ 0 < c < 1.$$

According to [2], a sequence (c_n), $c_0 = 0$ is regularly varying with index $\rho \in R$ if it has the form $c_n := n^\rho \ell_n$, $n \in N$ and $\ell_n = \ell(n)$ for some continuous $\ell \in K_0$. Then we also say that $c_n \in K_\rho$.

The theory of regular variation is well-developed and for more details see [1] and [4].

2. Proofs

We prove Theorem A in three steps.

First, we suppose that $\rho \in N$, $\ell(\cdot) := 1$, and prove the next proposition.

Proposition 1. *If* $E(X_n) \to \infty$ $(n \to \infty)$, *then*

$$E(X_n^m) := \frac{1}{P_n(c)} \sum_{k \le n} k^m p_{nk} c^k \sim (E(X_n))^m \quad (n \to \infty),$$

for each $m \in N$.

Denote by A the set of all polynomials with non-positive zeros.

To prove the last assertion, we need the following lemma.

Lemma 1. *If* $P_n(c) \in A$ *and* $E(X_n)$, $D^2(X_n)$ *are defined as above, then* $0 \le \dfrac{D^2(X_n)}{E(X_n)} < 1$ *for each* $c \in R^+$, $n \in N$.

Proof. Since $P_n(c) \in A$, it can be represented in the form

$$P_n(c) = p_{nn} \prod_{k \le n} (c + a_{nk}), \quad a_{nk} \ge 0.$$

Hence

$$E(X_n) = c\frac{d}{dc}(\log P_n(c)) = \sum_{k\le n} \frac{c}{c+a_{nk}};$$

$$D^2(X_n) = E(X_n^2) - E^2(X_n) = c\frac{d}{dc}(E(X_n)) = \sum_{k\le n} \frac{ca_{nk}}{(c+a_{nk})^2}.$$

Therefore,

$$D^2(X_n) = \sum_{k\le n} \frac{c}{c+a_{nk}} \cdot \frac{a_{nk}}{c+a_{nk}} < \sum_{k\le n} \frac{c}{c+a_{nk}} = E(X_n),$$

i.e., Lemma 1 is proved. ■

Consider now a sequence of polynomials $\{Q_m(c)\}$ generated from $P_n(c)$ by the recurrence relation

$$Q_m(c) := cQ'_{m-1}(c); \qquad Q_0(c) := P_n(c), \qquad m \in N.$$

It is easy to see that

$$Q_m(c) = \sum_{k\le n} k^m p_{nk} c^k = E(X_n^m) P_n(c) \quad m \in N.$$

Since $P_n(c) \in A$, by the classical result, its zeros are separated by the zeros of $P_n'(c)$. Hence, zeros of $Q_1(c) := cP_n'(c)$ are also non-positive.

By induction we obtain $Q_m(c) \in A$, $m \in N$. Therefore, we can apply Lemma 1 to the polynomial $Q = Q_m(c) \in A$ and obtain

$$0 \le T_m := \frac{D_Q^2(X_n)}{E_Q(X_n)} < 1, \quad m \in N.$$

But $T_m = E(X_n^{m+1})/E(X_n^m) - E(X_n^m)/E(X_n^{m-1})$, $m \in N$; hence

$$\frac{E(X_n^m)}{E(X_n^{m-1})} = E(X_n) + \sum_{k\le m} T_{k-1} = E(X_n) + O(m).$$

On the other hand,

$$\begin{aligned} E(X_n^m) &= \prod_{k\le m} E(X_n^k)/E(X_n^{k-1}) \\ &= \prod_{k\le m} (E(X_n) + O(k)) = E(X_n)^m + O(m^2)E(X_n)^{m-1}. \end{aligned}$$

Since $m \in N$ is fixed and $E(X_n) \to \infty$ $(n \to \infty)$, Proposition 1 is proved. ■

In the next step, we shall prove our assertion for real positive exponents i.e.,

Proposition 2. *If* $E(X_n) \to \infty$ $(n \to \infty)$ *then*

$$E(X_n^\rho) \sim (E(X_n))^\rho \qquad (n \to \infty),$$

for each $\rho \in R^+$.

Proof. For this we need the well-known Lyapunov's moments inequality

LEMMA 2. *For real $r > s > t$ we have*
$$(E(X_n^s))^{r-t} \le (E(X_n^r))^{s-t} \cdot (E(X_n^t))^{r-s}.$$

Let $m < \rho < m-1$, $m \in N$. Applying Lemma 2 and Proposition 1, we get
$$\begin{aligned} E(X_n^\rho) &\le (E(X_n^m))^{\rho-m+1} \cdot (E(X_n^{m-1}))^{m-\rho} \\ &= (E(X_n))^{m(\rho-m+1)+(m-1)(m-\rho)}(1+o(1)) = (E(X_n))^\rho(1+o(1)). \end{aligned}$$
Hence $\limsup\limits_{n\to\infty} E(X_n^\rho)/(E(X_n))^\rho \le 1$.

Putting now in Lyapunov's inequality $r := m+1;\ s := m;\ t := \rho$ we obtain
$$\begin{aligned} E(X_n^\rho) &\ge (E(X_n^m))^{m+1-\rho}/(E(X_n^{m+1}))^{m-\rho} \\ &= (E(X_n))^{m(m+1-\rho)-(m+1)(m-\rho)}(1+o(1)) = (E(X_n))^\rho(1+o(1)), \end{aligned}$$
i.e., $\liminf\limits_{n\to\infty} E(X_n^\rho)/(E(X_n))^\rho \ge 1$.

Therefore, Proposition 2 is proved. ∎

Now we are able to prove Theorem A. For this, we just need the following assertion which is fundamental in the Theory of Regular Variation ([1], [4]).

LEMMA 3. *For any slowly varying $\ell(\cdot)$, some $\mu \in R^+$ and $y \to \infty$, we have*
$$\text{(i)}\quad \sup_{x<y}(x^\mu \ell(x)) \sim y^\mu \ell(y); \qquad \text{(ii)}\quad \sup_{x>y} x^{-\mu}\ell(x) \sim y^{-\mu}\ell(y).$$

We shall estimate the expression T,
$$T := \frac{E(K_\rho(X_n))}{E(X_n^\rho)\ell(E(X_n))} - 1 = \frac{\sum_{k\le n} k^\rho p_{nk}(\ell(k)/\ell(E(X_n)) - 1)c^k}{\sum_{k\le n} k^\rho p_{nk} c^k}.$$
Now, for some σ, $0 < \sigma < 1$ we get
$$\begin{aligned} |T| &\le \frac{\sum_{k\le n} k^\rho p_{nk}|\ell(k)/\ell(E(X_n)) - 1|c^k}{\sum_{k\le n} k^\rho p_{nk} c^k} \\ &= \frac{1}{\sum_{k\le n} k^\rho p_{nk} c^k}\Big(\sum_{k<\sigma E(X_n)} + \sum_{\sigma E(X_n)\le k\le E(X_n)/\sigma} + \sum_{k>E(X_n)/\sigma}\Big) \\ &= T_1 + T_2 + T_3. \end{aligned}$$
Applying Lemma 3 (part (i)) and Proposition 2, we obtain
$$\begin{aligned} T_1 &= \frac{1}{\sum_{k\le n} k^\rho p_{nk} c^k} \sum_{k<\sigma E(X_n)} k^{\rho/2} p_{nk}|k^{\rho/2}\ell(k)/\ell(E(X_n)) - k^{\rho/2}|c^k \\ &\le \sup_{k\le \sigma E(X_n)} (k^{\rho/2}\ell(k)/\ell(E(X_n)) + k^{\rho/2}) \frac{E(X_n^{\rho/2})}{E(X_n^\rho)} \\ &\sim 2(\sigma E(X_n))^{\rho/2} \cdot (E(X_n))^{-\rho/2} \ll \sigma^{\rho/2}, \end{aligned}$$

and, analogously, using (ii) of Lemma 3,

$$\begin{aligned} T_3 &\leq \sup_{k > E(X_n)/\sigma} (k^{-\rho/2}\ell(k)/\ell(E(X_n)) + k^{-\rho/2}) \frac{E(X_n^{3\rho/2})}{E(X_n^{\rho})} \\ &\sim 2(E(X_n)/\sigma)^{-\rho/2} \cdot (E(X_n))^{\rho/2} \ll \sigma^{\rho/2}. \end{aligned}$$

We also have

$$\begin{aligned} T_2 &= \frac{\sum_{\sigma E(X_n) \leq k \leq E(X_n)/\sigma} k^{\rho} p_{nk} |\ell(k)/\ell(E(X_n)) - 1| c^k}{\sum_{k \leq n} k^{\rho} p_{nk} c^k} \\ &\leq \sup_{\sigma E(X_n) \leq k \leq E(X_n)/\sigma} |\ell(k)/\ell(E(X_n)) - 1| = o(1) \quad (E(X_n) \to \infty), \end{aligned}$$

by the Uniform Convergence Theorem ([1], pp. 6–11).

Therefore,

$$T \leq T_1 + T_2 + T_3 = O(\sigma^{\rho/2}) + o(1) \quad (n \to \infty).$$

Since $\rho > 0$ and σ can be taken arbitrarily small, we deduce that

$$E(K_{\rho}(E(X_n)) \sim E(X_n^{\rho})\ell(X_n) \sim (E(X_n))^{\rho}\ell(E(x_n)) = K_{\rho}(E(X_n)) \quad (n \to \infty),$$

i.e., operators E and K_{ρ} are asymptotically commutative, which was the content of Theorem A. Hence, the proof is done. ∎

Remark 1. In the previous proof, the sum T_3 may be empty. But then

$$\begin{aligned} T_2 &\leq \sup_{\sigma E(X_n) \leq k \leq n} |\ell(k)/\ell(E(X_n)) - 1| \\ &\leq \sup_{\sigma E(X_n) \leq k \leq E(X_n)/\sigma} |\ell(k)/\ell(E(X_n) - 1| = o(1) \quad (n \to \infty), \end{aligned}$$

by Uniform Convergence Theorem again.

Finally, we give some applications of Theorem A.

Example 1. Taking $P_n(c) := (1+c)^n$; $E(X_n) = \frac{c}{1+c} n$ $(n \to \infty)$ and putting $\frac{c}{1+c} := p$; $\frac{1}{1+c} := q$ we obtain an asymptotic formula for regularly varying moments of the Binomial Law:

$$\sum_{k \leq n} k^{\rho} \ell_k \binom{n}{k} p^k q^{n-k} \sim p^{\rho} n^{\rho} \ell_n, \quad \rho \in R^+ \quad (n \to \infty).$$

Example 2. Laguerre polynomials $L_n^{(a)}(c)$ of index $a > -1$ have all zeros real and positive. Hence $L_n^{(a)}(-c)$, $c > 0$, satisfy the condition of Theorem A. Using Perron's formula (cf.[3], p.197) we obtain $E(X_n) \sim \sqrt{cn}$ $(n \to \infty)$, i.e.,

$$\frac{1}{L_n^{(a)}(-c)} \sum_{k \leq n} k^{\rho} \ell_k \binom{n+a}{n-k} \frac{c^k}{k!} \sim c^{\rho/2} n^{\rho/2} \ell(\sqrt{n}), \quad c > 0,\ \rho \in R^+ \quad (n \to \infty).$$

Remark 2. Further considerations can show that Theorem A is also valid for negative values of exponent ρ (see [5]).

REFERENCES

[**1**] N.H. Bingham et al., *Regular Variation*, Cambridge University Press, 1989.

[**2**] R. Bojanic, E. Seneta, *A unified theory of regularly varying sequences*, Math. Zeitsch. **134** (1973), 91–106.

[**3**] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., Providence, R.I., 1959.

[**4**] E. Seneta, *Regularly Varying Functions*, Springer-Verlag, 1976.

[**5**] S. Simić, *Asymptotic behavior of some complex sequences*, Publ. Inst. Math. **39(53)** (1986), 119–128.

(received 10.01.2002)

Matematički institut SANU, Kneza Mihaila 35/I, Beograd, Serbia & Montenegro

E-mail: ssimic@turing.mi.sanu.ac.yu