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THE LOWER AND UPPER TOPOLOGIES AS A BITOPOLOGY

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Abstract. The importance of the theory of bitopological spaces is fully demonstrated by its natural relationship to the theory of ordered topological spaces. Using the parallels drawn by M. Canfell and T. McCallion between the theory of bitopological spaces and that of ordered topological spaces, we construct the dimension theory for ordered topological spaces and formulate and study the Baire-like properties of the latter spaces, thereby filling in the gap of the theory of ordered topological spaces. Further, based on these parallels, the relations between the separation axioms of ordered topological spaces and the corresponding bitopological spaces are established.

1. Introduction

The formation and progress of the theory of bitopological spaces, that is to say, sets on which are defined two arbitrary topologies, originated from [15], are not of isolated character. The theory acquires special importance in the light of applications of its results.

It is to be noted that distance functions, uniformity and proximity are the related notions in defining topology and, naturally, the situation treated in [15] is by no means the only way leading to a symmetric occurrence of two topologies on the same set: the investigations of quasi-uniformity [21], [33] and quasi-proximity [13], [24] also lead to a similar result. Proceeding from the symmetric generation of two topologies on a set, alongside with the above-mentioned cases, we can also consider ordered topological spaces i.e., sets, having a topology and a partial order [1], [6], [18], [19], [22], [25], partially ordered sets [2] and hence directed graphs [7], [8], semi-Boolean algebras [26], S-related topologies [34] and so on.

There are several hundred works dedicated to the investigation of bitopologies, i.e., pairs of topologies on the same set; most of them deal with the theory itself but very few with applications. The later papers have appeared after the late sixties (see, for example, [4]–[6], [8], [10], [11], [14], [17], [19], [25], [29]–[31]).

In the present paper the following abbreviations will be used: BS for a bitopological space, BsS for a bitopological subspace; BSs for bitopological spaces; similarly, TS for a topological space, OTS for an ordered topological space and so

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on. Always $i, j \in \{1, 2\}$ and $i \neq j$. Also note that in our discussion the letter "u" abbreviates the word "upper", the letter "l" abbreviates the word "lower" and the combination of the letters l.s.c. (u.s.c.) abbreviate the phrase "lower (upper) semicontinuous".

Let (X, τ_1, τ_2) be a BS and $\mathcal{P}$ be some topological property. Then (i, j)-$\mathcal{P}$ denotes the analogue of this property for τ_i with respect to τ_j, and p-$\mathcal{P}$ denotes the conjunction $(1, 2)$-$\mathcal{P} \wedge (2, 1)$-$\mathcal{P}$, i.e., p-$\mathcal{P}$ denotes the "absolute" bitopological analogue of $\mathcal{P}$, where p is the abbreviation for "pairwise". As we shall see below, sometimes $(1, 2)$-$\mathcal{P} \iff (2, 1)$-$\mathcal{P}$ (and thus $\iff p$-$\mathcal{P}$) so that it suffices to consider one of these three bitopological analogues. Also note that (X, τ_i) has a property $\mathcal{P} \iff (X, \tau_1, \tau_2)$ has a property i-$\mathcal{P}$, and d-$\mathcal{P} \iff 1$-$\mathcal{P} \wedge 2$-$\mathcal{P}$, where d is the abbreviation for "double".

By analogy, for an OTS $(X, \tau, \leq)$ we will consider the properties (l, u)-$\mathcal{P}$, $(u.l)$-$\mathcal{P}$ and O-$\mathcal{P} \iff (l, u)$-$\mathcal{P} \wedge (u, l)$-$\mathcal{P}$.

Let (X, τ_1, τ_2) be any BS and $A \subset X$ be its any subset. Then $\tau_i \operatorname{cl} A$ and $\tau_i \operatorname{int} A$ denote respectively the closure and the interior of A in the topology τ_i.

Though we expect the reader to be familiar with the basic notions of L. Nachbin's theory, nevertheless, we would like to recall some of them. Following [22], each subset $A \subset X$ determines in a unique fashion an increasing set $\mathrm{i}(A)$ (a decreasing set $\mathrm{d}(A)$) which is the smallest one among increasing (decreasing) sets containing A. A set $A \subset X$ is said to be convex if $A = \mathrm{i}(A) \cap \mathrm{d}(A)$. The smallest closed increasing set $\mathrm{I}(A)$ (the smallest closed decreasing set $\mathrm{D}(A)$) is also defined in a unique fashion. Hence the set $\mathrm{I}(A) \cap \mathrm{D}(A)$ is the smallest closed convex set containing A.

If $I = [0, 1]$ is the unit interval, then (I, ω') is the TsS of the natural TS $(\mathbb{R}, \omega)$. For the natural OTS $(\mathbb{R}, \omega, \leq)$ the binary relation $\leq$ is the natural order on $\mathbb{R}$ and $(I, \omega', \leq')$ is the OTsS of $(\mathbb{R}, \omega, \leq)$.

Finally, please note that all generalizations for bitopological or topology-order cases are constructed in the commonly accepted manner so that if the topologies coincide or a partial order on a set is discrete, one obtains the classical definitions and results from general topology.

2. Axioms of Separation

Definition 2.1. A function $f : (X, \tau_1, \tau_2) \to (I, \omega')$ is said to be (i, j)-l.u.semicontinuous (briefly, (i, j)-l.u.s.c.) if f is i-l.s.c. and j-u.s.c.

Using Definition 2.1 from [16] in our terms we come to

Definition 2.2. Let (X, τ_1, τ_2) be a BS and A, B be subsets of X. Then A is (i, j)-completely separated from B if there is an (i, j)-l.u.s.c. function $f : (X, \tau_1, \tau_2) \to (I, \omega')$ such that $f(A) = 0$ and $f(B) = 1$.

Since $f : (X, \tau_1, \tau_2) \to (I, \omega')$ is (i, j)-l.u.s.c. $\iff (1 - f) : (X, \tau_1, \tau_2) \to (I, \omega')$ is (j, i)-l.u.s.c., it is clear that A is (i, j)-completely separated from $B \iff B$ is (j, i)-completely separated from A.

DEFINITION 2.3. Let (X, τ_1, τ_2) be a BS. Then

(1) (X, τ_1, τ_2) is R-p-T_1 (i.e., p-T_1 in the sense of Reilly) if it is d-T_1 [27].

(2) (X, τ_1, τ_2) is p-T_2 if for each pair of distinct points x, $y \in X$ there exist disjoint a 1-open set U and a 2-open set V such that $x \in U$, $y \in V$ [15].

(3) (X, τ_1, τ_2) is (i, j)-regular if for each point $x \in X$ and each i-closed set $F \subset X$, $x \overline{\in} F$, there exist an i-open set $U \subset X$ and a j-open set $V \subset X$ such that $x \in U$, $F \subset V$ and $U \cap V = \varnothing$ [15].

(4) (X, τ_1, τ_2) is (i, j)-completely regular if every i-closed set $F \subset X$ is (i, j)-completely separated from each point $x \overline{\in} F$ [16], [20].

(5) (X, τ_1, τ_2) is p-normal if for every pair of disjoint sets A, B in X, where A is 1-closed and B is 2-closed, there exist a 2-open set $U \subset X$ and a 1-open set $V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$ [15].

(6) (X, τ_1, τ_2) is hereditarily p-normal if its every bitopological subspace is p-normal [9].

(7) (X, τ_1, τ_2) is (i, j)-extremally disconnected if $\tau_j \operatorname{cl} U = \tau_i \operatorname{int} \tau_j \operatorname{cl} U$ for every set $U \in \tau_i$ or, equivalently, $\tau_j \operatorname{cl} \tau_i \operatorname{int} A = \tau_i \operatorname{int} \tau_j \operatorname{cl} \tau_i \operatorname{int} A$ for every subset $A \subset X$ [3].

Using Lemma 0.2.1 from [11] it is not difficult to see that (X, τ_1, τ_2) is $(1, 2)$-extr. disconn. $\Longleftrightarrow$ (X, τ_1, τ_2) is $(2, 1)$-extr. disconn. $\Longleftrightarrow$ (X, τ_1, τ_2) is p-extr. disconn.

THEOREM 2.1. *A* BS (X, τ_1, τ_2) *is hereditarily p-normal if and only if it is p-completely normal in the sense of Patty, i.e., if and only if whenever A and B are subsets of X such that* $(\tau_1 \operatorname{cl} A \cap B) \cup (A \cap \tau_2 \operatorname{cl} B) = \varnothing$ *there exist a* 2*-open set U and a* 1*-open set V which are disjoint and for which $A \subset U$ and $B \subset V$* [23].

COROLLARY. *Every hereditarily p-normal* BS *is p-normal.*

In the topological case the complete regularity in internal terms, i.e., without using the notion of a function, was characterized by O. Frink [12], E. F. Steiner [32] and V. I. Zaĭcev [36]. Their modifications for BSs were studied, on the one hand, by M. J. Saegrove [28] using the generalization of Steiner's method and, on the other hand, by us with the aid of the generalized method of O. Frink and V. I. Zaĭcev (see Theorem 2.2).

A double family, i.e., a pair of families $\mathcal{Z} = \{\mathcal{Z}_1, \mathcal{Z}_2\}$, where $\mathcal{Z}_i$ is an i-closed base of a BS (X, τ_1, τ_2), is called a d-closed base and $\operatorname{co} \mathcal{Z} = \{\operatorname{co} \mathcal{Z}_1, \operatorname{co} \mathcal{Z}_2\}$, where $\operatorname{co} \mathcal{Z}_i$ is an i-open base, conjugate with $\mathcal{Z}_i$, is called a d-open base, conjugate with $\mathcal{Z} = \{\mathcal{Z}_1, \mathcal{Z}_2\}$.

DEFINITION 2.4. A d-closed base $\mathcal{Z} = \{\mathcal{Z}_1, \mathcal{Z}_2\}$ of a BS (X, τ_1, τ_2) is said to be a p-normal if the following conditions are satisfied:

(1) For every point $x \in X$ and its any neighbourhood $U(x) \in \operatorname{co} \mathcal{Z}_1$ ($U(x) \in \operatorname{co} \mathcal{Z}_2$) there exists a set $A \in \mathcal{Z}_2$ ($A \in \mathcal{Z}_1$) such that $x \in A \subset U(x)$.

(2) If $A \in \mathcal{Z}_1$, $B \in \mathcal{Z}_2$ and $A \cap B = \varnothing$, then there exist $U \in \operatorname{co}\mathcal{Z}_2$, $V \in \operatorname{co}\mathcal{Z}_1$ such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$.

THEOREM 2.2. *A* BS (X, τ_1, τ_2) *is p-completely regular if and only if it possesses at least one p-normal base* [11, *Theorem* 0.2.2].

In [6] M. Canfell draw the parallel between the theories of OTSs and BSs in the following manner: to each OTS $(X, \tau \le)$ there corresponds the BS (X, τ_1, τ_2), where $\tau_1 = \{U \in \tau : U = \mathrm{i}(U)\}$ and $\tau_2 = \{V \in \tau : V = \mathrm{d}(V)\}$ are respectively the upper and the lower topology with respect to the partial order $\le$ in terms of [6]. On the other hand, M. Canfell leaves the question open in what cases a BS can be treated as an OTS, i.e., whether (X, τ_1, τ_2) is a BS and whether $\tau = \sup(\tau_1, \tau_2)$, and what conditions one must have for the existence of a partial order $\le$ on X such that τ_1 would coincide with the upper topology and τ_2 with the lower topology of $(X, \tau, \le)$. According to H. A. Pristley [25, Proposition 10] the answer to this question is as follows: let (X, τ_1, τ_2) be a BS and $\tau = \sup(\tau_1, \tau_2)$ be compact. Then there exists a partial order $\le$ with a closed graph on X such that τ_1 and τ_2 are respectively the upper and the lower topology of $(X, \tau, \le)$ if and only if

(1) (X, τ_1, τ_2) is 1-T_0 (or a 2-T_0).

(2) (X, τ_1, τ_2) is p-regular.

The above duality seems essential for discussing different mutually beneficial relations between these two theories.

For example we can give a fact, which directly follows from this duality, (7) of Definition 2.3 and [25, p. 521]:

(X, τ_1, τ_2) is p-extremally disconnected $\Longleftrightarrow$ $(X, \tau, \le)$ is extremally order disconnected in $\tau_1 \Longleftrightarrow (X, \tau, \le)$ is extremally order disconnected in τ_2.

In the context with the above-said we shall consider the axioms of separation of OTSs, taking into account the axioms of separation of the corresponding BSs, also introduce and investigate the dimension functions and Baire-like properties for OTSs. Please take into consideration that owing to duality, the results constructed here are of quite a simple character.

DEFINITION 2.5. An OTS $(X, \tau, \le)$ is said to be upper (lower) T_1-ordered if for each pair of elements x, $y \in X$, $x \not\le y$, there exists a neighbourhood $U(y) = \mathrm{d}(U(y))$ $(U(x) = \mathrm{i}(U(x)))$ such that $x \,\overline{\in}\, U(y)$ $(y \,\overline{\in}\, U(x))$, and $(X, \tau, \le)$ is said to be T_1-ordered if it is both upper and lower T_1-ordered [19].

The concept of T_1-order coincides with those of semicontinuous partial order [35] and semiclosed partial order [22].

DEFINITION 2.6. An OTS $(X, \tau, \le)$ is said to be T_2-ordered if for each pair of elements $x, y \in X$, $x \not\le y$, there exist disjoint neighbourhoods $U(x) = \mathrm{i}(U(x))$ and $U(y) = \mathrm{d}(U(y))$ [19].

This concept coincides with those of continuous partial order and closed partial order in [35] and [22], respectively.

Definition 2.7. An OTS $(X, \tau, \leq)$ is said to be upper (lower) regularly ordered if for each set $F = I(F) \subset X$ $(F = D(F) \subset X)$ and each element $x \overline{\in} F$ there exist disjoint neighbourhoods $U(F) = \mathrm{i}(U(F))$ and $U(x) = \mathrm{d}(U(x))$ $(U(F) = \mathrm{d}(U(F))$ and $(U(x) = \mathrm{i}(U(x)))$.

$(X, \tau, \leq)$ is said to be regularly ordered if it is both upper and lower regularly ordered. $(X, \tau, \leq)$ is upper (lower) T_3-ordered if $(X, \tau, \leq)$ is both upper (lower) T_1-ordered and upper (lower) regularly ordered. $(X, \tau, \leq)$ is T_3-ordered if it is both T_1-ordered and regularly ordered [19].

Definition 2.8. An OTS $(X, \tau, \leq)$ is said to be normally ordered if for each pair of disjoint sets $F_1 = \mathrm{I}(F_1)$, $F_2 = \mathrm{D}(F_2)$ there exist disjoint neighbourhoods $U(F_1) = \mathrm{i}(U(F_1))$ and $U(F_2) = \mathrm{d}(U(F_2))$ [19], [22].

$(X, \tau, \leq)$ is said to be T_4-ordered if it is T_1-ordered and normally ordered [19].

The work [19] also contains the definitions of the strong T_k-order separation axioms for $k = \overline{1,4}$ obtained from the T_k-order separation axioms by using the term "open neighbourhood" instead of "neighbourhood". We denote the T_k-order (the strong T_k-order) separation axioms by $T_{k(0)}$ $(ST_{k(0)})$ for $k = \overline{1,4}$.

It is obvious that following implications hold:

$$\begin{array}{ccccccc}
ST_{4(0)} & \Longrightarrow & ST_{3(0)} & \Longrightarrow & ST_{2(0)} & \Longrightarrow & ST_{1(0)} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
T_{4(0)} & \Longrightarrow & T_{3(0)} & \Longrightarrow & T_{2(O)} & \Longrightarrow & T_{1(0)}.
\end{array}$$

The converse implications are not generally valid.

Out of the basic separation axioms only the axiom $T_{3\frac{1}{2}(0)}$ will be recalled.

Definition 2.9. An OTS $(X, \tau, \leq)$ is said to be completely regularly ordered if the following conditions are satisfied:

(1) For each point $x \in X$ and its every neighbourhood $U(x)$ there are two continuous real-valued functions f and g on X, where f is order preserving and g is order reversing such that $0 \leq f \leq 1$, $0 \leq g \leq 1$, $f(x) = 1 = g(x)$ and $\inf(f(y), g(y)) = 0$ if $y \in X \backslash U(x)$.

(2) If $x, y \in X$, $x \not\leq y$, then there exists an order preserving continuous real-valued function f such that $f(x) > f(y)$ [22].

$(X, \tau, \leq)$ is said to be $T_{3\frac{1}{2}(0)}$ ordered if it is T_1 and completely regularly ordered.

Let $(X, \tau, \leq)$ be an OTS. Then by [18] a set $A \subset X$ is called a decreasing (increasing) zero set in $(X, \tau, \leq)$ if there is an order preserving (order reversing) continuous function $f : (X, \tau, \leq) \to (\mathbb{R}, \omega, \leq)$ such that $A = \{x \in X : \ f(x) \leq 0\}$. The family of all dereasing (increasing) zero sets in $(X, \tau, \leq)$ is denoted by $\mathcal{A}_1$ $(\mathcal{A}_2)$. If $f : (X, \tau, \leq) \to (\mathbb{R}, \omega, \leq)$ is continuous and order preserving (order reversing), then by Proposition 1.1 from [18], $A \in \mathcal{A}_1$ $(A \in \mathcal{A}_2)$ defines a continuous order preserving (order reversing) function $f : (X, \tau, \leq) \to (\mathbb{R}, \omega, \leq)$ such that $A = \{x \in$

$X :\ f(x) = 0,\ f \geq 0\}$. Clearly, in both cases A is closed in $(X, \tau, \leq)$ and therefore $A = \mathrm{D}(A)$ for $A \in \mathcal{A}_1$ ($A = \mathrm{I}(A)$ for $A \in \mathcal{A}_2$).

Following [18], the family $\mathcal{A}_1(\mathcal{A}_2)$ is a base of closed sets of a topology $\tau_{\mathcal{A}_1}$ ($\tau_{\mathcal{A}_2}$) on X. Such topologies are characteristic of completely regularly ordered spaces.

Definition 2.10. Let $(X, \tau, \leq)$ be an OTS and $\mathcal{Z}_1$ ($\mathcal{Z}_2$) be a family of decreasing (increasing) closed sets of X. Then $\mathcal{Z}_1 \cup \mathcal{Z}_2$ is called a normally ordered subbase for $(X, \tau, \leq)$ if the following conditions are satisfied:

(1) $\mathcal{Z}_1$ ($\mathcal{Z}_2$) is a base for closed sets of the topology $\tau_{\mathcal{Z}_1}$ ($\tau_{\mathcal{Z}_2}$) on X such that $\sup(\tau_{\mathcal{Z}_1}, \tau_{\mathcal{Z}_2}) = \tau$ and $(\tau_{\mathcal{Z}_1}, \tau_{\mathcal{Z}_2})$ is an order defining pair, so that $x \in \tau_{\mathcal{Z}_1} \operatorname{cl}\{y\} \Longleftrightarrow x \leq y \Longleftrightarrow y \in \tau_{\mathcal{Z}_2} \operatorname{cl}\{x\}$.

(2) If $x \in X$, $F \in \operatorname{co}\tau_{\mathcal{Z}_1}$ ($F \in \operatorname{co}\tau_{\mathcal{Z}_2}$) and $x \,\overline{\in}\, F$, then there is a set $A \in \mathcal{Z}_2$ ($A \in \mathcal{Z}_1$) such that $x \in A$ and $A \cap F = \varnothing$.

(3) If $A_1 \in \mathcal{Z}_1$, $A_2 \in \mathcal{Z}_2$ and $A_1 \cap A_2 = \varnothing$, then there are sets $A_1' \in \mathcal{Z}_1$, $A_2' \in \mathcal{Z}_2$ such that $A_1 \subseteq A_1'$, $A_2 \subseteq A_2'$, $A_1 \cap A_2' = \varnothing = A_1' \cap A_2$ and $A_1' \cup A_2' = X$ [18].

We conclude the dicussion of the axioms of separation of OTSs by investigating their relations with the axioms of separation of the corresponding BSs, where the correspondence is of one of the two types mentioned above.

Theorem 2.3. *Let* $(X, \tau, \leq)$ *be an* OTS, (X, τ_1, τ_2) *and* $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ *be the corresponding* BS*s in the sense of M. Canfell* [6] *and T. McCallion* [18], *respectively. Then the following statements are valid:*

(1) (X, τ_1, τ_2) *is* R-p-$T_1 \Longrightarrow (X, \tau, \leq)$ *is* $ST_{1(0)}$.

(2) (X, τ_1, τ_2) *is* p-$T_2 \Longrightarrow (X, \tau, \leq)$ *is* $ST_{2(0)}$.

(3) (X, τ_1, τ_2) *is p-regular* $\Longleftrightarrow (X, \tau, \leq)$ *is strong regularly ordered.*

(4) $(X, \tau, \leq)$ *is completely regularly ordered* $\Longrightarrow (X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ *is p-completely regular and, conversely, if* $\sup(\tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) = \tau$, *then* $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ *is* p-$T_{3\frac{1}{2}} \Longrightarrow (X, \tau, \leq)$ *is* $T_{3\frac{1}{2}(0)}$.

(5) (X, τ_1, τ_2) *is p-normal* $\Longleftrightarrow (X, \tau, \leq)$ *is strong normally ordered.*

Proof. (1)–(3) and (5) are immediate consequences of the corresponding definitions. Hence it remains to prove only (4). We begin by assuming that $(X, \tau, \leq)$ is completely regularly ordered. Then by Theorem 1.3 from [18], $\mathcal{A}_1 \cup \mathcal{A}_2$ is a normally ordered subbase for $(X, \tau, \leq)$. Therefore on account of Definition 2.4, $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2\}$ is a p-normal base for the BS $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ and it remains to use Theorem 2.2.

Conversely, let $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ be p-$T_{3\frac{1}{2}}$ and $x \in X$. It is assumed that $U(x)$ is any τ-neighbourhood of x. Since $\sup(\tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) = \tau$, there are $U_1 \in \tau_{\mathcal{A}_1}$, $U_2 \in \tau_{\mathcal{A}_2}$ such that $x \in U_1 \cap U_2 \subset U(x)$. Thus $x \,\overline{\in}\, X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2)$ so that $x \,\overline{\in}\, X \setminus U_1$ and $x \,\overline{\in}\, X \setminus U_2$. Since $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ is p-completely regular, there are functions $f, g : (X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) \to (\mathrm{I}, \omega')$, where f is a (1,2)-l.u.s.c., g is a (2,1)-l.u.s.c. such that $f(x) = 1 = g(x)$, $f(X \setminus U_1) = g(X \setminus U_2) = 0$. But $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$

is p-completely regular and therefore it is p-regular. Hence by [15, p. 87], $f, g : (X,\tau,\leq) \to (\mathrm{I},\omega',\leq')$ are continuous and respectively order preserving and order reversing functions. Since $U_1 \cap U_2 \subset U(x) \Longrightarrow X\backslash U(x) \subset (X\backslash U_1) \cup (X\backslash U_2)$, we obtain $\inf\{f(y), g(y)\} = 0$ for $y \in X\backslash U(x)$. Thus (1) of Definition 2.9 is satisfied.

Finally, let $x, y \in X$, $x \leq y$ be false. Then $x \neq y$ and since $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ is R-p-T_1, we have $\{y\} \in \operatorname{co}\tau_1$ and therefore there is a $(1,2)$-l.u.s.c. function $f : (X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) \to (I, \omega')$ such that $f(x) = 1$, $f(y) = 0$. By virtue of the above reasoning, $f : (X, \tau, \leq\ ) \to (I, \omega', \leq)$ is also continuous and order preserving. ∎

Corollary 1. *Let* $(X,\tau,\leq)$ *be an* OTS *and* $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ *be* R-p-T_1. *If* $\sup(\tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) = \tau$, *then* $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ *is* p-$T_{3\frac{1}{2}} \Longleftrightarrow (X,\tau,\leq)$ *is* $T_{3\frac{1}{2}(0)}$.

Proof. Is an immediate consequence of (4) of Theorem 2.3. ∎

Corollary 2. *If* $\sup(\tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) = \tau$ *and* $(X, \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})$ *is* p-$T_{3\frac{1}{2}}$, *then* $(X,\tau,\leq)$ *has a normally ordered subbase.*

Proof. Indeed, by (4) of Theorem 2.3, $(X,\tau,\leq)$ is $T_{3\frac{1}{2}(0)}$ and therefore completely regularly ordered. Thus it remains to use Theorem 1.3 from [18]. ∎

3. Dimension functions

Let us define new operators which are also natural and necessary for further investigations. Every set A in $(X,\tau,\leq)$ determines uniquely the largest decreasing (increasing) set $d_1(A)$ ($i_1(A)$) contained in A and the largest open decreasing set $\mathrm{D}_1(A)$ (increasing set $\mathrm{I}_1(A)$) contained in A. It is obvious that $d_1 = X \backslash \mathrm{i}(X\backslash A)$ ($i_1(A) = X \backslash \mathrm{d}(X\backslash A)$), $\mathrm{D}_1(A) = X \backslash \mathrm{I}(X\backslash A)$ ($I_1(A) = X \backslash \mathrm{D}(X\backslash A)$). Moreover, if A is closed and decreasing (increasing), then $A = d(A) = \mathrm{D}(A) = d_1(A)$ ($A = i(A) = \mathrm{I}(A) = i_1(A)$), and if A is open and decreasing (increasing), then $A = d_1(A) = \mathrm{D}_1(A) = \mathrm{d}(A)$ ($A = \mathrm{i}_1(A) = I_1(A) = \mathrm{i}(A)$).

If $(X,\tau,\leq)$ is an OTS and (X,τ_1,τ_2) is the corresponding BS in Canfell's sense, then it is clear that $\mathrm{D}(A) = \tau_1 \operatorname{cl} A$ ($I_1(A) = \tau_1 \operatorname{int} A$), $\mathrm{I}(A) = \tau_2 \operatorname{cl} A$ ($\mathrm{D}_1(A) = \tau_2 \operatorname{int} A$) and thus A is closed and convex in $(X,\tau,\leq)$, i.e., $A = \mathrm{D}(A) \cap \mathrm{I}(A) \Longleftrightarrow A$ is p-closed in (X,τ_1,τ_2), i.e., $A = \tau_1 \operatorname{cl} A \cap \tau_2 \operatorname{cl} A$. Due to these relations we have $\mathrm{D}(A\cup B) = \mathrm{D}(A)\cup\mathrm{D}(B)$, $\mathrm{D}(\mathrm{D}(A)) = \mathrm{D}(A)$, $\mathrm{I}(A\cup B) = \mathrm{I}(A)\cup\mathrm{I}(B)$, $\mathrm{I}(\mathrm{I}(A)) = \mathrm{I}(A)$, $\mathrm{D}_1(A\cap B) = \mathrm{D}_1(A)\cap\mathrm{D}_1(B)$, $\mathrm{D}_1(\mathrm{D}_1(A)) = \mathrm{D}_1(A)$ and $\mathrm{I}_1(A\cap B) = \mathrm{I}_1(A)\cap\mathrm{I}_1(B)$, $\mathrm{I}_1(\mathrm{I}_1(A)) = \mathrm{I}_1(A)$.

Also note that u-$\mathcal{B}d(X) = \{A \in 2^X : \mathrm{I}_1(A) = \varnothing\}$, l-$\mathcal{B}d(X) = \{A \in 2^X : \mathrm{D}_1(A) = \varnothing\}$, u-$\mathrm{D}(X) = \{A \in 2^X : \mathrm{I}(A) = X\}$ and l-$\mathrm{D}(X) = \{A \in 2^X : \mathrm{D}(A) = X\}$.

Definition 3.1. For a subset A of an OTS $(X,\tau,\leq)$ the (l,u)- and (u,l)-boundaries are respectively the sets (l,u)-$\operatorname{Fr} A = \mathrm{D}(A) \cap \mathrm{I}(X\backslash A)$, (u,l)-$\operatorname{Fr} A = \mathrm{I}(A) \cap \mathrm{D}(X\backslash A)$.

Theorem 3.1. *In an* OTS $(X,\tau,\leq)$ *the* (l,u)- *and* (u,l)-*boundaries have the properties as follows:*

(1) (l,u)-$\operatorname{Fr} A = B_1 \cup B_2$, *where* $B_1 \in l$-$\mathcal{B}d(X)$ *and* $B_2 \in u$-$\mathcal{B}d(X)$.
(u,l)-$\operatorname{Fr} A = C_1 \cup C_2$, *where* $C_1 \in u$-$\mathcal{B}d(X)$ *and* $C_2 \in l$-$\mathcal{B}d(X)$.

(2) $\mathrm{D}(A) = \mathrm{D}_1(A) \cup (l,u)$-$\operatorname{Fr} A$, $\mathrm{I}(A) = I_1(A) \cup (u,l)$-$\operatorname{Fr} A$.

(3) $\mathrm{D}_1(A) = A \backslash (l,u)$-$\operatorname{Fr} A$, $I_1(A) = A \backslash (u,l)$-$\operatorname{Fr} A$.

(4) (l,u)-$\operatorname{Fr} A = (u,l)$-$\operatorname{Fr}(X \backslash A)$.

(5) $X = I_1(A) \cup (u,l)$-$\operatorname{Fr} A \cup \mathrm{D}_1(X\backslash A) = \mathrm{D}_1(A) \cup (l,u)$-$\operatorname{Fr} A \cup I_1(X \backslash A)$.

(6) (l,u)-$\operatorname{Fr} \mathrm{D}(A) \cup (l,u)$-$\operatorname{Fr} \mathrm{D}_1(A) \subset (l,u)$-$\operatorname{Fr} A$.
(u,l)-$\operatorname{Fr} \mathrm{I}(A) \cup (u,l)$-$\operatorname{Fr} \mathrm{I}_1(A) \subset (u,l)$-$\operatorname{Fr} A$.

(7) $A = \mathrm{D}(A) \Longleftrightarrow (l,u)$-$\operatorname{Fr} A = A \backslash \mathrm{D}_1(A)$.
$A = \mathrm{I}(A) \Longleftrightarrow (u,l)$-$\operatorname{Fr} A = A \backslash I_1(A)$.

(8) $A = \mathrm{D}_1(A) \Longleftrightarrow (l,u)$-$\operatorname{Fr} A = \mathrm{D}(A) \backslash A$.
$A = I_1(A) \Longleftrightarrow (u,l)$-$\operatorname{Fr} A = \mathrm{I}(A) \backslash A$.

(9) $A = \mathrm{D}_1(A) = \mathrm{D}(A) \Longleftrightarrow (l,u)$-$\operatorname{Fr} A = \varnothing$.
$A = I_1(A) = \mathrm{I}(A) \Longleftrightarrow (u,l)$-$\operatorname{Fr} A = \varnothing$.

(10) (l,u)-$\operatorname{Fr} A \cup (l,u)$-$\operatorname{Fr} B = (l,u)$-$\operatorname{Fr}(A \cup B) \cup (l,u)$-$\operatorname{Fr}(A \cap B) \cup ((l,u)$-$\operatorname{Fr} A \cap (u,l)$-$\operatorname{Fr} B) \cup ((l,u)$-$\operatorname{Fr} B \cap (u,l)$-$\operatorname{Fr} A)$.
(u,l)-$\operatorname{Fr} A \cup (u,l)$-$\operatorname{Fr} B = (u,l)$-$\operatorname{Fr}(A \cup B) \cup (u,l)$-$\operatorname{Fr}(A \cap B) \cup ((u,l)$-$\operatorname{Fr} A \cap (l,u)$-$\operatorname{Fr} B) \cup ((u,l)$-$\operatorname{Fr} B \cap (l,u)$-$\operatorname{Fr} A)$.

Proof. The theorem is proved by simple calculations, taking into account Canfell's duality and Theorem 1.3.1 from [11]. ■

Remark 3.1. Following [25, p. 509] if A is a subset of an OTS $(X, \tau, \leq)$, A may be made into an OTS $(A, \tau', \leq')$, where τ' is the induced topology and $\leq'$ the induced order, so that if x, $y \in A$, then $x \leq' y \Longleftrightarrow x \leq y$. On the other hand, A may be regarded as a BsS (A, τ_1', τ_2'), τ_1' and τ_2' being the topologies induced on A by the upper and lower topologies τ_1 and τ_2 on X. It is clear, that every member of τ_1' is τ'-open and increasing in A and every member of τ_2' is τ'-open and decreasing in A. Define $(A, \tau', \leq')$ to be an order subspace of $(X, \tau, \leq)$, if τ_1' and τ_2' coincide with the upper and lower topologies of $(A, \tau', \leq')$. The same idea is considered by S. D. McCartan [19], where the term "τ-compatibly ordered" is used.

In the sequel we shall consider only an order subspace of an OTS $(X, \tau, \leq)$.

Definition 3.2. An OTS $(X, \tau, \leq)$ is said to be hereditarily strong normally ordered if its any order subspace is strong normally ordered.

To characterize such spaces we shall make use of the following notions.

Definition 3.3. If A and B are subsets of an OTS $(X, \tau, \leq)$, then we write $A <' B$ to indicate that $(A \cap \mathrm{I}(B)) \cup (\mathrm{D}(A) \cap B) = \varnothing$. In addition, if $A <' B$ and there exist disjoint neighbourhoods $U(A) = \mathrm{D}_1(U(A))$ and $U(B) = I_1(U(B))$, then we write $A \ll' B$.

Proposition 3.1. *The following conditions are satisfied in an* OTS $(X, \tau, \leq)$*:*

(1) $(X, \tau, \leq)$ *is hereditarily strong normally ordered if and only if* $A <' B \Longrightarrow A \ll' B$*, for every pair of subsets* $A, B \subset X$.

(2) If $(X, \tau, \leq)$ *is strong regularly ordered and* $A \subset X$*, then* $(A, \tau', \leq')$ *is also strong regularly ordered.*

(3) If $(X, \tau, \leq)$ *is strong normally ordered and* $A = \mathrm{D}(A) \cap \mathrm{I}(A) \subset X$*, then* $(A, \tau', \leq')$ *is also strong normally ordered.*

Proof. Using Canfell's duality and Remark 3.1, the proof immediately follows from Theorem 2.1, Propositions 3.1.1 and 3.2.1 from [11]. ■

COROLLARY. *Every hereditarily strong normally ordered space is strong normally ordered.*

Proof. Follows immediately from (1) of Proposition 3.1. ■

DEFINITION 3.4. Let $(X, \tau, \leq)$ be an OTS and n denote a nonnegative integer. We say that

$(1)_1$ (u,l)-$\operatorname{ind} X = -1 \Longleftrightarrow X = \varnothing$.

$(2)_1$ (u,l)-$\operatorname{ind} X \leq n$ if for every point $x \in X$ and any neighbourhood $U(x) = I_1(U(x))$ there exists a neighbourhood $V(x) = I_1(V(x))$ such that $\mathrm{I}(V(x)) \subset U(X)$ and (u,l)-$\operatorname{ind}((l,u)$-$\operatorname{Fr} V(x)) \leq n-1$.

$(3)_1$ (u,l)-$\operatorname{ind} X = n$ if (u,l)-$\operatorname{ind} X \leq n$ and the inequality (u,l)-$\operatorname{ind} X \leq n-1$ does not hold.

$(4)_1$ (u,l)-$\operatorname{ind} X = \infty$ if the inequality (u,l)-$\operatorname{ind} X \leq n$ does not hold for any n.

Similarly,

$(1)_2$ (l,u)-$\operatorname{ind} X = -1 \Longleftrightarrow X = \varnothing$.

$(2)_2$ (l,u)-$\operatorname{ind} X \leq n$ if for every point $x \in X$ and any neighbourhood $U(x) = \mathrm{D}_1(U(x))$ there exists a neighbourhood $V(x) = \mathrm{D}_1(V(x))$ such that $\mathrm{D}(V(x)) \subset U(X)$ and (l,u)-$\operatorname{ind}((u,l)$-$\operatorname{Fr} V(x)) \leq n-1$.

$(3)_2$ (l,u)-$\operatorname{ind} X = n$ if (l,u)-$\operatorname{ind} X \leq n$ and the inequality (l,u)-$\operatorname{ind} X \leq n-1$ does not hold.

$(4)_2$ (l,u)-$\operatorname{ind} X = \infty$ if the inequality (l,u)-$\operatorname{ind} X \leq n$ does not hold for any n.
O-$\operatorname{ind} X \leq n \Longleftrightarrow (u,l)$-$\operatorname{ind} X \leq n \wedge (l,u)$-$\operatorname{ind} X \leq n$.

DEFINITION 3.5. Let $(X, \tau, \leq)$ be an OTS and n denote a nonnegative integer. We say that

$(1)_1$ (u,l)-$\operatorname{Ind} X = -1 \Longleftrightarrow X = \varnothing$.

$(2)_1$ (u,l)-$\operatorname{Ind} X \leq n$ if for every set $F = \mathrm{I}(F)$ and any neighbourhood $U(F) = I_1(U(F))$ there exists a neighbourhood $V(F) = I_1(V(F))$ such that $\mathrm{I}(V(F)) \subset U(F)$ and (u,l)-$\operatorname{Ind}((l,u)$-$\operatorname{Fr} V(F)) \leq n-1$.

$(3)_1$ (u,l)-$\operatorname{Ind} X = n$ if (u,l)-$\operatorname{Ind} X \leq n$ and the inequality (u,l)-$\operatorname{Ind} X \leq n-1$ does not hold.

$(4)_1$ (u,l)-$\operatorname{Ind} X = \infty$ if the inequality (u,l)-$\operatorname{Ind} X \leq n$ does not hold for any n.

Similarly,

$(1)_2$ (l,u)-$\operatorname{Ind} X = -1 \Longleftrightarrow X = \varnothing$.

$(2)_2$ (l,u)-$\operatorname{Ind} X \leq n$ if for every set $F = \mathrm{D}(F)$ and any neighbourhood $U(F) = \mathrm{D}_1(U(F))$ there exists a neighbourhood $V(F) = \mathrm{D}_1(V(F))$ such that $\mathrm{D}(V(F)) \subset U(F)$ and (l,u)-$\operatorname{Ind}((u,l)$-$\operatorname{Fr} V(F)) \leq n-1$.

$(2)_3$ (l,u)-$\operatorname{Ind} X = n$ if (l,u)-$\operatorname{Ind} X \leq n$ and the inequality (l,u)-$\operatorname{Ind} X \leq n-1$ does not hold.

$(2)_4$ (l,u)-$\operatorname{Ind} X = \infty$ if the inequality (l,u)-$\operatorname{Ind} X \leq n$ does not hold for any n.

O-$\operatorname{Ind} X \leq n \Longleftrightarrow (u,l)$-$\operatorname{Ind} X \leq n \wedge (l,u)$-$\operatorname{Ind} X \leq n$.

Definition 3.6. Let $(X, \tau, \leq)$ be an OTS and n denote a nonnegative integer. We say that

$(1)_1$ (u,l)-$\dim X = -1 \Longleftrightarrow X = \varnothing$.

$(2)_1$ (u,l)-$\dim X \leq n$ if for all families of sets $\{U_s = I_1(U_s) : s = \overline{1,k}\}$ and $\{F_s = \mathrm{I}(F_s) : s = \overline{1,k}\}$, where $F_s \subset U_s$ for each $s = \overline{1,k}$, there exists a family of sets $\{V_s = I_1(V_s) : s = \overline{1,k}\}$ such that $F_s \subset V_s \subset U_s$ for each $s = \overline{1,k}$ and $\operatorname{ord}\{(l,u)$-$\operatorname{Fr} V_s : s = \overline{1,k}\} \leq n$.

$(3)_1$ (u,l)-$\dim X = n$ if (u,l)-$\dim X \leq n$ and the inequality (u,l)-$\dim X \leq n-1$ does not hold.

$(4)_1$ (u,l)-$\dim X = \infty$ if the inequality (u,l)-$\dim X \leq n$ does not hold for any n.

Similarly,

$(1)_2$ (l,u)-$\dim X = -1 \Longleftrightarrow X = \varnothing$.

$(2)_2$ (l,u)-$\dim X \leq n$ if for all families of sets $\{U_s = \mathrm{D}_1(U_s) : s = \overline{1,k}\}$ and $\{F_s = \mathrm{D}(F_s) : s = \overline{1,k}\}$, where $F_s \subset U_s$ for each $s = \overline{1,k}$, there exists a family of sets $\{V_s = \mathrm{D}_1(V_s) : s = \overline{1,k}\}$ such that $F_s \subset V_s \subset U_s$ for each $s = \overline{1,k}$ and $\operatorname{ord}\{(u,l)$-$\operatorname{Fr} V_s : s = \overline{1,k}\} \leq n$.

$(3)_2$ (l,u)-$\dim X = n$ if (l,u)-$\dim X \leq n$ and the inequality (l,u)-$\dim X \leq n-1$ does not hold.

$(4)_2$ (l,u)-$\dim X = \infty$ if the inequality (l,u)-$\dim X \leq n$ does not hold for any n.

O-$\dim X \leq n \Longleftrightarrow (u,l)$-$\dim X \leq n \wedge (l,u)$-$\dim X \leq n$.

If $(X, \tau, \leq)$ is an OTS and (X, τ_1, τ_2) is the corresponding BS in the sense of Canfell, then it is obvious that

(u,l)-$\operatorname{ind}(X, \tau, \leq) = (1,2)$-$\operatorname{ind}(X, \tau_1, \tau_2)$,
(l,u)-$\operatorname{ind}(X, \tau, \leq) = (2,1)$-$\operatorname{ind}(X, \tau_1, \tau_2)$,
O-$\operatorname{ind}(X, \tau, \leq) = p$-$\operatorname{ind}(X, \tau_1, \tau_2)$;

(u,l)-$\operatorname{Ind}(X, \tau, \leq) = (1,2)$-$\operatorname{Ind}(X, \tau_1, \tau_2)$,
(l,u)-$\operatorname{Ind}(X, \tau, \leq) = (2,1)$-$\operatorname{Ind}(X, \tau_1, \tau_2)$,
O-$\operatorname{Ind}(X, \tau, \leq) = p$-$\operatorname{Ind}(X, \tau_1, \tau_2)$;

(u,l)-$\dim(X, \tau, \leq) = (1,2)$-$\dim(X, \tau_1, \tau_2)$,
(l,u)-$\dim(X, \tau, \leq) = (2,1)$-$\dim(X, \tau_1, \tau_2)$,
O-$\dim(X, \tau, \leq) = p$-$\dim(X, \tau_1, \tau_2)$.

Thus one can prove that for the OTS $(\mathbb{R},\omega,\leq)$ the values of nine ordered dimension functions coincide with integer 1.

All results presented below concerning the ordered dimension function are the immediate corollaries of the results discussed in Chapter III of [11] with Remark 3.1 taken into account. One can easily verify that if two OTSs $(X,\tau,\leq)$ and $(Y,\gamma,\leq')$ are both homeomorphic and order isomorphic in the sense of [11], then $O\text{-}\mathrm{ind}\, X = O\text{-}\mathrm{ind}\, Y$, $O\text{-}\mathrm{Ind}\, X = O\text{-}\mathrm{Ind}\, Y$ and $O\text{-}\dim X = O\text{-}\dim Y$.

For the sake of simplicity all results are formulated for the dimension functions $O\text{-}\mathrm{ind}\, X$, $O\text{-}\mathrm{Ind}\, X$ and $O\text{-}\dim X$.

Theorem 3.2. *The following conditions are satisfied in an* OTS $(X,\tau,\leq)$*:*

(1) If $O\text{-}\mathrm{ind}\, X$ ($(u,l)\text{-}\mathrm{Ind}\, X$ *or* $(l,u)\text{-}\mathrm{Ind}\, X$) *is finite, then* $(X,\tau,\leq)$ *is strong regularly ordered (strong normally ordered).*

(2) If $(A,\tau',\leq')$ *is an order subspace of* $(X,\tau,\leq)$ $(A = \mathrm{D}(A)\cap \mathrm{I}(A))$*, then* $O\text{-}\mathrm{ind}\, A \leq O\text{-}\mathrm{ind}\, X$ ($O\text{-}\mathrm{Ind}\, A \leq O\text{-}\mathrm{Ind}\, X$ *and* $O\text{-}\dim A \leq O\text{-}\dim X$).

(3) The equalities $O\text{-}\mathrm{Ind}\, X = 0$ *and* $O\text{-}\dim X = 0$ *are equivalent.*

(4) If $(X,\tau,\leq)$ *is a strong normally ordered space and* $\{X_m\}_{m=1}^{\infty}$ *is a sequence of subsets in* X *such that* $X = \bigcup\limits_{m=1}^{\infty} X_m$, $X_m = \mathrm{D}(X_m)=\mathrm{I}(X_m)$ *and* $O\text{-}\mathrm{Ind}\, X_m = 0$ *(or, equivalently,* $O\text{-}\dim X_m = 0$*) for each* $m=\overline{1,\infty}$*, then* $O\text{-}\mathrm{Ind}\, X = 0$ *(or, equivalently,* $O\text{-}\dim X = 0$*).*

It is clear that (4) *remains valid if* $X_m = \bigcup\limits_{n=1}^{\infty} F_n^m$*, where* $F_n^m = D(F_n^m) = I(F_n^m)$ *and* $O\text{-}\mathrm{Ind}\, F_n^m = 0$ *(or, equivalently,* $O\text{-}\dim F_n^m = 0$*) for each* $m=\overline{1,\infty}$*,* $n=\overline{1,\infty}$*.*

Theorem 3.3. *The following conditions are satisfied in a hereditarily strong normally* OTS $(X,\tau,\leq)$*:*

(1) If $M_0, M_1, \dots, M_n$ *are any subsets of* X*, then* $O\text{-}\mathrm{ind}(M_0\cup M_1\cup\cdots\cup M_n) \leq O\text{-}\mathrm{ind}\, M_0 + O\text{-}\mathrm{ind}\, M_1 + \cdots + O\text{-}\mathrm{ind}\, M_n + n$ *and thus if* $X = \bigcup\limits_{k=0}^{n} M_k$*, where* $O\text{-}\mathrm{ind}\, M_k = 0$ *for each* $k=\overline{0,n}$*, then* $O\text{-}\mathrm{ind}\, X \leq n$*.*

(2) If $X_m = \mathrm{D}_1(X_m) = I_1(X_m)$ *for each* $m=\overline{1,\infty}$*,* $X_{m+1}\subset X_m$*,* $X_1 = X$ *and* $\bigcap\limits_{m=1}^{\infty} X_m = \varnothing$*, then* $O\text{-}\mathrm{Ind}(X_m\backslash X_{m+1}) \leq n$ *for each* $m=\overline{1,\infty}$ *implies that* $O\text{-}\mathrm{Ind}\, X \leq n$*.*

Therefore if $A = \mathrm{D}(A) = \mathrm{I}(A)$*, then* $O\text{-}\mathrm{Ind}\, A\leq n$ *and* $O\text{-}\mathrm{Ind}(X\backslash A)\leq n$ *imply that* $O\text{-}\mathrm{Ind}\, X\leq n$*.*

(3) If $\{D_m\}_{m=1}^{\infty}$ *is a disjoint sequence of sets covering* X *such that* $F_s = \bigcup\limits_{m\leq s} \mathrm{D}_m = \mathrm{D}(F_s) = \mathrm{I}(F_s)$ *for each* $s=\overline{1,\infty}$*, then* $0\text{-}\mathrm{Ind}\,\mathrm{D}_m \leq n$ *for each* $m=\overline{1,\infty}$ *implies that* $0\text{-}\mathrm{Ind}\, X\leq n$*.*

(4) If $X = P\cup Q$*, where* $O\text{-}\mathrm{Ind}\, P\leq n$*,* $O\text{-}\mathrm{Ind}\, Q\leq 0$*, then* $O\text{-}\mathrm{Ind}\, X\leq n+1$*.*

Thus if $X = \bigcup\limits_{m=0}^{n} X_m$*, where* $O\text{-}\mathrm{Ind}\, X_m \leq 0$ *(or, equivalently,* $0\text{-}\dim X_m\leq 0$*) for each* $m=\overline{0,n}$*, then* $O\text{-}\mathrm{Ind}\, X\leq n$*.*

It is clear that many other results from [11, Chapter III], which we have not included in Theorems 3.2 and 3.3, since the objective was the demonstration of the M. Canfell duality for dimension functions, remain valid.

4. Baire-like properties

Definition 4.1. A subset A of an OTS $(X, \tau, \leq)$ is u-nowhere dense (also called u-rare) in X if $I_1(\mathrm{I}(A)) = \varnothing$ and A is l-nowhere dense (also called l-rare) in X if $\mathrm{D}_1(\mathrm{D}(A)) = \varnothing$.

The family of all u-nowhere dense (l-nowhere dense) subsets of X is denoted by u-$\mathcal{ND}(X)$ (l-$\mathcal{ND}(X)$).

Proposition 4.1. *Let $(X, \tau, \leq)$ be an OTS. Then the following conditions are satisfied:*

(1) $A \in u\text{-}\mathcal{ND}(X)$ $((A \in l\text{-}\mathcal{ND}(X)) \Longleftrightarrow \mathrm{I}(A) \subset \mathrm{D}(X \setminus \mathrm{I}(A)$ $(\mathrm{D}(A) \subset \mathrm{I}(X \setminus \mathrm{D}(A))$, *for any subset* $A \subset X$.

(2) If $(Y, \tau', \leq')$ *is an order subspace and* $A \subset Y$, *then* $A \in u\text{-}\mathcal{ND}(Y)$ $((A \in l\text{-}\mathcal{ND}(Y)) \Longleftrightarrow \mathrm{I}'(A) \subset \mathrm{D}(Y \setminus \mathrm{I}'(A)$ $(\mathrm{D}'(A) \subset \mathrm{I}(Y \setminus \mathrm{D}'(A))$.

Proof. Immediately follows from Propositions 1.1.1 and 1.5.2 from [11], Remark 3.1 and using Canfell's duality. ∎

Definition 4.2. A subset A of an OTS $(X, \tau, \leq)$ is of u-first (l-first) category (also called u-meager (l-meager), u-exhaustible (l-exhaustible)) in X if $A = \bigcup\limits_{n=1}^{\infty} A_n$, where $A_n \in u\text{-}\mathcal{ND}(X)$ ($A_n \in l\text{-}\mathcal{ND}(X)$) for every $n = \overline{1, \infty}$ and A is of u-second (l-second) category (also called u-nonmeager (l-nonmeager), u-inexhaustible (l-inexhaustible)) in X if it is not of u-first (l-first) category in X.

The family of all sets of u-first (l-first) category in X is denoted by u-$\mathcal{C}atg_{\mathrm{I}}(X)$ (l-$\mathcal{C}atg_{\mathrm{I}}(X)$), while the family of all sets of u-second (l-second) category in X is denoted by u-$\mathcal{C}atg_{\mathrm{II}}(X)$ (l-$\mathcal{C}atg_{\mathrm{II}}(X)$).

A subset A of an OTS $(X, \tau, \leq)$ is of u-$\mathcal{C}atg\,\mathrm{I}$ (u-$\mathcal{C}atg\,\mathrm{II}$) (respectively l-$\mathcal{C}atg\,\mathrm{I}$ (l-$\mathcal{C}atg\,\mathrm{II}$)) if A is of u-first (u-second) (respectively l-first (l-second)) category in itself considered as on order subspace.

We introduce the following notations: $u\text{-}\mathcal{G}_\delta(X) = \{A \subset X : A = \bigcap\limits_{n=1}^{\infty} A_n$, where $A_n = I_1(A_n)$, $\forall n = \overline{1, \infty}\}$, $l\text{-}\mathcal{G}_\delta(X) = \{A \subset X : A = \bigcap\limits_{n=1}^{\infty} A_n$, where $A_n = \mathrm{D}_1(A_n)$, $\forall n = \overline{1, \infty}\}$, $u\text{-}\mathcal{F}_\sigma(X) = \{A \subset X : A = \bigcup\limits_{n=1}^{\infty} A_n$, where $A_n = \mathrm{I}(A_n)$, $\forall n = \overline{1, \infty}\}$ and $l\text{-}\mathcal{F}_\sigma(X) = \{A \subset X : A = \bigcup\limits_{n=1}^{\infty} A_n$, where $A_n = \mathrm{D}(A_n)$, $\forall n = \overline{1, \infty}\}$.

By analogy with Theorem 1.1.3 from [11] we have

Theorem 4.1. *The following statements hold in an* OTS, $(X, \tau, \leq)$*:*

(1) The family $u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(l\text{-}\mathcal{C}atg_{\mathrm{I}}(X))$ *is a* σ*-ideal so that* $A_n \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(A_n \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X))$ *for every* $n = \overline{1,\infty} \Longrightarrow \bigcup\limits_{n=1}^{\infty} A_n \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(\bigcup\limits_{n=1}^{\infty} A_n \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X))$, *and* $B \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(B \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X))$, $A \subset B \Longrightarrow A \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X))$.

(2) $u\text{-}\mathcal{F}_\sigma(X) \cap u\text{-}\mathcal{B}d(X) \subset u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ *and* $l\text{-}\mathcal{F}_\sigma(X) \cap l\text{-}\mathcal{B}d(X) \subset l\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$.

(3) For every set $A \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X))$ *there is a set* $B \in u\text{-}\mathcal{F}_\sigma(X) \cap u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(B \in l\text{-}\mathcal{F}_\sigma(X) \cap l\text{-}\mathcal{C}atg_{\mathrm{I}}(X))$ *such that* $A \subset B$.

(4) The family $u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(l\text{-}\mathcal{C}atg_{\mathrm{II}}(X))$ *is closed under arbitrary unions and* $A \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X))$, $A \subset B \Longrightarrow B \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(B \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X))$.

(5) If $X \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(X \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X))$ *and for a subset* $A \subset X$ *there is a set* $B \in l\text{-}\mathcal{G}_\delta(X) \cap l\text{-}\mathcal{D}(X)$ $(B \in u\text{-}\mathcal{G}_\delta(X) \cap u\text{-}\mathcal{D}(X))$, $B \subset A$, *then* $A \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X))$.

(6) $X \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(X \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X)) \Longleftrightarrow$ *the intersection of any sequence* $\{U_n\}_{n=1}^{\infty}$, *where* $U_n = \mathrm{D}_1(U_n) \in l\text{-}\mathcal{D}(X)$ $(U_n = I_1(U_n) \in u\text{-}\mathcal{D}(X))$ *for each* $n = \overline{1,\infty}$, *is nonempty.*

Proposition 4.2. *Let* $(X, \tau, \leq)$ *be an* OTS, (X, τ_1, τ_2) *be the corresponding* BS *in the Canfell's Sense,* $(Y, \tau', \leq')$ *be an order subspace, where* $Y \in \tau_1$ $(Y \in \tau_2)$ *and* $A \subset X$. *Then* $A \in u\text{-}\mathcal{ND}(X)$ $(A \in l\text{-}\mathcal{ND}(X)) \Longrightarrow A \cap Y \in u\text{-}\mathcal{ND}(Y)$ $(A \cap Y \in l\text{-}\mathcal{ND}(Y))$ *so that* $A \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X)) \Longrightarrow A \cap Y \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(Y)$ $(A \cap Y \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(Y))$ *and* $A \cap Y \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(Y)$ $(A \cap Y \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(Y)) \Longrightarrow A \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X))$.

Corollary. *Let* $(X, \tau, \leq)$ *be an* OTS *and* $(Y, \tau', \leq')$ *be an order subspace, where* $Y \in \tau_1$ $(Y \in \tau_2)$ *and* $A \subset Y$. *Then* $A \in u\text{-}\mathcal{ND}(X)$ $(A \in l\text{-}\mathcal{ND}(X)) \Longrightarrow A \in u\text{-}\mathcal{ND}(Y)$ $(A \in l\text{-}\mathcal{ND}(Y))$ *so that* $A \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X)) \Longrightarrow A \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(Y)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(Y))$ *and* $A \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(Y)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(Y)) \Longrightarrow A \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(A \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X))$.

Proof. Follows directly from Canfell's duality and Theorem 1.5.1 from [11]. ■

We know that the main result (Corollary of Theorem 1.5.2 from [11]) holds only for BSs of the type $(X, \tau_1 < \tau_2)$, i.e., for $\tau_1 \subset \tau_2$, while for the BS (X, τ_1, τ_2) associated with an OTS $(X, \tau, \leq)$ in Canfell's sense, the topologies τ_1 and τ_2 are not comparable by the set-theoretic operation inclusion. Therefore it is clear that for an OTS $(X, \tau, \leq)$ and according to Canfell associated BS (X, τ_1, τ_2) we have:

if $U \in \tau_1$ $(U \in \tau_2)$ and $(U, \tau', \leq')$ is an order subspace, then U is of $u\text{-}\mathcal{C}atg\,\mathrm{II}$ $(l\text{-}\mathcal{C}atg\,\mathrm{II}) \Longrightarrow U \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ $(U \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X))$ and $U \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ $(U \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X) \Longrightarrow U$ is of $u\text{-}\mathcal{C}atg\,\mathrm{I}$ $(l\text{-}\mathcal{C}atg\,\mathrm{I})$.

Let $(X, \tau, \leq)$ be an I-space in the sense of [25, p. 508], so that $U \in \tau \Longrightarrow i(U), d(U) \in \tau$. Then by Proposition 5 in [25], $Y \in \tau$ implies that $(Y, \tau', \leq')$ is the order subspace. But for Canfell's BS (X, τ_1, τ_2) we have $\tau_i \subset \tau$ and therefore for I-space $(X, \tau, \leq)$ the above formulated results has the form:

if $U \in \tau_1$ ($U \in \tau_2$), then
U is of u-$\mathcal{C}atg\,\mathrm{II}$ (l-$\mathcal{C}atg\,\mathrm{II}$) $\Longrightarrow U \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ ($l\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$) and
$U \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ ($U \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$) $\Longrightarrow U$ is of u-$\mathcal{C}atg\,\mathrm{I}$ (l-$\mathcal{C}atg\,\mathrm{I}$). ■

Definition 4.3. An upper (lower) Baire space or briefly u-BrS (l-BrS) is an OTS $(X, \tau, \leq)$ such that $U = \mathrm{I}_1(U) \neq \varnothing$ ($U = \mathrm{D}_1(U) \neq \varnothing$) $\Longrightarrow U$ is of u-$\mathcal{C}atg\,\mathrm{II}$ (U is of l-$\mathcal{C}atg\,\mathrm{II}$).

It is clear that we can introduce also

Definition 4.4. An almost upper (lower) Baire space or, briefly, A-u-BrS (A-l-BrS) is an OTS $(X, \tau, \leq)$ such that $U = I_1(U) \neq \varnothing$ ($U = \mathrm{D}_1(U) \neq \varnothing$) $\Longrightarrow U \in u\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$ ($U \in l\text{-}\mathcal{C}atg_{\mathrm{II}}(X)$).

It is obvious, that if any $U \in \tau_1 \setminus \{\varnothing\}$ ($U \in \tau_2 \setminus \{\varnothing\}$) is an order subspace of an OTS $(X, \tau, \leq)$, then $(X, \tau, \leq)$ is u-BrS (l-BrS) $\Longrightarrow (X, \tau, \leq)$ A-u-BrS (an A-l-BrS) so that, if $(X, \tau, \leq)$ is an I-space, then $(X, \tau, \leq)$ is an u-BrS (l-BrS) $\Longrightarrow (X, \tau, \leq)$ is A-u-BrS (A-l-BrS).

Theorem 4.2. *The following conditions are equivalent in an* OTS $(X, \tau, \leq)$*:*

(1) $(X, \tau, \leq)$ *is* A-u-BrS (A-l-BrS).

(2) If $\{U_n\}_{n=1}^{\infty}$ *is any countable family of subsets in* X*, where* $U_n = \mathrm{D}_1(U_n) \in l\text{-}\mathcal{D}(X)$ ($U_n = I_1(U_n) \in u\text{-}\mathcal{D}(X)$) *for each* $n = \overline{1, \infty}$*, then* $\bigcap_{n=1}^{\infty} U_n \in l\text{-}\mathcal{D}(X)$ ($\bigcap_{n=1}^{\infty} U_n \in u\text{-}\mathcal{D}(X)$).

(3) $A \in u\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$ ($A \in l\text{-}\mathcal{C}atg_{\mathrm{I}}(X)$) $\Longrightarrow X \setminus A \in l\text{-}\mathcal{D}(X)$ ($X \setminus A \in u\text{-}\mathcal{D}(X)$).

(4) If $\{F_n\}_{n=1}^{\infty}$ *is any countable family of subsets in* X*, where* $F_n = \mathrm{I}(F_n) \in u\text{-}\mathcal{B}d(X)$ ($F_n = \mathrm{D}(F_n) \in l\text{-}\mathcal{B}d(X)$) *for each* $n = \overline{1, \infty}$*, then* $\bigcup_{n=1}^{\infty} F_n \in u\text{-}\mathcal{B}d(X)$ ($\bigcup_{n=1}^{\infty} F_n \in l\text{-}\mathcal{B}d(X)$).

The proof of this theorem repeats that of Theorem 4.1.2 from [11], taking into account (3) of Theorem 4.1. and

Lemma 4.1. *The following equivalences are correct in an* OTS $(X, \tau, \leq)$*:*

$A \in l\text{-}\mathcal{D}(X)$ ($A \in u\text{-}\mathcal{D}(X)$) $\Longleftrightarrow$ *every set* $U = I_1(U) \neq \varnothing$ ($U = \mathrm{D}_1(U) \neq \varnothing$) *intersects* $A \Longleftrightarrow X \setminus A \in u\text{-}\mathcal{B}d(X)$ ($X \setminus A \in l\text{-}\mathcal{B}d(X)$).

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