МАТЕМАТИЧКИ ВЕСНИК
55 (2003), 53–57

UDK 514.752.2
оригинални научни рад
research paper

INTEGRAL INEQUALITIES FOR MAXIMAL SPACE-LIKE SUBMANIFOLDS IN THE INDEFINITE SPACE FORM

Liu Ximin

Abstract. In this note, we give two intrinsic integral inequalities for compact maximal space-like submanifolds in the indefinite space form and a sufficent and necessary condition for such submanifolds to be totally geodesic.

1. Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an indefinite space form of index p and simply a space form when $p=0$. If $c>0$, we call it a de Sitter space of index p. Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. As the semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of M^n, M^n is called a space-like submanifold. A space-like submanifold with vanishing mean curvature is called a maximal space-like submanifold. Kobayashi [5] gave the Weierstrass-Enneper representation formulas for maximal space-like surfaces in 3-dimensional Minkowski space and exhibited various examples. In particular, he determined the maximal space-like surfaces which are rotation surfaces or ruled surfaces. Montiel [6] gave an integral inequality for compact space-like hypersurfaces in a de Sitter space and by use of this integral inequality, he studied the constant mean curvature space-like hypersurfaces. Also, Akutagawa [1] and Ramanathan [8] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfies $H^2 \le c$ when $n=2$ and $n^2H^2 < 4(n-1)c$ when $n \ge 3$. Later, Cheng [3] generalized this result to general submanifolds in a de Sitter space.

In this paper, we study compact maximal space-like submanifolds in the indefinite space form with flat normal bundle and obtain two intrinsic integral inequalities for such submanifolds. We also give a sufficent and necessary condition for such submanifolds to be totally geodesic. We will prove the following

AMS Subject Classification: 53 C 40, 53 C 42, 53 C 50.

Keywords and phrases: Maximal space-like submanifold, indefinite space form, flat normal bundle.

This work is partially supported by the National Natural Science Foundation of China

THEOREM 1. *Let M^n be an n-dimensional compact maximal space-like submanifold in $M_p^{n+p}(c)$ with flat normal bundle. Then*

$$\int_{M^n}\Big\{\frac{1}{2}\sum R_{mijk}^2+\sum R_{mj}^2-ncR\Big\}*1\le 0.$$

THEOREM 2. *Let M^n be an n-dimensional compact maximal space-like submanifold in $M_p^{n+p}(c)$ with flat normal bundle. Then*

$$\int_{M^n}\Big\{\frac{1}{2}\sum R_{mijk}^2+\tfrac{1}{n}S^2+(n-2)cS-n(n-1)c^2\Big\}*1\le 0.$$

THEOREM 3. *Let M^n be an n-dimensional compact maximal space-like submanifold in $M_p^{n+p}(c)$ with flat normal bundle. Then M^n is totally geodesic if and only if*

$$\int_{M^n}\Big\{\frac{1}{2}\sum R_{mijk}^2+(n-2)cS-n(n-1)c^2\Big\}*1=0.$$

In the above theorems, $\sum R_{mijk}^2$ is the square length of the Riemannian curvature tensor, $\sum R_{mj}^2$ the square length of the Ricci curvature tensor, S the square length of the second fundamental form, R the scalar curvature. All these are intrinsic properties of M^n.

2. Preliminaries

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional semi-Riemannian manifold of constant curvature c whose index is p. Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1,\dots,e_{n+p}$ in $M_p^{n+p}(c)$ such that at each point of M^n, $e_1,\dots,e_n$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1\le A,B,C,\dots\le n+p;\quad 1\le i,j,k,\dots\le n;\quad n+1\le\alpha,\beta,\gamma\le n+p.$$

Let $\omega_1,\dots,\omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by $d\bar{s}^2=\sum_i\omega_i^2-\sum_\alpha\omega_\alpha^2=\sum_A\epsilon_A\omega_A^2$, where $\epsilon_i=1$ and $\epsilon_\alpha=-1$. Then the structure equations of $M_p^{n+p}(c)$ are given by

$$d\omega_A=\sum_B\epsilon_B\omega_{AB}\wedge\omega_B,\quad \omega_{AB}+\omega_{BA}=0,$$

$$d\omega_{AB}=\sum_C\epsilon_C\omega_{AC}\wedge\omega_{CB}-\frac{1}{2}\sum_{C,D}K_{ABCD}\omega_C\wedge\omega_D,$$

$$K_{ABCD}=c\,\epsilon_A\epsilon_B(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}).$$

Restricting these forms to M^n, we obtain $\omega_\alpha=0$, $n+1\le\alpha\le n+p$, and the Riemannian metric of M^n is written as $ds^2=\sum_i\omega_i^2$. From Cartan's lemma we can write

$$\omega_{\alpha i}=\sum_j h_{ij}^\alpha\omega_j,\quad h_{ij}^\alpha=h_{ji}^\alpha.$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} K_{ijkl} \omega_k \wedge \omega_l,$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{1}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

For details on indefinite Riemannian manifolds we refer to O'Neill [7].

We call

$$h = \sum_\alpha h_\alpha e_\alpha = \sum_{i,j,\alpha} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha$$

the second fundamental form of M^n and the square length of the second fundamental form is defined by

$$S = \sum_\alpha tr(h_\alpha)^2 = \sum_{\alpha,i,j} (h^\alpha_{ij})^2.$$

The mean curvature vector N of M^n is defined by

$$N = \frac{1}{n} \sum_\alpha tr(h_\alpha) e_\alpha = \frac{1}{n} \sum_\alpha (\sum_i h^\alpha_{ii}) e_\alpha,$$

and it is well known that N is independent of the choice of unit normal vectors $e_{n+1}, \cdots, e_{n+p}$ to M^n. The length of the mean curvature vector is called the mean curvature of M^n, denoted by H.

If M^n is maximal, then

$$\sum_i h^\alpha_{ii} = 0, \quad \alpha = n+1, \cdots, n+p. \tag{2}$$

Define the first and the second covariant derivatives of $\{h^\alpha_{ij}\}$, say $\{h^\alpha_{ijk}\}$ and $\{h^\alpha_{ijkl}\}$, by

$$\sum_k h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj} \omega_{ki} + \sum_k h^\alpha_{ik} \omega_{kj} + \sum_\beta h^\beta_{ij} \omega_{\beta\alpha},$$

$$\sum_l h^\alpha_{ijkl} \omega_l = dh^\alpha_{ijk} + \sum_m h^\alpha_{mjk} \omega_{mi} + \sum_m h^\alpha_{imk} \omega_{mj} + \sum_m h^\alpha_{ijm} \omega_{mk} + \sum_\beta h^\beta_{ijk} \omega_{\beta\alpha}.$$

Then we have

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{3}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im} R_{mjkl} + \sum_m h^\alpha_{jm} R_{mikl} + \sum_\beta h^\beta_{ij} R_{\alpha\beta kl}, \tag{4}$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor of M^n, that is

$$R_{\alpha\beta kl} = \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}).$$

If $R_{\alpha\beta kl}=0$ at a point x of M^n we say that the normal connection of M^n is flat at x, and it is well known [2] that $R_{\alpha\beta kl}=0$ at x if and only if h_α are simultaneously diagonalizable at x.

The Laplacian Δh_{ij}^α of the fundamental form h_{ij}^α is defined to be $\sum_k h_{ijkk}^\alpha$, and hence, if M^n has flat normal bundle, from (3) and (4) we have

$$\begin{aligned}\Delta h_{ij}^\alpha &= \sum_k (h_{ijkk}^\alpha - h_{ikjk}^\alpha) + \sum_k (h_{ikjk}^\alpha - h_{ikkj}^\alpha) + \sum_k (h_{ikkj}^\alpha - h_{kkij}^\alpha)\\ &= \sum_{m,k} h_{im}^\alpha R_{mkjk} + \sum_{m,k} h_{mk}^\alpha R_{mijk} \end{aligned} \tag{5}$$

3. Proofs of the Theorems

Proof of Theorem 1. From (1), (2) and (5), we have

$$\begin{aligned}\sum h_{ij}^\alpha \Delta h_{ij}^\alpha &= \sum h_{ij}^\alpha h_{mk}^\alpha R_{mijk} + \sum h_{ij}^\alpha h_{im}^\alpha R_{mkjk}\\ &= \frac{1}{2}\sum (h_{ij}^\alpha h_{mk}^\alpha - h_{mj}^\alpha h_{ik}^\alpha) R_{mijk} + \sum (h_{ij}^\alpha h_{im}^\alpha - h_{ii}^\alpha h_{jm}^\alpha) R_{mj}\\ &= \frac{1}{2}\sum [c(\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik}) - R_{imjk}] R_{mijk}\\ &\quad + \sum [c(\delta_{ij}\delta_{im} - \delta_{ii}\delta_{jm}) + R_{ijim}] R_{mj}\\ &= \frac{1}{2}\sum R_{mijk}^2 + \sum R_{mj}^2 - ncR.\end{aligned}$$

Since $\int_{M^n} \{\sum h_{ij}^\alpha \Delta h_{ij}^\alpha\} * 1 \le 0$, we have

$$\int_{M^n} \Big\{ \frac{1}{2}\sum R_{mijk}^2 + \sum R_{mj}^2 - ncR \Big\} * 1 \le 0. \tag{6}$$

Theorem 1 is proved. ■

In order to prove Theorem 2, we need the following algebraic lemma

Lemma. *Let $a_1, \dots, a_n$ be real numbers. Then*

$$\sum (a_i)^2 \ge \frac{1}{n}(\sum a_i)^2, \tag{7}$$

and the equality holds if and only if $a_1 = \cdots = a_n$.

In fact,

$$n\sum (a_i)^2 - (\sum a_i)^2 = \sum (a_i - a_j)^2, \tag{8}$$

and the Lemma follows immediately from (8).

Proof of Theorem 2. From (1), we have

$$R_{mj} = (n-1)c\delta_{mj} + \sum h_{mi}^\alpha h_{ij}^\alpha, \quad \text{and} \quad R = n(n-1)c + S.$$

Since M^n has flat normal bundle, we can diagonalize the second fundamental form simultaneously so that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, and then from (7), we have

$$\begin{aligned} R_{mj} &= (n-1)c\delta_{mj} + \sum(\lambda_j^\alpha)^2\delta_{mj}, \\ \sum R_{mj}^2 &= n(n-1)^2c^2 + 2(n-1)cS + \sum(\lambda_j^\alpha)^4 \\ &\geq n(n-1)^2c^2 + 2(n-1)cS + \frac{1}{n}\{\sum(\lambda_j^\alpha)^2\}^2 \\ &= n(n-1)^2c^2 + 2(n-1)cS + \frac{1}{n}S^2. \end{aligned}$$

Therefore, from (6) we have

$$\int_{M^n}\Big\{\frac{1}{2}\sum R_{mijk}^2 + \tfrac{1}{n}S^2 + (n-2)cS - n(n-1)c^2\Big\} * 1 \leq 0. \tag{9}$$

Theorem 2 is proved. ∎

Proof of Theorem 3. From (9) we have

$$\int_{M^n}\Big\{\frac{1}{2}\sum R_{mijk}^2 + (n-2)cS - n(n-1)c^2\Big\} * 1 \leq 0. \tag{10}$$

If M^n is totally geodesic, i.e., $S=0$, $h_{ij}^\alpha = 0$, then from (1) we have

$$R_{mijk} = c(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}), \quad \sum R_{mijk}^2 = 2n(n-1)c^2,$$

and in this case, (10) becomes an equality.

Conversely, if (10) becomes an equality, then $S=0$, and M^n is totally geodesic. ∎

Acknowledgements. This note was written during the author's stay at the Max-Planck-Institut für Mathematik in Bonn. The author would like to express his thanks to the institute for its hospitality and support.

REFERENCES

[**1**] K. Akutagawa, *On space-like hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z., **196** (1987), 13–19.

[**2**] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.

[**3**] Q. M. Cheng, *Complete space-like submanifolds in a de Sitter space with parallel mean curvature vector*, Math. Z., **206** (1991), 333–339.

[**4**] S. Y. Cheng and S. T. Yau, *Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces*, Ann. Math., **104** (1976), 407–419.

[**5**] O. Kobayashi, *Maximal surfaces in the 3-dimensional Minkowski space L^3*, Tokyo J. Math., **6** (1983), 297–309.

[**6**] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J., **37** (1988), 909–917.

[**7**] B. O'Neill, *Semi-Riemannian Geometry*, New York, Academic Press, 1983.

[**8**] J. Ramanathan, *Complete space-like hypersurfaces of constant mean curvature in the de Sitter space*, Indiana Univ. Math. J., **36** (1987), 349–55.

(received 10.05.2001)

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
E-mail: xmliu@dlut.edu.cn