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ON GNOMONS

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Abstract. A gnomon is a shape which, when added to a figure, yields a figure that is similar to the original one. Gazalé [4] conjectured that if the gnomon is a regular polygon and the figure to which it is added is a finite polygon then the gnomon is a triangle or a square. We prove this conjecture to be correct.

1. Introduction

A gnomon is a shape which, when added to a figure, yields a figure that is similar to the original. A precise mathematical definition follows.

DEFINITION 1. Suppose that V and W are closed subsets in the plane whose interiors are non-empty and disjoint. Then W is called a *gnomon* of V if $W \cup V$ is similar to V, i.e., there exists a bijection $f\colon W \cup V \to V$ and a real number λ with $0 < \lambda < 1$ such that $d(x,y) = \lambda d(f(x), f(y))$.

Fig. 1. W is a gnomon of V

Figure 2 shows the examples that are presented in Gazalé's book [4] and in a column of Ian Stewart in the Scientific American [8].

In Figure 2, the left hand figure is the spiral that is associated to the golden number, or the golden ratio. Here the gnomon W is a square and V is a rectangle, the sides of which have a ratio that is equal to the golden ratio, the positive root of $x^2 + x - 1$. The similarity between $W \cup V$ and V is the composition of a clockwise

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Fig. 2. The square and triangle as gnomons

rotation over $\pi/2$ and a contraction by the golden ratio. The figure on the right shows an equilateral triangle as a gnomon of a pentagon V. The ratio between the lengths of successive sides of V is equal to the positive root of $x^3 + x^2 - 1$. This root is called the silver number by Gazalé. The similarity transformation between $W \cup V$ and V is a clockwise rotation over $\pi/3$ combined with a contraction by the silver number. The figure on the left is called the golden spiral and the figure on the right is called the silver spiral. In [4, page 143] it is stated that "there is (in all likelihood) no finitary polygon whose gnomon is a regular polygon other than the golden rectangle and the silver pentagon". We shall show this to be true.

THEOREM 2. *Suppose that W is a gnomon of V, that V is a finite polygon and that W is a regular polygon. Then V and W constitute either the golden spiral or the silver spiral.*

In this theorem the orientation of the spiral is irrelevant. In other words, spirals that are mirror images of each other are considered the same.

2. Similarity transformations

We assume from now on that W is a gnomon of V, that both W and V are finite polygons and that W is convex. Suppose that $\varphi: W \cup V \to V$ is a similarity transformation. As a similarity transformations is determined by its action on three non-collinear points, we may assume that φ is defined on all of $\mathbf{R}^2$. By the Banach contraction theorem, φ has a unique fixed point. It is known that contracting similarity transformations of the plane of which the origin is a fixed point can be of three types [3]:

1. A *pure* contraction $x \to \lambda x$ for some λ such that $0 < \lambda < 1$.
2. A composition of a contraction and a rotation.
3. A composition of a contraction and a reflection.

The gnomon in figure 1 has a similarity of type 1 or type 3. The gnomons in figure 2 have a similarity of type 2. The *orbit space* of $\mathbf{R}^2 \setminus \{0\}$ is the quotient space under the equivalence relation generated by $x \sim \varphi(x)$. The orbit space is a torus for similarity transformations of type 1 or 2. It is a Klein bottle for similarity transformations of type 3.

Lemma 3. *Suppose that $\varphi: W \cup V \to V$ is of type 1 or 3, that W is convex and that V is a finite polygon. Then W is a quadrilateral.*

Proof. First note that $V \cap W \neq \emptyset$, for otherwise V is the disjoint union of $\varphi(W)$, $\varphi^2(W)$, $\varphi^3(W)$, $\ldots$ together with the limit point, whence not a polygon. Let φ be of type 1. Then the only iterate $\varphi^i(W)$ that has non-empty intersection with W is $\varphi(W)$. By convexity W and $\varphi(W)$ intersect in a single side, which, since V is a finite polygon, has to be a mutual side of V and W. This accounts for two sides of W, one being mapped onto the other under the similarity transformation. The gnomon has to be the convex hull of these two sides, tiling a triangle under the iteration of the similarity transformation.

Now let φ be a similarity transformation of type 3. If W and $\varphi(W)$ have non-empty intersection, then by the same argument as above W is a quadrilateral and V is a triangle. If W and $\varphi(W)$ are disjoint, then W intersects $\varphi^2(W)$ and by the same argument V must be a union of two triangles whose intersection is the fixed point of the similarity transformation. As the resulting figure is not a polygon, the last case can not occur. ■

From now on we shall consider only gnomons with similarities of type 2, namely contracting rotations; we may assume as well that angles of the rotation are positive and less than π.

3. Convex gnomons

Let W be a gnomon for V and let $q \in V$ be the fixed point of the contracting rotation φ. We may assume that the rotation has angle less than π. It is easily seen that q must be an interior point of V. It follows that the copies of W under the iteration of φ form a tessellation of $V \setminus \{q\}$.

$$V \setminus \{q\} = \varphi(W) \cup \varphi^2(W) \cup \varphi^3(W) \cup \cdots$$

By extending the tessellation so as to include negative iterates, we get a tessellation of the punctured plane $\mathbf{R}^2 \setminus \{q\}$ with sets $\varphi^i(W)$, i an integer. For each point x the orbit of x under the action of φ is a set with one limit point only, namely q. The orbit space of $\mathbf{R}^2 \setminus \{q\}$ is obtained as a quotient space by identifying each orbit to a single point. It can be represented by W with identification of points in the boundary ∂W. To describe this identification properly, we distinguish between geometric sides and combinatorial edges of a gnomon.

Definition 4. Suppose that W is a gnomon of V with contracting rotation $\varphi: V \cup W \to V$. We say that an arc $I \subset \partial W$ is a combinatorial edge of W if $I = W \cap \varphi^i(W)$ for some (positive or negative) i. We shall say that the end points of a combinatorial edge are combinatorial vertices.

The combinatorial edges of a gnomon are subsets of the geometric sides, and they are identified pairwise by the orbit quotient map. The number of combinatorial edges is equal to the number of neighbors of W in the tessellation $\{\varphi^i(W) : i \in \mathbf{Z}\}$

of the punctured plane. A combinatorial vertex is the common intersection of at least three tiles. Figure 2 showes that combinatorial vertices are the intersection of three tiles in the golden spiral and that they are the intersection of four tiles in the silver spiral.

LEMMA 5. *Suppose that W is a convex gnomon of V and that the similarity transformation is a contractive rotation. Then W has either four or six combinatorial edges. In the first case, each combinatorial vertex is the intersection of four tiles. In the second case, each combinatorial vertex is the intersection of three tiles.*

Proof. This is a common observation for tessellations of the plane, see e.g. [3, page 63]. Let n be the number of combinatorial edges of W. Then n is equal to the number of combinatorial vertices, since ∂W is a simple closed curve. The orbit map identifies the tessellation to a triangulation of a torus, with 1 face and $n/2$ edges. As each combinatorial vertex belongs to at least 3 tiles, the number of combinatorial vertices is $\leq n/3$. Since the torus has zero Euler characteristic, the number of combinatorial vertices is $n/2 - 1$. It follows that $n \leq 6$. Since n is even, n is equal to 4 or 6. In the first case all combinatorial vertices are identified to one point. In the second case they are identified to two points in the orbit space. ■

Fig. 3. The Voronoi-diagram of an orbit

COROLLARY 6. *A convex gnomon has at most six geometric sides.*

Proof. A geometric side contains at least one combinatorial edge. ■

One may wonder whether convex gnomons with six edges really exist. Figure 3 shows that they do. The figure is constructed by drawing the Voronoi cells of a contracting rotation over 53 degrees.

4. Regular gnomons

We shall say that a gnomon is *regular* if it is a regular polygon.

LEMMA 7. *A regular gnomon is either a triangle or a square.*

Proof. Let W be a regular gnomon and let φ be a contracting rotation. Since W is a regular polygon, all its sides are of equal length. As was indicated above combinatorial edges of ∂W are pairwise identified and in each pair one combinatorial edge is a geometric side. If there are four combinatorial edges, the gnomon is a triangle. If there are six combinatorial edges, the gnomon is a square. ■

LEMMA 8. *Suppose that W is a regular gnomon with contracting rotattion φ. Then $\varphi(W) \cap W$ is non-empty.*

Proof. We give the proof in the case W is a triangle only. This case is more difficult than the case that W is a square. Each vertex of W equals the intersection of four tiles, three of which have the vertex as a corner point. We label the three corner points of W as v_1, v_2, v_3. They are chosen in such a way that v_1 the intersection with three other tiles, all of which are larger than W. The second vertex v_2 is the intersection with two larger tiles and one smaller tile. The third vertex v_3 is the intersection with one larger tile and two smaller tiles. Let $V \cup W$ be the polygon such that $V \cup W$ is similar to V. Then v_1 is a corner point of $V \cup W$ with angle $\pi/3$, v_2 is a vertex with angle $2\pi/3$ and v_3 is on a side of $V \cup W$. All interior angles in V therefore are equal to $\pi/3$ or $2\pi/3$. The exterior angles of V (for each corner point only one) add up to 2π, so there are two possibilities for V: one angle of V is $\pi/3$ and four angles are $2\pi/3$, or two angles are $\pi/3$ and two angles are $2\pi/3$. In the latter case V has a diamond shape which is clearly impossible. So v_1 has a unique interior angle in $V \cup W$ and v_2 has a unique interior angle in V, This implies that $\varphi(v_1) = v_2$ and the result follows. ■

Proof of Theorem 2. Suppose that W is regular gnomon and that φ is its contracting rotation with $d(\varphi(x), \varphi(y)) = \lambda d(x, y)$. First we consider the case that W is a triangle, so it has four combinatorial edges, two of which are geometric sides. In the boundary of W there is one unique combinatorial vertex v that is not a corner point of W. It is the intersection of four copies of W, one of which meets W in v only. Label the vertices of W as a, b, c such that ab the edge that contains v and b is in $W \cap \varphi(W)$. It follows that $\varphi(c) = b$ and $\varphi(a) = v$. So φ rotates over $\pi/3$ and φ^6 is a pure contraction. This implies that $\{v\} = W \cap \varphi^6(W)$, in particular $\varphi^6(c) = v$. It follows that $\varphi^5(b) = v$ and $\varphi^5(c) = a$, so $\lambda + \lambda^5 = 1$. The last equation has the silver number as its only positive real root. This statement follows from the fact that $\lambda^3 + \lambda^2 - 1$ is a factor of $\lambda^5 + \lambda - 1$, the other factor being $\lambda^2 - \lambda + 1$ (which has no real zeroes).

Now we consider the case that W is a square and apply the same arguments. W has six combinatorial edges, three of which are geometric sides. Let v and w be the two combinatorial vertices that are not corner points. Since $W \cap \varphi(W)$ is non-empty, the rotation has to be over $\pi/2$. So ϕ^4 is a proper contraction, which implies that $\{v, w\} = W \cap \varphi^4(W)$. Label the corner points of W as a, b, c, d with ab the edge that contains $\{v, w\}$ and b in $W \cap \varphi(W)$. We may assume that $\varphi(a) = v$. Then $\varphi(d) = b$, $\varphi^4(c) = v$, $\varphi^4(d) = w$. We have $\varphi^3(c) = a$ and $\varphi^3(b) = w$, so $\lambda + \lambda^3 + \lambda^4 = 1$. The last equation has the golden number as its only positive real root.

5. Concluding remarks

Gnomons were introduced by Gazalé in a metaphysical book on shapes that occur in biology and architecture. The silver number has appeared in works on architecture, in particular that of the Dutch architect and Benedictine monk Dom van der Laan, [2, 5, 6, 7, 8].

We have seen that convex gnomons can have six sides. We are unable, however, to decide whether there exist gnomons that have five sides.

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