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SPECTRAL PROBLEMS FOR PARABOLIC EQUATIONS WITH CONDITIONS OF CONJUGATION AND DYNAMICAL BOUNDARY CONDITIONS

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Abstract. The initial boundary value problems for the heat equation with discontinuous heat flow and concentrated heat capacity in the interior points or at the boundary are considered. The corresponding spectral problems, where the eigenvalues appear in the boundary or interface conditions, are derived and studied.

1. Abstract setting of the problem

Let H be a real separable Hilbert space endowed with the inner product $(\cdot,\cdot)$, and norm $\|\cdot\|$, and A is a linear selfadjoint unbounded positive definite linear operator with domain $D(A)$ dense in H. The product $(u,v)_A = (Au,v)$, $(u,v \in D(A))$ satisfies the inner product axioms. Reinforcing $D(A)$ in the norm $\|u\|_A = (u,u)_A^{1/2}$, we obtain a Hilbert space $H_A \subset H$. Operator A extends to mapping $A\colon H_A \to H_{A^{-1}}$, where $H_{A^{-1}}$ is the adjoint space for H_A, and $H_A \subset H \subset H_{A^{-1}}$ form a Gel'fand triple. We also define the Sobolev spaces $W_2^s(a,b;H)$, $W_2^0(a,b;H) = L_2(a,b;H)$ of the functions $u=u(t)$ mapping the interval $(a,b) \subset \mathbf{R}$ into H. The problems solved below can ve written as an abstract Cauchy problem

$$\frac{du}{dt} + Au = f(t), \qquad 0<t<T; \quad u(0)=u_0, \tag{1.1}$$

where u_0 is a given element of H, $f(t) \in L_2(0,T;H_{A^{-1}})$ and $u(t)$ is the unknown function from $(0,T)$ into H_A. Let us define λ by

$$\frac{1}{\lambda} = \sup_{U\in H_A} \frac{\|U\|^2}{\|U\|_A^2}. \tag{1.2}$$

The solution of the variational problem is the solution of the following spectral problem [2]:

$$AU = \lambda U. \tag{1.3}$$

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The spectrum of (1.3) is discrete, all eigenvalues $\lambda = \lambda_n$, $n = 1,2,\dots$ are positive: $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, $\lambda_n \to \infty$, while the normalized eigenfunctions $U = U_n$, $n = 1,2,\dots$ satisfy the condition of orthogonality $(U_n, U_m) = \delta_{nm}$ and represent a basis of the spaces H and H_A. Thus, the solution of the problem (1.1) can be obtained in the form

$$u(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\left[c_n + \int_0^t e^{\lambda_n \tau} f_n(\tau)\,d\tau\right] U_n, \tag{1.4}$$

where $c_n = (u_0, U_n)$, $f_n(t) = (f(t), U_n)$.

Setting $f(t) = 0$ in (1.1), an a priori estimate can be done. We take inner product in H of (1.1) with u and apply the inequality $\lambda_1(x,x) \leqslant (Ax,x)$ to obtain

$$\|u(t)\| \leqslant e^{-\lambda_1 t}\|u_0\|. \tag{1.5}$$

Now, let B be a linear selfadjoint positive definite operator with domain $D(B)$ dense in H, A is a linear selfadjoint unbounded positive definite linear operator with domain $D(A)$ dense in H_B and $A \geqslant B$. We consider the abstract Cauchy problem

$$B\frac{du}{dt} + Au = f(t), \qquad t \in (0,T); \quad u(0) = u_0, \tag{1.6}$$

where u_0 is a given element of H_B, $f(t) \in L_2(0,T; H_{A^{-1}})$ and $u(t)$ is the unknown function from $(0,T)$ into H_A. It is easy to see that problem (1.6) can be written in the form

$$\frac{d\tilde{u}}{dt} = \tilde{A}\tilde{u} = \tilde{f}, \qquad t \in (0,T); \quad \tilde{u}(0) = \tilde{u}_0,$$

if one sets $\tilde{u} = B^{1/2}u$, $B^{-1/2}AB^{-1/2} = \tilde{A}$ and $B^{-1/2}f(t) = \tilde{f}$. Then, the following spectral problem can be obtained:

$$\tilde{A}\tilde{U} = \tilde{\lambda}\tilde{U}.$$

Operator $\tilde{A}$ is positive definite and unbounded in H. Therefore, its spectrum is discrete, all eigenvalues $\tilde{\lambda}_n$, $n = 1,2,\dots$ are positive, $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots$, $\tilde{\lambda}_n \to \infty$, while the eigenfunctions U_n, $n = 1,2,\dots$ satisfy the condition of orthogonality $(\tilde{U}_j, \tilde{U}_k) = \delta_{jk}$ and represent a basis of the space H. If we set $\tilde{U} = B^{1/2}U$, $\tilde{U}_n = B^{1/2}U_n$, the spectral problem takes the form

$$AU = \tilde{\lambda}BU. \tag{1.7}$$

In such a way, the spectrum of (1.7) is discrete, all eigenvalues are positive, $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots$, $\tilde{\lambda}_n \to \infty$, while the eigenfunctions U_n, $n = 1,2,\dots$ satisfy the condition of orthogonality

$$(U_j, U_k)_B = (BU_j, U_k) = (\tilde{U}_j, \tilde{U}_k) = \delta_{jk}$$

and represent a basis of the space H_B.

The solution of problem (1.6) can be written in the form

$$u(t) = \sum_{n=1}^{\infty} e^{-\tilde{\lambda}_n t}\left[c_n + \int_0^t e^{\tilde{\lambda}_n \tau} f_n(\tau)\,d\tau\right] U_n,$$

where $c_n = (u_0, U_n)_B$, $f_n(t) = (f(t), U_n)$.

In order to obtain an energy estimate for the solution of the problem, when $f(t) = 0$, we take an inner product of (1.6) with $2u$,

$$\frac{d}{dt}[(BU, u)] + 2\|u\|_A^2 = 0.$$

Takimg into account the inequality $\|u\|_A^2 \geqslant \tilde{\lambda}_1 \|u\|_B^2$, we get

$$\frac{d(\|u\|_B^2)}{\|u\|_B^2} \leqslant -2\tilde{\lambda}_1\, dt.$$

Integrating the result from 0 to t , the following estimate can be done:

$$\|u(t)\|_B \leqslant \|u_0\|_B e^{-\tilde{\lambda}_1 t}.$$

2. Heat equation with concentrated capacity

2.1. The first boundary value problem

Let us consider the first initial boundary value problem for the heat equation with concentrated capacity at the interior point $x = \xi$:

$$[1 + K\delta(x - \xi)]\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad (x,t) \in (0,1) \times (0,T), \tag{2.1}$$

$$u(0,t) = u(1,t) = 0, \qquad 0 < t < T, \tag{2.2}$$

$$u(x,0) = u_0(x), \qquad x \in (0,1), \tag{2.3}$$

where $K > 0$ and $\delta(x)$ is the Dirac distribution. Similar problems are already mentioned in [3], [4]. The derivations in (2.1) are taken in the sense of the distribution theory. It follows from (2.1), that the solution of this problem satisfies at $(x,t) \in (0,\xi) \times (0,T)$ and $(x,t) \in (\xi,1) \times (0,T)$ the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad x \in (0,\xi) \cup (\xi,1), \quad t \in (0,T),$$
$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = u_0(x),$$

and at $x = \xi$ the conditions of conjugation

$$[u]_{x=\xi} \equiv u(\xi + 0, t) - u(\xi - 0, t) = 0, \qquad K\frac{\partial u}{\partial t}(\xi,t) = \left[\frac{\partial u}{\partial x}\right]_{x=\xi}.$$

The initial boundary value problem (2.1)–(2.3) can be reduced to the form (1.6) letting $H = L_2(0,1)$,

$$Au = -\frac{\partial^2 u}{\partial x^2}, \qquad Bu = [1 + K\delta(x - \xi)]u$$

and

$$(u,v)_B = \int_0^1 u(x)v(x)\, dx + Ku(\xi)v(\xi),$$

while $H_A = \overset{\circ}{W}{}_2^1(0,1)$. Thus, the following spectral problem can be obtained:

$$-\frac{d^2U}{dx^2} = \lambda[1 + K\delta(x-\xi)]U(x), \qquad x \in (0,1),$$
$$U(0) = U(1) = 0,$$

or

$$-\frac{d^2U}{dx^2} = \lambda U(x), \qquad x \in (0,\xi) \cup (\xi,1),$$
$$U(0) = U(1) = 0, \tag{2.4}$$
$$[U]_{x=\xi} \equiv U(\xi+0) - U(\xi-0) = 0, \qquad -\left[\frac{dU}{dx}\right]_{x=\xi} = \lambda K U(\xi).$$

The solution of this problem can be written in the following explicit form:

$$U(x) = \begin{cases} A\sin\alpha x, & x \in (0,\xi), \\ B\sin\alpha(1-x), & x \in (\xi,1), \end{cases}$$

It is obvious that $U(x)$ satisfies the boundary conditions. The values of the constants A and B can be obtained using the first condition of conjugation, and we get $A = C\sin\alpha(1-\xi)$, $B = C\sin\alpha\xi$, where C is a multiplicative constant, so we can set $C = 1$. From (2.4) and the second condition of conjugation, we find that

$$\alpha = \frac{1}{K}[\cot\alpha(1-\xi) + \cot\alpha\xi]. \tag{2.5}$$

i	α_i	λ_i
1	1.07687	1.15965
2	3.64360	13.27582
3	6.57833	43.27442
4	9.62956	92.72843
5	12.72230	161.85692

Table 1

If $\xi = 1/2$, the equation (2.5) takes the form $\alpha = \dfrac{2}{K}\cot\dfrac{\alpha}{2}$. There exists a countable set of its solutions: α_n, $n = 1,2,\ldots$. Using the condition $\lambda = \alpha^2$, we obtain the eigenvalues $\lambda_n = \alpha_n^2$, $n = 1,2,\ldots$. The graphical solution of this equation is shown in Fig. 1, and the numerical values of α_n and λ_n are shown in Table 1. In Fig. 2 the first three eigenfunctions U_1, U_2 and U_3 are presented.

Figure 1

Figure 2

If $\xi \neq 1/2$ the right-hand side of (2.5) is a sum of two periodical functions which have different periods. Transcendent equation (2.5) has a countable set of solutions $\alpha = \alpha_n$, $n = 1, 2, \ldots$. So we can obtain the eigenvalues

$$\lambda = \lambda_n = \alpha_n^2, \qquad 0 < \lambda_1 < \lambda_2 < \cdots, \quad \lambda_n \to \infty,$$

and the corresponding system of eigenfunctions $U = U_n(x)$, $n = 1, 2, \ldots$.

2.2. The second boundary value problem

Let us consider the heat equation with concentrated capacity (2.1) with inital value (2.3) and Neumann's boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \qquad 0 < t < T. \tag{2.6}$$

The problem (2.1), (2.3), (2.6) can be written in the form

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad x \in (0,\xi) \cup (\xi,1), \quad t \in (0,T), \\
\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \qquad u(x,0) = u_0(x), \\
[u]_{x=\xi} = 0, \qquad \left[\frac{\partial u}{\partial x}\right]_{x=\xi} = K\frac{\partial u}{\partial t}(\xi,t).
\end{gathered}$$

Using the same procedure as it was done in the section 2.1, the following spectral problem can be obtained

$$\begin{gathered}
-\frac{d^2U}{dx^2} = \lambda U(x), \qquad x \in (0,\xi) \cup (\xi,1), \\
U'(0) = U'(1) = 0, \\
[U]_{x=\xi} = 0, \qquad -\left[\frac{dU}{dx}\right]_{x=\xi} = \lambda K U(\xi).
\end{gathered}$$

The solution of this spectral problem can be written in the form

$$U(x) = \begin{cases} A\cos\alpha x, & x \in (0,\xi), \\ B\cos\alpha(1-x), & x \in (\xi,1). \end{cases}$$

It is obvious that it automatically satisfies the boundary conditions. We obtain the values of the constants A and B using the first condition of conjugation: $A = C\cos\alpha(1-\xi)$, $B = C\cos\alpha\xi$. Here, C is a multiplicative constant, and we can set $C = 1$. By the second condition of conjugation, taking into account that $\lambda = \alpha^2$, we obtain

$$\alpha = -\frac{1}{K}[\tan\alpha\xi + \tan\alpha(1-\xi)]. \tag{2.7}$$

i	α_i	λ_i
1	4.05752	16.46347
2	9.82636	96.55735
3	15.95730	254.63542
4	22.17110	491.55768
5	28.41490	807.40654

Table 2

If $\xi = 1/2$, the equation (2.7) takes the form $\alpha = -\dfrac{2}{K}\tan\dfrac{\alpha}{2}$ and has a countable set of solutions α_n, $n = 1, 2, \dots$. Now, using the condition $\lambda = \alpha^2$, we can obtain the eigenvalues $\lambda_n = \alpha_n^2$, $n = 1, 2, \dots$. The graphical solution of equation (2.7) is shown in Fig. 3, and the numerical values of α_n and λ_n are shown in Table 2. In Fig. 4 the first three eigenfunctions U_1, U_2 and U_3 are presented.

Figure 3

Figure 4

If $\xi \neq 1/2$ the right-hand side of (2.7) is a sum of two periodical functions which have different periods. Again, there exists a countable set of solutions $\alpha = \alpha_n$, $n = 1, 2, \dots$. From here we obtain the eigenvalues $\lambda_n = \alpha_n^2$, $0 < \lambda_1 < \lambda_2 < \cdots$, $\lambda_n \to \infty$, and the corresponding eigenfunctions $U = U_n(x)$, $n = 1, 2, \dots$.

2.3. The third boundary value problem

We consider the heat equation with concentrated capacity (2.1) with inital value (2.3) and the third boundary conditions

$$-\frac{\partial u}{\partial x}(0,t) + \alpha u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) + bu(1,t) = 0, \qquad 0 < t < T. \tag{2.8}$$

The problem (2.1), (2.3), (2.8) can be written in the form

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad x \in (0,\xi) \cup (\xi,1), \quad t \in (0,T), \\
-\frac{\partial u}{\partial x}(0,t) + au(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) + bu(1,t) = 0, \\
u(x,0) = u_0(x), \\
[u]_{x=\xi} = 0, \qquad \left[\frac{\partial u}{\partial x}\right]_{x=\xi} = K\frac{\partial u}{\partial t}(\xi,t).
\end{gathered}$$

Thus, the following spectral problem can be obtained

$$
\begin{gathered}
-\frac{d^2U}{dx^2} = \lambda U(x), \qquad x \in (0,\xi) \cup (\xi,1), \\
-U'(0) + aU(0) = 0, \qquad U'(1) + bU(1) = 0, \\
[U]_{x=\xi} = 0, \qquad -\left[\frac{dU}{dx}\right]_{x=\xi} = \lambda K U(\xi).
\end{gathered} \tag{2.9}
$$

The solution of this spectral problem can be written in the form

$$
U(x) = \begin{cases} A\sin(\alpha x + \beta), & x \in (0,\xi), \\ B\sin[\alpha(1-x)+\gamma], & x \in (\xi,1). \end{cases}
$$

It is easy to see that $\lambda = \alpha^2$. By the boundary conditions we obtain

$$
\tan\beta = \frac{\alpha}{a}, \quad \tan\gamma = \frac{\alpha}{b}.
$$

Using the first condition of conjugation we get the values of the constants: $A = D\sin[\alpha(1-\xi)+\gamma]$ and $B = D\sin(\alpha\xi+\beta)$. Here, D is a multiplicative constant, and we can set $D = 1$. Takimg into account the second condition of conjugation we obtain

$$
\alpha = \frac{1}{K}\left[\frac{1-\frac{\alpha}{a}\tan\alpha\xi}{\frac{\alpha}{a}+\tan\alpha\xi} + \frac{1-\frac{\alpha}{b}\tan\alpha(1-\xi)}{\frac{\alpha}{b}+\tan\alpha(1-\xi)}\right]. \tag{2.10}
$$

If we consider the special case $\xi = 1/2$ and $a = b$, the equation (2.10) takes the form $\alpha = \dfrac{2}{K}F(\alpha)$, $F(\alpha) = \dfrac{a-\alpha\tan\frac{\alpha}{2}}{\alpha+a\tan\frac{\alpha}{2}}$. It is easy to see that there exists a countable set of solutions α_n, $n = 1,2,\ldots$. Now, using the condition $\lambda = \alpha^2$, we can obtain the eigenvalues $\lambda_n = \alpha_n^2$, $n = 1,2,\ldots$. The graphical solution of this equation for $\alpha = 1$ is shown in Fig. 5, and the numerical values of α_n and λ_n are shown in Table 3. In Fig. 6 the first three eigenfunctions U_1, U_2 and U_3 are presented.

i	α_i	λ_i
1	0.87638	0.76804
2	4.43297	19.65122
3	10.01790	100.35832
4	16.07970	258.55675
5	22.26010	495.51205

Table 3

Figure 5

Figure 6

i	α_i	λ_i
1	1.13814	1.29536
2	3.97213	15.77782
3	5.15003	26.52281
4	9.78116	95.67109
5	10.54330	111.16117

Table 4

If $\xi = 1/2$ and $a \neq b$, the equation (2.10) takes the form

$$\alpha = \frac{1}{K}\left[\frac{a-\alpha\tan\frac{\alpha}{2}}{\alpha+a\tan\frac{\alpha}{2}}+\frac{b-\alpha\tan\frac{\alpha}{2}}{\alpha+b\tan\frac{\alpha}{2}}\right].$$

Its graphical and numerical sopultions, setting $a = 1$, $b = 4$, are presented in Fig. 7 and Tab. 4 respectively. In Fig. 8 the first three eigenfunctions U_1, U_2 and U_3 are presented.

Figure 7

Figure 8

If $\xi \neq 1/2$ on the right-hand side of (2.10) we have a sum of two functions that have the same form as $F(\alpha)$. Once again, there exists a countable set of solutions $\alpha = \alpha_n$, $n = 1, 2, \ldots$. So, we can obtain the eigenvalues

$$\lambda_n = \alpha_n^2, \qquad 0 < \lambda_1 < \lambda_2 < \cdots, \quad \lambda_n \to \infty,$$

and the corresponding eigenfuctions $U = U_n(x)$, $n = 1, 2, \ldots$.

3. Heat equation with dynamical boundary condition

Let us consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad (x,t) \in (0,1)\times(0,T), \tag{3.1}$$

with initial value (2.3) and dynamical boundary conditions at the point $x = 0$ (see [1], [4]):

$$K\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial x}(0,t), \qquad 0 < t < T, \tag{3.2}$$

where $K > 0$. At $x = 1$ we consider one of the standard boundary conditions:

$$u(1,t) = 0, \qquad 0 < t < T, \tag{3.3}$$

$$\frac{\partial u}{\partial x}(1,t) = 0, \qquad 0 < t < T, \tag{3.4}$$

or

$$\frac{\partial u}{\partial x}(1,t) + bu(1,t) = 0, \qquad 0 < t < T. \tag{3.5}$$

Using odd extension of the solution and the input data these problems can be reduced to the above mentioned. Indeed, setting

$$\tilde{f}(x,t) = f(-x,t), \quad \tilde{u}_0(x) = u_0(-x), \quad \tilde{u}(x) = u(-x), \qquad x \in (-1,0)$$

the problem (3.1), (2.3), (3.2), (3.3) reduces to

$$\begin{gathered} \frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f}(x,t), \qquad x \in (-1,0) \cup (0,1), \quad t \in (0,T), \\ \tilde{u}(-1,t) = \tilde{u}(1,t) = 0, \qquad \tilde{u}(x,0) = \tilde{u}_0(x), \\ [\tilde{u}]_{x=0} = 0, \qquad 2K\frac{\partial \tilde{u}}{\partial t}(0,t) = \left[\frac{\partial \tilde{u}}{\partial x}\right]_{x=0}, \end{gathered}$$

or equivalently

$$\begin{gathered} [1 + 2K\delta(x)]\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f}(x,t), \qquad (x,t) \in (-1,1) \times (0,T); \\ \tilde{u}(-1,t) = \tilde{u}(1,t) = 0, \qquad 0 < t < T, \\ \tilde{u}(x,0) = \tilde{u}_0(x), \qquad x \in (-1,1). \end{gathered}$$

Analogously, (3.1), (2.3), (3.2), (3.4) reduces to

$$\begin{gathered} [1 + 2K\delta(x)]\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f}(x,t), \qquad (x,t) \in (-1,1) \times (0,T); \\ \frac{\partial \tilde{u}}{\partial x}(-1,t) = \frac{\partial \tilde{u}}{\partial x}(1,t) = 0, \qquad 0 < t < T, \\ \tilde{u}(x,0) = \tilde{u}_0(x), \qquad x \in (-1,1), \end{gathered}$$

while (3.1), (2.3), (3.2), (3.5) reduces to

$$\begin{gathered} [1 + 2K\delta(x)]\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f}(x,t), \qquad (x,t) \in (-1,1) \times (0,T); \\ -\frac{\partial \tilde{u}}{\partial x}(-1,t) + b\tilde{u}(-1,t) = \frac{\partial \tilde{u}}{\partial x}(1,t) + b\tilde{u}(1,t) = 0, \\ \tilde{u}(x,0) = \tilde{u}_0(x), \qquad x \in (-1,1). \end{gathered}$$

The corresponding spectral problems

$$-\frac{d^2U}{dx^2} = \lambda U, \;\; x \in (0,1), \qquad -\frac{dU}{dx}(0) = \lambda K U(0), \qquad U(1) = 0, \tag{3.6}$$

$$-\frac{d^2U}{dx^2} = \lambda U, \;\; x \in (0,1), \qquad -\frac{dU}{dx}(0) = \lambda K U(0), \qquad \frac{dU}{dx}(1) = 0, \tag{3.7}$$

and

$$-\frac{d^2U}{dx^2} = \lambda U, \;\; x \in (0,1), \qquad -\frac{dU}{dx}(0) = \lambda K U(0), \qquad \frac{dU}{dx}(1) + bU(1) = 0 \quad (3.8)$$

reduce respectively to

$$-\frac{d^2\tilde{U}}{dx^2} = \lambda[1 + 2K\delta(x)]\tilde{U}, \;\; x \in (-1,1), \qquad \tilde{U}(-1) = \tilde{U}(1) = 0, \qquad (3.9)$$

$$-\frac{d^2\tilde{U}}{dx^2} = \lambda[1 + 2K\delta(x)]\tilde{U}, \;\; x \in (-1,1), \qquad \frac{d\tilde{U}}{dx}(-1) = \frac{d\tilde{U}}{dx}(1) = 0, \quad (3.10)$$

and

$$\begin{aligned} -\frac{d^2\tilde{U}}{dx^2} &= \lambda[1 + 2K\delta(x)]\tilde{U}, \qquad x \in (-1,1), \\ -\frac{d\tilde{U}}{dx}(-1) + b\tilde{U}(-1) &= \frac{d\tilde{U}}{dx}(1) + b\tilde{U}(1) = 0. \end{aligned} \qquad (3.11)$$

Note that the singular point in all three cases is the midpoint of the interval $(-1,1)$.

From the previous results we immediately obtain eigenvalues and eigenfunctions of (3.6) in the form

$$\lambda_n = \alpha_n, \qquad U_n(x) = \sin \alpha_n(1-x), \qquad n = 1, 2, \ldots,$$

where α_n are solutions of the equation $\alpha = \dfrac{1}{K}\cot\alpha$. Analogously, eigenvalues and eigenfunctions of (3.7) are

$$\lambda_n = \alpha_n, \qquad U_n(x) = \cos \alpha_n(1-x), \qquad n = 1, 2, \ldots,$$

where α_n are solutions of the equation $\alpha = -\dfrac{1}{K}\tan\alpha$, while eigenvalues and eigenfunctions of (3.8) are

$$\lambda_n = \alpha_n, \qquad U_n(x) = \sin\left[\alpha_n(1-x) + \arctan\frac{\alpha_n}{b}\right], \qquad n = 1, 2, \ldots,$$

where α_n are solutions of the equation $\alpha = -\dfrac{1}{K}\dfrac{b - \alpha\tan\alpha}{\alpha + b\tan\alpha}$.

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