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# UNIFICATION OF SOME CONCEPTS SIMILAR TO THE LINDELÖF PROPERTY

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**Abstract.** In this paper the $\varphi_{1,2}$-Lindelöf property is defined and studied with the aim of unifying various concepts related to the Lindelöf property in General Topology.

## 1. Introduction

Unifications of various concepts in general topology have been studied in [1, 10–16,18, 21, 24–28]. Although the unifications discussed in [11] and [13] using operations were given for fuzzy topological spaces, it was pointed out that they apply equally well to topological spaces.

A principle aim of unification theory is to reduce the confusion caused by a plethora of very similar properties. For the most part we will not repeat here the definitions of the various properties used to illustrate the results obtained, although these will usually be clear from the particular choice of operations involved.

In a topological space $(X,\tau)$ we will use int, cl, scl, etc., to stand for the interior, closure and semi-closure operations, and so on. We will also use $A^o$, $\overline{A}$ to stand for the interior and closure, respectively, of a subset $A$ of $X$.

Definition 1.1. Let $(X,\tau)$ be a topological space. A mapping $\varphi\colon P(X)\to P(X)$ is called an *operation on* $(X,\tau)$ if $A^o\subset\varphi(A)$ for all $A\in P(X)$ and $\varphi(\emptyset)=\emptyset$.

The class of all operations on a topological space $(X,\tau)$ will be denoted by $O(X,\tau)$.

A partial order "$\leq$" on $O(X,\tau)$ is defined by $\varphi_1\leq\varphi_2\Leftrightarrow\varphi_1(A)\subset\varphi_2(A)$ for each $A\in P(X)$. An operation $\varphi\in O(X,\tau)$ is called *monotonous* if $\varphi(A)\subset\varphi(B)$ whenever $A\subset B$, $(A,B\in P(X))$.

Definition 1.2. Let $\varphi\in O(X,\tau)$ and $A,B\subset X$. Then $A$ is called *$\varphi$-open* if $A\subset\varphi(A)$. Likewise, $B$ is called *$\varphi$-closed* if $X\setminus B$ is $\varphi$-open. An operation

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$\tilde{\varphi} \in O(X,\tau)$ is called the *dual operation of* $\varphi$ if $\tilde{\varphi}(A) = X \setminus \varphi(X \setminus A)$ for each $A \in P(X)$.

If $\varphi$ is monotonous, then the family of all $\varphi$-open sets is a supratopology ($\mathcal{U} \subset P(X)$) is a supratopology on $X$ means that $\emptyset \in \mathcal{U}$, $X \in \mathcal{U}$ and $\mathcal{U}$ is closed under arbitrary unions [2]).

Let $(X,\tau)$ be a topological space, $\varphi \in O(X,\tau)$, $\mathcal{U} \subset P(X)$ and $x \in X$. We will use the following notation.

$$\begin{aligned}
\mathcal{U}(x) &= \{\, U : x \in U \in \mathcal{U} \,\},\\
\varphi O(X) &= \{\, U : U \subset X,\ U \text{ is } \varphi\text{-open} \,\},\\
\varphi C(X) &= \{\, K : K \subset X,\ K \text{ is } \varphi\text{-closed} \,\},\\
\varphi O(X,x) &= \{\, U : U \in \varphi O(X),\ x \in U \,\},\\
\mathcal{N}(\mathcal{U},x) &= \{\, N : N \subset X \text{ and there exists a } U \in \mathcal{U}(x) \text{ such that } U \subset N \,\}.
\end{aligned}$$

Definition 1.3. Take $\varphi \in O(X,\tau)$ and $\mathcal{U} \subset P(X)$. Then $\varphi$ is called *regular with respect to* (shortly w.r.t.) $\mathcal{U}$ if $x \in X$ and given $U, V \in \mathcal{U}(x)$ there exists $W \in \mathcal{U}(x)$ such that $\varphi(W) \subset \varphi(U) \cap \varphi(V)$.

For any operation $\varphi \in O(X,\tau)$, $\tau \subset \varphi O(X)$ and $X, \emptyset$ are both $\varphi$-open and $\varphi$-closed.

Definition 1.4. Let $\varphi_1, \varphi_2 \in O(X,\tau)$, $A \subset X$. Then

a) $x \in \varphi_{1,2} \operatorname{int} A \Leftrightarrow$ there exists $U \in \varphi_1 O(X,x)$ such that $\varphi_2(U) \subset A$.

b) $x \in \varphi_{1,2} \operatorname{cl} A \Leftrightarrow$ for each $U \in \varphi_1 O(X,x)$, $\varphi_2(U) \cap A \neq \emptyset$.

c) $A$ is $\varphi_{1,2}$-open $\Leftrightarrow A \subset \varphi_{1,2} \operatorname{int} A$.

d) $A$ is $\varphi_{1,2}$-closed $\Leftrightarrow \varphi_{1,2} \operatorname{cl} A \subset A$.

If $A \subset B$ then $\varphi_{1,2} \operatorname{int} A \subset \varphi_{1,2} \operatorname{int} B$ and $\varphi_{1,2} \operatorname{cl} A \subset \varphi_{1,2} \operatorname{cl} B$. Clearly, for any set $A$ we have $X \setminus \varphi_{1,2} \operatorname{int} A = \varphi_{1,2} \operatorname{cl}(X \setminus A)$ and $A$ is $\varphi_{1,2}$-open iff $X \setminus A$ is $\varphi_{1,2}$-closed.

We will use $\varphi_{1,2}O(X)$ ($\varphi_{1,2}C(X)$) to denote the family of all $\varphi_{1,2}$-open subsets (the family of all $\varphi_{1,2}$-closed subsets) of $X$.

Theorem 1.5. ([13]) *Let* $\varphi_1, \varphi_2 \in O(X,\tau)$.

a) $\varphi_{1,2}O(X)$ *is a supratopology on* $X$.

b) *If* $\varphi_2$ *is regular w.r.t.* $\varphi_1 O(X)$ *then* $\varphi_{1,2}O(X)$ *is a topology on* $X$ *and a subset* $K$ *of* $X$ *is closed w.r.t. this topology iff* $\varphi_{1,2} \operatorname{cl} K \subset K$.

c) *If* $\varphi_2$ *is regular w.r.t.* $\varphi_1 O(X)$ *and if* $\varphi_2 \geq \imath$ *or* $\varphi_2 \geq \varphi_1$ *then* $\varphi_{1,2}O(X)$ *is a topology on* $X$ *and a set* $K$ *is closed w.r.t. this topology iff* $\varphi_{1,2} \operatorname{cl} K = K$.

In [17], conditions were obtained under which the operator $k\colon P(X) \to P(X)$ defined by $k(A) = \varphi_{1,2} \operatorname{cl} A$ for each $A \in P(X)$ is a Kuratowski closure operator.

Example 1.6. Let the following operations be defined on a topological space $(X,\tau)$: $\varphi_1 = \operatorname{int}$, $\varphi_2 = \operatorname{cl} \circ \operatorname{int}$, $\varphi_3 = \operatorname{cl}$, $\varphi_4 = \operatorname{scl}$, $\varphi_5 = \imath$ ($\imath$ is the identity operation), $\varphi_6 = \operatorname{int} \circ \operatorname{cl}$. Then:

$\varphi_1 \leq \varphi_2 \leq \varphi_3$, $\varphi_1 \leq \varphi_5 \leq \varphi_4 \leq \varphi_3$ and $\varphi_1 \leq \varphi_6 \leq \varphi_4$.

$\varphi_1 O(X) = \tau$, $\varphi_2 O(X) = SO(X) =$ the family of semi-open sets.

$\varphi_3 O(X) = \varphi_5 O(X) = \varphi_4 O(X) = P(X) =$ the power set of $X$.

$\varphi_6 O(X) = PO(X) =$ the family of pre-open sets.

$\varphi_{1,3} O(X) = \tau_\theta =$ the topology of all $\theta$-open sets.

$\varphi_{2,4} O(X) = S\theta O(X) =$ the family of semi-$\theta$-open sets.

$\varphi_{1,6} O(X) = \tau_s =$ the semi regularization topology of $X$. This is the topology with base $RO(X)$ consisting of the regular open sets = the family of $\delta$-open sets.

$\varphi_{2,3} O(X) = \theta SO(X) =$ the family of all $\theta$-semi-open sets.

$\varphi_1, \varphi_3$ ($\varphi_2, \varphi_6$) are pairs of dual operations. Also, $\varphi_2$, $\varphi_3$, $\varphi_4$, $\varphi_5$ and $\varphi_6$ are regular w.r.t. $\varphi_1 O(X)$.

We use $SC(X)$ ($PC(X)$, $RC(X)$, $S\theta C(X)$ and $\theta SC(X)$, respectively) to denote the family of semi-closed (pre-closed, regular closed, semi-$\theta$-closed, and $\theta$-semi-closed) sets.

$SR(X) = SO(X) \cap SC(X) =$ the family of semi-regular sets.

Clearly if $\varphi_1$ is monotonous and $\varphi_2 = \imath$ for $\varphi_1, \varphi_2 \in O(X,\tau)$, then $\varphi_{1,2} O(X) = \varphi_1 O(X)$ and $\varphi_{1,2} C(X) = \varphi_1 C(X)$.

## 2. Filters

Throughout this section the operations $\varphi_i$, $i = 1, 2, \dots$ are defined on a topological space $(X, \tau)$.

Definition 2.1. ([26]) Let $\mathcal{F}$ be a filter (or filter-base) in $(X,\tau)$ and $a \in X$. Then $\mathcal{F}$ is said to:

a) $\varphi_{1,2}$-accumulate to $a$ if $a \in \bigcap\{\varphi_{1,2} \operatorname{cl} F : F \in \mathcal{F}\}$;

b) $\varphi_{1,2}$-converge to $a$ if for each $U \in \varphi_1 O(X, a)$, there exists an $F \in \mathcal{F}$ such that $F \subset \varphi_2(U)$.

Theorem 2.2. ([26]) 1) *A filter-base $\mathcal{F}_b$ $\varphi_{1,2}$-accumulates ($\varphi_{1,2}$-converges) to $a$ iff filter generated by $\mathcal{F}_b$ $\varphi_{1,2}$-accumulates ($\varphi_{1,2}$-converges) to $a$.*

2) *If $\varphi_2$ is monotonous, we can take the family $\mathcal{N}(\varphi_1 O(X), a)$ instead of $\varphi_1 O(X, a)$ in the above definitions.*

3) *A filter $\mathcal{F}$ $\varphi_{1,2}$-converges to $a$ iff $\{\varphi_2(U) : U \in \varphi_1 O(X, a)\} \subset \mathcal{F}$.*

4) *If $\mathcal{F}$ $\varphi_{1,2}$-converges to $a$ then $\mathcal{F}$ $\varphi_{1,2}$-accumulates to $a$.*

5) *Let $\mathcal{F} \subset \mathcal{F}'$ for the filters $\mathcal{F}$ and $\mathcal{F}'$*

a) *If $\mathcal{F}'$ $\varphi_{1,2}$-accumulates to $a$, then $\mathcal{F}$ $\varphi_{1,2}$-accumulates to $a$.*

b) *If $\mathcal{F}$ $\varphi_{1,2}$-converges to $a$ then $\mathcal{F}'$ $\varphi_{1,2}$-converges to $a$.*

6) *If $\varphi_1' O(X) \subset \varphi_1 O(X)$ and $\varphi_2' \geq \varphi_2$ for the operations $\varphi_1$, $\varphi_2$, $\varphi_1'$, $\varphi_2' \in O(X,\tau)$, then a filter (or a filter-base) $\mathcal{F}$ $\varphi_{1,2}'$-accumulates ($\varphi_{1,2}'$-converges) to $a$ whenever $\mathcal{F}$ $\varphi_{1,2}$-accumulates ($\varphi_{1,2}$-converges) to $a$.*

A space $(X,\tau)$ is called $\varphi_{1,2}$-$T_2$ if for each $x,y \in X$ $(x \neq y)$ there are $\varphi_1$-open sets $U_x$ and $U_y$ such that $x \in U_x$, $y \in U_y$ and $\varphi_2(U_x) \cap \varphi_2(U_y) = \emptyset$ [13, 26].

THEOREM 2.3. ([26]) *Let $(X,\tau)$ be a $\varphi_{1,2}$-$T_2$ space. If a filter $\mathcal{F}$ $\varphi_{1,2}$-converges to some point $a \in X$ and $\varphi_{1,2}$-accumulates to some point $b \in X$ then $a = b$.*

EXAMPLE 2.4. Let $a \in X$ and $\mathcal{F}$ be a filter in $(X,\tau)$.

a) Let $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, $\varphi_2 = \mathrm{cl}$. $\mathcal{F}$ $\varphi_{1,2}$-converges ($\varphi_{1,2}$-accumulates) to $a$ iff $\mathcal{F}$ rc-converges (rc-accumulates) to $a$ since $\{\overline{V} : V \in \tau, x \in \overline{V}\} = \{\overline{U} : x \in U \in SO(X)\}$ [9].

b) Let $\varphi_1 = \mathrm{int}$, $\varphi_2 = \mathrm{cl}$. $\mathcal{F}$ $\varphi_{1,2}$-converges to $a$ iff $\mathcal{F}$ r-converges [8] (or equivalently almost converges [4]) to $a$. $(X,\tau)$ is $\varphi_{1,2}$-$T_2$ iff $(X,\tau)$ is Urysohn.

c) Let $\varphi_1 = \mathrm{int}$, $\varphi_2 = \imath$. $\mathcal{F}$ $\varphi_{1,2}$-converges to $a$ iff $\mathcal{F}$ converges to $a$ in $(X,\tau)$. $(X,\tau)$ is $\varphi_{1,2}$-$T_2$ iff $(X,\tau)$ is Hausdorff.

A filter $\mathcal{F}$ is called a *$\sigma$-filter* if $\mathcal{F}$ is closed under countable intersections and a filter-base $\mathcal{F}_b$ is called a *$\sigma$-filter-base* if $\mathcal{F}_b$ is a base of a $\sigma$-filter. Clearly in any topological space $(X,\tau)$, for any $\varphi \in O(X,\tau)$, $\mathcal{F} = \{X\}$ is a $\sigma$-filter and $X \in \varphi O(X)$.

LEMMA 2.5. *If $\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$, then $\varphi_1 O(X) \subset \varphi_2 O(X)$. The converse is false.*

EXAMPLE 2.6. Let $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, $\varphi_2 = \text{semi-}\mathrm{int}$ be defined on $\mathbb{R}$ with the usual topology. For $A = (0,1]$, $\varphi_1(A) = [0,1]$, $\varphi_2(A) = (0,1]$ and $\varphi_1(A) \not\subset \varphi_2(A)$. i.e. $\varphi_1 \not\leq \varphi_2$.

For the set of rational numbers $\mathbb{Q}$, $\mathbb{Q} \not\subset \varphi_2(\mathbb{Q}) = \emptyset$. i.e. $\varphi_2 \not\geq \imath$. But $\varphi_1 O(\mathbb{R}) = SO(\mathbb{R}) = \varphi_2 O(\mathbb{R})$.

It can be easily seen that if $\varphi_2$ is regular w.r.t. $\varphi_1 O(X)$ and $\varphi_1 O(X) \subset \varphi_2 O(X)$ then $\varphi_{1,2} O(X)$ is the same topology as that given in Theorem 1.5 (c).

THEOREM 2.7. *Let $A \subset X$. Then:*

1) *If there exists a $\sigma$-filter $\mathcal{F}$ containing $A$ which $\varphi_{1,2}$-accumulates to $a$, then $a \in \varphi_{1,2} clA$.*

2) *If $\varphi_1 O(X)$ is closed under countable intersections and $\varphi_2$ is monotonous, then,*

a) *$a \in \varphi_{1,2}\, \mathrm{cl}\, A$ iff there exists a $\sigma$-filter containing $A$ which $\varphi_{1,2}$-converges to $a$.*

b) *$A$ is $\varphi_{1,2}$-closed iff whenever there exists a $\sigma$-filter which contains $A$ and $\varphi_{1,2}$-converges to a point $a$ in $X$, then $a \in A$.*

For the remainder of this section we will assume that $\varphi_1 O(X)$ is closed under countable intersections.

THEOREM 2.8. *If $\varphi_2$ is monotonous then a $\sigma$-filter $\mathcal{F}$ $\varphi_{1,2}$-accumulates to a point $a$ iff there exists a $\sigma$-filter $\mathcal{F}'$ such that $\mathcal{F} \subset \mathcal{F}'$ and $\mathcal{F}'$ $\varphi_{1,2}$-converges to $a$.*

THEOREM 2.9. a) *If* $\varphi_1 O(X) \subset \varphi_2 O(X)$ *then* $\Phi_a = \varphi_1 O(X, a)$ *is* $\sigma$*-filterbase for each* $a \in X$ *and* $\Phi_a$ $\varphi_{1,2}$*-converges to* $a$.

b) *If* $\varphi_2$ *is monotonous and* $\varphi_1 O(X) \subset \varphi_2 O(X)$ *then* $\Phi_a = \{\varphi_2(U) : U \in \varphi_1 O(X, a)\}$ *is a* $\sigma$*-filter-base that* $\varphi_{1,2}$*-converges to* $a$.

If $\varphi_2$ is monotonous, then since $\varphi_1 O(X)$ is closed under countable intersections, clearly $\varphi_2$ is regular w.r.t. $\varphi_1 O(X)$.

THEOREM 2.10. *Let* $\varphi_2$ *be monotonous and* $\varphi_1 O(X)$ *closed under countable intersections. For any* $A \in P(X)$, *let us define*

$$\operatorname{cl}^* A = \{\, x : \text{ there exists a } \sigma\text{-filter which contains } A \text{ and } \varphi_{1,2}\text{-converges to } x \,\}.$$

*Clearly* $\operatorname{cl}^* A = \varphi_{1,2} \operatorname{cl} A$.

a) *The* $\operatorname{cl}^*$ *operator defines the topology* $\tau^*$ *given by*

$$\begin{aligned}\tau^* &= \{\, U \subset X : (X \setminus U)^* \subset X \setminus U \,\} \\ &= \{\, U \subset X : \varphi_{1,2} \operatorname{cl}(X \setminus U) \subset X \setminus U \,\} = \varphi_{1,2} O(X).\end{aligned}$$

b) *If* $\varphi_1 O(X) \subset \varphi_2 O(X)$ *then,*

$$\begin{aligned}\tau^* &= \{\, U \subset X : (X \setminus U)^* = X \setminus U \,\} \\ &= \{\, U \subset X : \varphi_{1,2} \operatorname{cl} A = A \,\} = \varphi_{1,2} O(X).\end{aligned}$$

c) *If* $\varphi_1 O(X) \subset \varphi_2 O(X)$ *and* $\{\varphi_2(U) : U \in \varphi_1 O(X)\} \subset \varphi_{1,2} O(X)$ *then* $\operatorname{cl}^*$ *and* $\varphi_{1,2} \operatorname{cl}$ *are Kuratowski closure operators, since in this case* $\varphi_{1,2} \operatorname{cl}(\varphi_{1,2} \operatorname{cl} A) = \varphi_{1,2} \operatorname{cl} A$, *for each* $A \in P(X)$.

EXAMPLE 2.11. Let $\varphi_1 = \operatorname{int}$, $\varphi_2 = \operatorname{cl}$. Then:

$\varphi_1 O(X) = \tau$. $\varphi_2$ is monotonous, regular w.r.t. $\varphi_1 O(X)$ and $\varphi_2 \geq \varphi_1$. Also, $\varphi_{1,2} \operatorname{cl} A = \theta \operatorname{cl} A$ for any $A \in P(X)$ and $\operatorname{cl}^* A = \{x : \text{there exists a } \sigma\text{-filter } \mathcal{F} \text{ such that } A \in \mathcal{F} \text{ and } \mathcal{F}\ \varphi_{1,2}\text{-converges to x}\}$.

If $(X, \tau)$ is a P space (i.e. each $G_\delta$ set is open), then a set $A$ is $\theta$-closed iff $\theta \operatorname{cl} A = A$ iff whenever $\sigma$-filter contains $A$ and $\varphi_{1,2}$-converges to a point $a$ in $X$, then $a \in A$.

## 3. The Lindelöf property

DEFINITION 3.1. Take $\varphi_1, \varphi_2 \in O(X, \tau)$, $X \in \mathcal{A} \subset P(X)$ and $A \subset X$.

a) If $A \subset \bigcup \mathcal{U}$ for a subfamily $\mathcal{U}$ of $\mathcal{A}$, then $\mathcal{U}$ is called an $\mathcal{A}$-cover of $A$. If an $\mathcal{A}$-cover $\mathcal{U}$ of $A$ is countable (finite) then we call $\mathcal{U}$ a countable $\mathcal{A}$-cover (finite $\mathcal{A}$-cover) of $A$.

b) If each $\mathcal{A}$-cover $\mathcal{U}$ of $A$ has a countable subfamily $\mathcal{U}'$ such that $A \subset \bigcup\{\varphi_2(U) : U \in \mathcal{U}'\}$, then we say that $A$ is ($\mathcal{A}$-$\varphi_2$)-Lindelöf relative to $X$ (shortly, a ($\mathcal{A}$-$\varphi_2$)-L. set)).

c) We call a ($\mathcal{A}$-$\imath$)-Lindelöf set relative to $X$ a $\mathcal{A}$-L. set for short.

d) If we take $\mathcal{A} = \varphi_1 O(X)$ in (b), then we say $A$ is $\varphi_{1,2}$-Lindelöf relative to $X$ (shortly, a $\varphi_{1,2}$-L. set).

If we take $\mathcal{A} = \varphi_{1,2}O(X)$ in (c) we get the definition of a $\varphi_{1,2}O(X)$-L. set.

If $X$ is a $\varphi_{1,2}$-L. set relative to itself, then $X$ will be called a $\varphi_{1,2}$-L. space.

If $X$ is a $\varphi_{1,2}O(X)$-L. set relative to itself, then $X$ will be called a $\varphi_{1,2}O(X)$-L. space.

In the sequel it will be assumed that all operations $\varphi_i$, $i = 1, 2, \ldots$ are defined on $(X, \tau)$ whenever they are used.

Theorem 3.2. *If $X \in \mathcal{A}' \subset \mathcal{A} \subset P(X)$ and $\varphi_2 \leq \varphi_2'$ then each $(\mathcal{A}$-$\varphi_2)$-L. set is an $(\mathcal{A}'$-$\varphi_2')$-L. set. So if $\varphi_1' O(X) \subset \varphi_1 O(X)$ and $\varphi_2 \leq \varphi_2'$ then a $\varphi_{1,2}$-L. set is a $\varphi_{1,2}'$-L. set.*

Corollary 3.3. *If $\varphi_1' \leq \varphi_1$ and $\varphi_2 \leq \varphi_2'$, then each $\varphi_{1,2}$-L. set is a $\varphi_{1,2}'$-L. set.*

Remark 3.4. If $\varphi_2$ is regular w.r.t. $\varphi_1 O(X)$, then a subset $A$ of $X$ is a $\varphi_{1,2}O(X)$-L. set iff $A$ is Lindelöf relative to $X$ in the topological space $(X, \varphi_{1,2}O(X))$.

Example 3.5. Let $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, $\varphi_2 = \mathrm{scl}$, $\varphi_1' = \mathrm{int}$, $\varphi_2' = \mathrm{int} \circ \mathrm{cl}$, $\varphi_1'' = \mathrm{int}$ and $\varphi_2'' = \mathrm{cl}$. For a subset $A$, we have:

$A$ is a $\varphi_{1,2}$-L. set iff $A$ is a $(SO(X)$-scl$)$-L. set iff $A$ is an SR-L. set (see Example 3.10 below).

An SR-L. space was called rs-Lindelöf in [7].

$A$ is a $\varphi_{1,2}O(X)$-L. set iff $A$ is a $S\theta O(X)$-L. set iff $A$ is Lindelöf relative to $X$ in the supratopological space $(X, S\theta O(X))$.

Note that $\varphi_{1,2}'$-L. sets were defined for $H(P)$ spaces under the name $C_1$-closed relative to $X$ [23] and $\varphi_{1,2}'$-L. spaces were defined for $H(P)$ spaces under the name strongly $H(P)$-closed spaces [23].

$A$ is a $\varphi_{1,2}'O(X)$-L. set iff $A$ is Lindelöf relative to $X$ in $(X, \tau_s)$.

The set $A$ is a $\varphi_{1,2}''O(X)$-L. set iff $A$ is Lindelöf relative to $X$ in $(X, \tau_\theta)$.

$H(P)$-closedness was defined for $H(P)$ spaces. An $H(P)$ space is called $H(P)$-closed iff it is a $\varphi_{1,2}''$-L. space [20].

Sets which are $H(P)$-closed relative to $X$ were defined in $H(P)$ spaces in [19]. So, in an $H(P)$ space, a set $A$ is $H(P)$-closed relative to $X$ iff $A$ is a $\varphi_{1,2}''$-L. set.

A $\varphi_{1,2}''$-L. space was called weakly Lindelöf in [5].

Since $\varphi_1'' \leq \varphi_1$ and $\varphi_2 \leq \varphi_2''$, each $\varphi_{1,2}$-L. set is a $\varphi_{1,2}''$-L. set. Hence each SR-L. set is a $(\tau$-cl$)$-L. set and each rs-Lindelöf space is a weakly Lindelöf space.

Theorem 3.6. *If $\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$, and if $\varphi_1$ is monotonous, then each $\varphi_1 O(X)$-L. set is a $\varphi_{1,2}O(X)$-L. set.*

*Proof.* It can be easily seen that $\varphi_{1,2}O(X) \subset \varphi_1 O(X)$ under the given conditions [13], from which the proof is clear. ∎

THEOREM 3.7. *Let* $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$. *Then:*

a) *If* $\varphi_2(U) \in \varphi_1 O(X)$ *and* $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$ *for each* $U \in \varphi_1 O(X)$ *then* $\mathcal{B} \subset \varphi_{1,2} O(X) \cap \varphi_1 O(X)$.

b) *If* $\varphi_1 O(X) \subset \varphi_2 O(X)$ *and* $\mathcal{B} \subset \varphi_{1,2} O(X)$, *then* $\mathcal{B}$ *is a base for the supratopology* $\varphi_{1,2} O(X)$.

c) *If* $\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq \imath$, *and if* $\mathcal{B} \subset \varphi_{1,2} O(X)$, *then* $\mathcal{B}$ *is a base for the supratopology* $\varphi_{1,2} O(X)$ *[24].*

d) *If* $\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq \imath$, *and if* $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$ *for each* $U \in \varphi_1 O(X)$, *then* $\mathcal{B}$ *is a base for the supratopology* $\varphi_{1,2} O(X)$ *[24].*

*Proof.* a) Let $U \in \varphi_1 O(X)$ and $x \in \varphi_2(U)$. Then $x \in \varphi_2(U) \in \varphi_1 O(X)$ and $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$, so $x \in \varphi_{1,2} \operatorname{int} \varphi_2(U)$. We have $\varphi_2(U) \subset \varphi_{1,2} \operatorname{int} \varphi_2(U)$. Hence $\varphi_2(U) \in \varphi_{1,2} O(X)$ for each $U \in \varphi_1 O(X)$ and $\mathcal{B} \subset \varphi_{1,2} O(X) \cap \varphi_1 O(X)$.

b) Let $A \in \varphi_{1,2} O(X)$ and $x \in A$. There exists a $U \in \varphi_1 O(X, x)$ such that $\varphi_2(U) \subset A$. Hence we have $x \in U \subset \varphi_2(U) \subset A$, $\varphi_2(U) \in \varphi_{1,2} O(X)$ and $\varphi_2(U) \in \mathcal{B}$.

The proofs of (c) and (d) are clear from Lemma 2.5 and (a), (b). ■

THEOREM 3.8. *The following results are valid.*

1) *If* $A$ *is a* $\varphi_{1,2}$*-L. set, then it is a* $\varphi_{1,2} O(X)$*-L. set.*

2) *Let* $\mathcal{B} \subset P(X)$. *Then:*

a) *If* $X \in \mathcal{B} \subset \varphi_{1,2} O(X)$, *then each* $\varphi_{1,2} O(X)$*-L. set is a* $\mathcal{B}$*-L. set.*

b) *If* $\mathcal{B}$ *is a base of* $\varphi_{1,2} O(X)$ *then a set is a* $\mathcal{B}$*-L. set iff it is a* $\varphi_{1,2} O(X)$*-L. set.*

3) *Let* $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$. *Then:*

a) *If* $\varphi_1 O(X) \subset \varphi_2 O(X)$ *then each* $\mathcal{B}$*-L. set is a* $\varphi_{1,2}$*-L. set.*

b) *If* $\varphi_1 O(X) \subset \varphi_2 O(X)$ *and* $\mathcal{B} \subset \varphi_{1,2} O(X)$ *then a set is a* $\varphi_{1,2}$*-L. set iff it is a* $\varphi_{1,2} O(X)$*-L. set iff it is a* $\mathcal{B}$*-L. set.*

c) *If* $\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq \imath$, *and if* $\mathcal{B} \subset \varphi_{1,2} O(X)$, *then a set is a* $\mathcal{B}$*-L. set iff it is a* $\varphi_{1,2} O(X)$*-L. set iff it is a* $\varphi_{1,2}$*-L. set.*

d) *If* $\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq \imath$, *and if* $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$ *for each* $U \in \varphi_1 O(X)$, *then a set is a* $\mathcal{B}$*-L. set iff it is a* $\varphi_{1,2} O(X)$*-L. set iff it is a* $\varphi_{1,2}$*-L. set.*

THEOREM 3.9. *If* $\varphi_2(U) = \varphi_3(U)$ *for each* $U \in \varphi_1 O(X)$, *then for a subset* $A$, *we have:*

a) $A$ *is a* $\varphi_{1,2}$*-L. set iff it is a* $\varphi_{1,3}$*-L. set.*

b) $A$ *is a* $\varphi_{1,2} O(X)$*-L. set iff it is a* $\varphi_{1,3} O(X)$*-L. set.*

EXAMPLE 3.10. a) Let $\varphi_1 = \operatorname{cl} \circ \operatorname{int}$, $\varphi_2 = \operatorname{scl}$. Then:

$\varphi_1 O(X) = SO(X)$ and $\varphi_{1,2} O(X) = S\theta O(X)$. For $U \in \varphi_1 O(X) = SO(X)$ we have $\varphi_2(U) = \operatorname{scl} U \in SR(X) \subset SO(X)$. Also, $\varphi_2(\varphi_2(U)) = \operatorname{scl}(\operatorname{scl} U) =$

$\operatorname{scl} U = \varphi_2(U)$ and $\varphi_2 \geq \imath$. Thus, $\mathcal{B} = \{\operatorname{scl} U : U \in \varphi_1 O(X)\} = SR(X)$, whence $\mathcal{B} = SR(X)$ is a base for the supratopology $\varphi_{1,2}O(X) = S\theta O(X)$.

So, a set is a SR-L. set iff it is a $(SO(X)$-scl)-L. set iff it is a $S\theta O(X)$-L. set.

b) Let $\varphi_1 = \operatorname{int}$, $\varphi_2 = \operatorname{int} \circ \operatorname{cl}$, $\varphi_3 = \operatorname{scl}$. Then:

$\varphi_1 O(X) = \tau$, $\varphi_{1,2}O(X) = \varphi_{1,3}O(X) = \tau_s$, $\varphi_2 \geq \varphi_1$ and $\varphi_3 \geq \varphi_1$. For each $U \in \varphi_1 O(X) = \tau$ we have $\varphi_2(U) = \overline{U}^o = U \cup \overline{U}^o = \operatorname{scl} U$ [3], and $\varphi_3(U) = \varphi_2(U) \in \varphi_1 O(X) = \tau$, $\varphi_2(\varphi_2(U)) = \overline{(\overline{U}^o)}^o = \overline{U}^o = \varphi_2(U)$. Then, $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = RO(X)$ and $RO(X)$ is a base for $\tau_s$.

So, a set is a $RO(X)$-L. set iff it is a ($\tau$-int$\circ$cl)-L. set iff it is a $\tau_s$-L. set iff a ($\tau$-scl)-L. set.

Remark 3.11. a) For $\varphi_2 = \imath$, the $\varphi_{1,2}$-L. property relative to $X$ coincides with the $\varphi_1 O(X)$-L. property relative to $X$.

b) If $\varphi_1$ is monotonous and $\varphi_2 = \imath$, then for $A \subset X$ we have:

The set $A$ is a $\varphi_{1,2}$-L. set iff $A$ is a $\varphi_1 O(X)$-L. set iff $A$ is a $\varphi_{1,2}O(X)$-L. set.

*Proof.* a) Clear.

b) If $\varphi_1$ is monotonous then $\varphi_1 O(X)$ is a supratopology and for $\varphi_2 = \imath$ we have $\varphi_1 O(X) = \varphi_{1,2}O(X)$, from which the result follows easily. ■

Theorem 3.12. *Take $X \in \mathcal{A} \subset P(X)$, $A \subset X$ and let $\mathcal{K} = \{X \setminus U : U \in \mathcal{A}\}$. Then the following are equivalent:*

a) *$A$ is an $\mathcal{A}$-L. set.*

b) *If $\mathcal{W}$ is any subfamily of $\mathcal{K}$ such that for each countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ we have $A \cap (\bigcap \mathcal{W}') \neq \emptyset$, then $A \cap (\bigcap \mathcal{W}) \neq \emptyset$.*

c) *If $\mathcal{W}$ is any subfamily of $\mathcal{K}$ with $A \cap (\bigcap \mathcal{W}) = \emptyset$ then there exists a countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ such that $A \cap (\bigcap \mathcal{W}') = \emptyset$.*

Theorem 3.13. *If $\emptyset$, $X \in \mathcal{A} \subset P(X)$, then $X$ is an $\mathcal{A}$-L. space iff for each $U \in \mathcal{A}$, $X \setminus U$ is a $\mathcal{A}$-L. set.*

If we take $\mathcal{A} = \varphi_{1,2}O(X)$ in Theorem 3.12 and Theorem 3.13, then $\mathcal{K} = \varphi_{1,2}C(X)$ and we obtain equivalent conditions for a set to be a $\varphi_{1,2}O(X)$-L. set, and equivalent conditions for the space $X$ to be a $\varphi_{1,2}O(X)$-L. space.

Theorem 3.14. *If $\widetilde{\varphi_2}$ is the dual of $\varphi_2$ then the following are equivalent.*

a) *$A$ is a $\varphi_{1,2}$-L. set.*

b) *For any family $\Phi$ of $\varphi_1$-closed sets with $A \cap (\bigcap \Phi) = \emptyset$, there exists a countable subfamily $\Phi'$ of $\Phi$ such that $A \cap (\bigcap \{\widetilde{\varphi_2}(F) : F \in \Phi'\}) = \emptyset$.*

c) *If $\Phi$ is a family of $\varphi_1$-closed sets such that for each countable subfamily $\Phi'$ of $\Phi$ we have $A \cap (\bigcap \{\widetilde{\varphi_2}(F) : F \in \Phi'\}) \neq \emptyset$, then $A \cap (\cap \Phi) \neq \emptyset$.*

Theorem 3.15. *Take $A \subset X$ and $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$. If $\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$, and if $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$ for each $U \in \varphi_1 O(X)$, then the following are equivalent:*

a) *$A$ is a $\varphi_{1,2}$-L. set.*

b) *$A$ is a $\mathcal{B}$-L. set.*

c) *$A$ is a $\varphi_{1,2}O(X)$-L. set.*

d) *If $\mathcal{W}$ is any subfamily of $\{X \setminus \varphi_2(U) : U \in \varphi_1 O(X)\}$ such that for each countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ we have $A \cap (\bigcap \mathcal{W}') \neq \emptyset$ then $A \cap (\bigcap \mathcal{W}) \neq \emptyset$.*

e) *If $\mathcal{W}$ is any subfamily of $\{X \setminus \varphi_2(U) : U \in \varphi_1 O(X)\}$ with $A \cap (\bigcap \mathcal{W}) = \emptyset$ then there exists a countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ such that $A \cap (\bigcap \mathcal{W}') = \emptyset$.*

f) *If $\mathcal{W}$ is any subfamily of $\{X \setminus U : U \in \varphi_{1,2}O(X)\}$ such that for each countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ we have $A \cap (\bigcap \mathcal{W}') \neq \emptyset$, then $A \cap (\cap \mathcal{W}) \neq \emptyset$.*

g) *If $\mathcal{W}$ is any subfamily of $\{X \setminus U : U \in \varphi_{1,2}O(X)\}$ such that $A \cap (\bigcap \mathcal{W}) = \emptyset$, then there exists a countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ such that $A \cap (\bigcap \mathcal{W}') = \emptyset$.*

*If $\widetilde{\varphi_2}$ is the dual of $\varphi_2$ then the following statements (h) and (i) are equivalent to each one of the above statements.*

h) *For any family of $\Phi$ of $\varphi_1$-closed sets with $A \cap (\bigcap \Phi) = \emptyset$, there exists a countable subfamily $\Phi'$ of $\Phi$ such that $A \cap (\bigcap \{\widetilde{\varphi_2}(F) : F \in \Phi'\}) = \emptyset$.*

i) *If $\Phi$ is any family of $\varphi_1$-closed sets such that for each countable subfamily $\Phi'$ of $\Phi$ we have $A \cap (\bigcap \{\widetilde{\varphi_2}(F) : F \in \Phi'\}) \neq \emptyset$, then $A \cap (\bigcap \Phi) \neq \emptyset$.*

Now, using the equality $\varphi_{1,2}C(X) = \{X \setminus U : U \in \varphi_{1,2}O(X)\}$ and the Theorems 3.13 and 3.15 we obtain the following theorem.

Theorem 3.16. *Under the hypotheses of Theorem 3.15 the following are equivalent.*

a) *$X$ is a $\varphi_{1,2}$-L. space.*

b) *$X$ is a $\mathcal{B}$-L. space.*

c) *$X$ is a $\varphi_{1,2}O(X)$-L. space.*

d) *For each $U \in \varphi_1 O(X)$, $X \setminus \varphi_2(U)$ is a $\varphi_{1,2}$-L. set.*

e) *For each $U \in \varphi_1 O(X)$, $X \setminus \varphi_2(U)$ is a $\varphi_{1,2}O(X)$-L. set.*

f) *For each $U \in \varphi_1 O(X)$, $X \setminus \varphi_2(U)$ is a $\mathcal{B}$-L. set.*

g) *Each $\varphi_{1,2}$-closed set is a $\varphi_{1,2}$-L. set.*

h) *Each $\varphi_{1,2}$-closed set is a $\varphi_{1,2}O(X)$-L. set.*

i) *Each $\varphi_{1,2}$-closed set is a $\mathcal{B}$-L. set.*

*Other equivalent expressions may be obtained by taking $X$ in place of $A$ in* (d–i) *of Theorem 3.15.*

Example 3.17. Let $\varphi_1 = \text{semi-int}$, $\varphi_2 = \text{scl}$. Then:

The conditions of Theorem 3.15 are satisfied and $\varphi_2$ is the dual of $\varphi_1$.

$\varphi_1 O(X) = SO(X)$, $\varphi_1 C(X) = SC(X)$, $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = \{\text{scl}\, U : U \in SO(X)\} = SR(X)$.

$\{X \setminus \varphi_2(U) : U \in \varphi_1 O(X)\} = SR(X)$. $\varphi_{1,2}O(X) = S\theta O(X)$ and $\varphi_{1,2}C(X) = S\theta C(X)$.

Now we obtain equivalent conditions for a set to be a ($SO(X)$-scl)-L. set by using Theorem 3.15, and equivalent conditions for the space $(X, \tau)$ to be a ($SO(X)$-scl)-L. space (equivalently, SR-Lindelöf) by using Theorem 3.16.

THEOREM 3.18. *Let $(X, \tau)$ be a topological space and $A \subset X$. Then the following are equivalent.*

a) *For each $SO(X)$-cover $\mathcal{U}$ of $A$, there exists a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $A \subset \cup\{\operatorname{scl} U : U \in \mathcal{U}'\}$.*

b) *For each $SR(X)$-cover $\mathcal{U}$ of $A$ there exists a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $A \subset \bigcup \mathcal{U}'$.*

c) *For each $S\theta O(X)$-cover $\mathcal{U}$ of $A$, there exists a countable subfamily $\mathcal{U}'$ of $A$ such that $A \subset \bigcup \mathcal{U}'$.*

d) *If $\mathcal{W}$ is a subfamily of $SR(X)$ such that for each countable subfamily $\mathcal{W}'$ of $\mathcal{W}$, we have $A \cap (\bigcap \mathcal{W}') \neq \emptyset$, then $A \cap (\bigcap \mathcal{W}) \neq \emptyset$.*

e) *If $\mathcal{W}$ is any subfamily of $SR(X)$ such that $A \cap (\bigcap \mathcal{W}) = \emptyset$, then there exists a countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ such that $A \cap (\bigcap \mathcal{W}') = \emptyset$.*

f) *If $\mathcal{W}$ is a subfamily of $S\theta C(X)$ such that for each countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ we have $A \cap (\bigcap \mathcal{W}') \neq \emptyset$, then $A \cap (\bigcap \mathcal{W}) \neq \emptyset$.*

g) *If $\mathcal{W}$ is any subfamily of $S\theta C(X)$ with $A \cap (\bigcap \mathcal{W}) = \emptyset$ then there exists a countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ such that $A \cap (\bigcap \mathcal{W}') = \emptyset$.*

h) *For any subfamily $\Phi$ of semi-closed sets with $A \cap (\bigcap \Phi) = \emptyset$ there exists a countable subfamily $\Phi'$ of $\Phi$ such that $A \cap (\bigcap \{\textit{semi-}\operatorname{int}(F) : F \in \Phi'\}) = \emptyset$.*

i) *If $\Phi$ is any family of semi-closed sets such that for each countable subfamily $\Phi'$ of $\Phi$, we have $A \cap (\bigcap \{\textit{semi-}\operatorname{int}(F) : F \in \Phi'\}) \neq \emptyset$, then $A \cap (\bigcap \Phi) \neq \emptyset$.*

*For a topological space $(X, \tau)$ to be rs-Lindelöf [7] (equivalently, a ($SO(X)$-scl)-L. space), we can add the following equivalent statements to those given in Theorem 3.18, by taking $X$ instead of $A$ there.*

j) *Each semi-regular set is a $SR(X)$-L. set.*

k) *Each semi-regular set is a $S\theta O(X)$-L. set.*

l) *Each semi-regular set is a ($SO(X)$-scl)-L. set.*

m) *Each $\varphi_{1,2}$-closed set is a $SR(X)$-L. set.*

n) *Each $\varphi_{1,2}$-closed set is a $S\theta O(X)$-L. set.*

o) *Each $\varphi_{1,2}$-closed set is a ($SO(X)$-scl)-L. set.*

EXAMPLE 3.19. a) Let $\varphi_1 = \operatorname{cl} \circ \operatorname{int}$, $\varphi_2 = \operatorname{cl}$. Then:

The conditions of Theorem 3.15 are satisfied. $\widetilde{\varphi_2} = \operatorname{int}$ is the dual of $\varphi_2$. $\varphi_1 O(X) = SO(X)$ and $\varphi_1 C(X) = SC(X)$.

$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = \{\operatorname{cl} U : U \in SO(X)\} = RC(X)$.

$\varphi_{1,2} O(X) = \theta SO(X)$ and $\varphi_{1,2} C(X) = \theta SC(X) = \{X \setminus U : U \in \varphi_{1,2} O(X)\}$. $\{X \setminus B : B \in \mathcal{B}\} = RO(X)$.

A set $A$ is a $\varphi_{1,2}$-L. set iff it is a $(SO(X)$-cl$)$-L. set iff it is a $\theta SO(X)$-L. set iff it is $RC(X)$-L. set.

Hence we obtain, $(X,\tau)$ is a $\varphi_{1,2}$-L. space iff $X$ is rc-Lindelöf (rc-Lindelöf spaces were defined by D. Janković and Ch. Konstadilaki, as mentioned in [5]).

Again, we can get equivalent conditions for a set to be a $(SO(X)$-cl$)$-L. set in a topological space $(X,\tau)$, and equivalent conditions for a space to be rc-Lindelöf by using Theorems 3.15 and 3.16.

b) Let $\varphi_1 = \operatorname{int}$, $\varphi_2 = \operatorname{int}\circ\operatorname{cl}$ on $(X,\tau)$. Then:

The conditions of Theorem 3.15 are satisfied and $\widetilde{\varphi_2} = \operatorname{cl}\circ\operatorname{int}$.

$\varphi_1 O(X) = \tau$, $\varphi_1 C(X) =$ family of closed sets. $\varphi_{1,2}O(X) = \tau_s =$ family of $\delta$-open sets, $\{X\setminus U : U \in \varphi_{1,2}O(X)\} = \varphi_{1,2}C(X) =$ family of $\tau_s$-closed sets $=$ family of $\delta$-closed sets.

$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = \{\overline{U}^o : U \in \tau\} = RO(X)$. $\{X\setminus B : B \in \mathcal{B}\} = \{X\setminus \varphi_2(U) : U \in \varphi_1 O(X)\} = RC(X)$.

Now, using Theorems 3.15 and 3.16 we can obtain equivalent conditions for a set to be a $(\tau$-$\operatorname{int}\circ\operatorname{cl})$-L. set, and equivalent conditions for a space to be a $(\tau$-$\operatorname{int}\circ\operatorname{cl})$-L. space. Also, in $H(P)$ spaces, equivalent conditions for a set to be $C_1$-closed relative to the space and equivalent conditions for a space to be strongly $H(P)$-closed.

Some equalities related to different types of closure can be obtained using operations. Some of them were given in [24] and some of them are given below. In a topological space $(X,\tau)$ we have:

1) $\overline{T} = \theta\text{-}clT = \tau_s\text{-}\operatorname{cl}T$, for each $T \in \tau$.

2) $\theta$-semi-$\operatorname{cl}T = \operatorname{scl}T =$ semi-$\theta$-$\operatorname{cl}T$, for each $T \in \tau$.

3) $\overline{T} = \tau_s\text{-}\operatorname{cl}T = \tau^\alpha\text{-}\operatorname{cl}T$, for each $T \in \tau^\alpha$, $(\tau^\alpha = \{U : U \subset X, U \subset \overline{U^o}^o\})$.

4) semi-$\theta$-$\operatorname{cl}T = \operatorname{scl}T$, for each $T \in \tau^\alpha$.

Example 3.20. Let $\varphi_1 = \operatorname{int}$, $\varphi_2 = \tau$-cl, $\varphi_3 = \theta$-cl, $\varphi_4 = \tau_s$-cl. Then:

$\varphi_1 O(X) = \tau$. For each $U \in \tau$ we have $\varphi_2(U) = \varphi_3(U) = \varphi_4(U)$.

$\widetilde{\varphi_2} = \tau$-int, for each $\varphi_1$-closed set $K$, $\operatorname{int}K = \theta\text{-}\operatorname{int}K = \tau_s\text{-}\operatorname{int}K$.

$\varphi_{1,2}O(X) = \varphi_{1,3}O(X) = \varphi_{1,4}O(X) = \tau_\theta$.

$\{X\setminus U : U \in \varphi_{1,2}O(X)\} = \varphi_{1,2}C(X) = \varphi_{1,3}C(X) = \varphi_{1,4}C(X) =$ family of $\theta$-closed sets.

Now, by using Theorems 3.9, 3.12, 3.13, 3.14 and 3.16 we can add some more equivalent conditions for a set to be a $\varphi_{1,2}$-L. set ($\varphi_{1,2}O(X)$-L. set), and for a space $(X,\tau)$ to be a $\varphi_{1,2}$-L. space ($\varphi_{1,2}O(X)$-L. space) for some special choices of the operations $\varphi_1$ and $\varphi_2$.

Lemma 3.21. ([25]) *For each $U \in \varphi_1 O(X)$ we have $U \subset \varphi_{1,2}\operatorname{int}\varphi_2(U)$.*

Theorem 3.22. *The following are equivalent for any subset $A$ of $X$.*

a) *$A$ is a $\varphi_{1,2}$-L. set.*

b) *Each $\sigma$-filter-base in $A$ $\varphi_{1,2}$-accumulates in $X$ to some point in $A$.*

c) *Every $\sigma$-filter-base in $X$ which meets $A$ $\varphi_{1,2}$-accumulates in $X$ to some point of $A$.*

d) *For any family of non-empty sets $\mathcal{W}$ with $A \cap (\bigcap\{\varphi_{1,2} \operatorname{cl} F : F \in \mathcal{W}\}) = \emptyset$, there exists a countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ such that $A \cap (\bigcap\{F : F \in \mathcal{W}'\}) = \emptyset$.*

e) *For any family $\mathcal{W}$ of non-empty sets such that for each countable subfamily $\mathcal{W}'$ of $\mathcal{W}$ we have $A \cap (\bigcap\{F : F \in \mathcal{W}'\}) \neq \emptyset$, it holds that $A \cap (\bigcap\{\varphi_{1,2} \operatorname{cl} F : F \in \mathcal{W}\}) \neq \emptyset$.*

f) *If $\mathcal{F}$ is a $\sigma$-filter-base such that $A \cap (\bigcap\{\varphi_{1,2} \operatorname{cl} F : F \in \mathcal{F}\}) = \emptyset$, then there exists an $F \in \mathcal{F}$ such that $F \cap A = \emptyset$.*

*Proof.* $(a \Rightarrow b)$. Let $A$ be a $\varphi_{1,2}$-L. set. Let us accept that some $\sigma$-filter-base $\mathcal{F}_b$ in $A$ does not accumulate to any point in $A$.

Then, for each $a \in A$ there exists a $U_a \in \varphi_1 O(X, a)$ and an $F_a \in \mathcal{F}_b$ such that $F_a \cap \varphi_2(U_a) = \emptyset$. Since $A$ is a $\varphi_{1,2}$-L. set there is a countable subset $C$ of $A$ such that $A \subset \bigcup_{a\in C} \varphi_2(U_a)$. Hence $(\bigcap_{a\in C} F_a) \cap (\bigcup_{a\in C} \varphi_2(U_a)) = \emptyset$. Hence there exists an $F \in \mathcal{F}_b$ such that $F \subset \bigcap_{a\in C} F_a$. Thus $F \cap A = \emptyset$. This contradiction completes the proof.

$(b \Rightarrow c)$. If $\mathcal{F}_b$ is a $\sigma$-filter-base in $X$ which meets $A$, then $\mathcal{F}'_b = \{A \cap F : F \in \mathcal{F}_b\}$ is a $\sigma$-filter-base in $A$ and hence in $X$.

The $\sigma$-filter $\mathcal{F}'$ generated by $\mathcal{F}'_b$ contains $A$ and $\mathcal{F}_b \subset \mathcal{F}'$. From (a), $\mathcal{F}'_b$ $\varphi_{1,2}$-accumulates to some point $a$ in $A$. So $\mathcal{F}_b$ $\varphi_{1,2}$-accumulates to $a$.

$(c \Rightarrow d)$. Let $\{F_i : i \in I\}$ be a family of non-empty sets with $(\cap\{\varphi_{1,2} \operatorname{cl} F_i : i \in I\}) \cap A = \emptyset$. Let us accept that for each countable subset $J$ of $I$ we have $(\bigcap_{j\in J} F_j) \cap A \neq \emptyset$. Then $\mathcal{F}_b = \{\bigcap_{j\in J} F_j : J \subset I, J \text{ countable }\}$ is a $\sigma$-filter-base in $X$ which meets $A$. Hence there exists a point $a$ in $A$ such that $\mathcal{F}_b$ $\varphi_{1,2}$-accumulates to $a$.

For each $U \in \varphi_1 O(X, a)$ and for each $F \in \mathcal{F}_b$, $F \cap \varphi_2(U) \neq \emptyset$. So for each $U \in \varphi_1 O(X, a)$ and for each $i \in I$, $F_i \cap \varphi_2(U) \neq \emptyset$. This gives $a \in (\bigcap_{i\in I} \varphi_{1,2} \operatorname{cl} F_i) \cap A \neq \emptyset$. This contradiction completes the proof.

$(d \Rightarrow a)$. Let $A \subset \cup\mathcal{U}$ where $\mathcal{U} \subset \varphi_1 O(X)$. Then $(\bigcap\{X \setminus U : U \in \mathcal{U}\}) \cap A = \emptyset$.

Let $\mathcal{K} = \{X \setminus \varphi_2(U) : U \in \mathcal{U}\}$. If there exists a $U \in \mathcal{U}$ such that $X \setminus \varphi_2(U) = \emptyset$, then $A \subset \varphi_2(U) = X$. On the other hand, if for each $U \in \mathcal{U}$, $X \setminus \varphi_2(U) \neq \emptyset$ let us show that $(\bigcap\{\varphi_{1,2} \operatorname{cl}(X \setminus \varphi_2(U)) : U \in \mathcal{U}\}) \cap A = \emptyset$.

Firstly, let us see that for each $U \in \mathcal{U}$ we have $\varphi_{1,2} \operatorname{cl}(X \setminus \varphi_2(U)) \subset X \setminus U$. Let $U \in \mathcal{U}$ and $x \in \varphi_{1,2} \operatorname{cl}(X \setminus \varphi_2(U)$. From Lemma 3.20, $U \subset \varphi_{1,2} \operatorname{int} \varphi_2(U)$. Now $x \in \varphi_{1,2} \operatorname{cl}(X \setminus \varphi_2(U)) = (X \setminus \varphi_{1,2} \operatorname{int} \varphi_2(U)) \subset X \setminus U$. Hence

$$(\bigcap\{\varphi_{1,2} \operatorname{cl}(X \setminus \varphi_2(U) : U \in \mathcal{U}\}) \cap A \subset (\bigcap\{X \setminus U : U \in \mathcal{U}\}) \cap A = \emptyset.$$

From (d) there exists a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $(\bigcap\{X \setminus \varphi_2(U) : U \in \mathcal{U}'\}) \cap A = \emptyset$. Hence $A \subset \cup\{\varphi_2(U) : U \in \mathcal{U}'\}$, so $A$ is a $\varphi_{1,2}$-L. set.

$d \Leftrightarrow e$, $d \Rightarrow f$ and $f \Rightarrow c$ are clear. ∎

THEOREM 3.23. *If* $(X, \tau)$ *is* $\varphi_{1,2}$-$T_2$, $\varphi_1 O(X)$ *is closed under countable intersections and* $\varphi_2$ *is monotonous then each* $\varphi_{1,2}$*-L. set is* $\varphi_{1,2}$*-closed.*

*Proof.* Let $x \in \varphi_{1,2} \operatorname{cl} A$. Then there exists a $\sigma$-filter $\mathcal{F}$ such that $A \in \mathcal{F}$ and $\mathcal{F}$ $\varphi_{1,2}$-converges to $x$. Since $\mathcal{F}$ meets $A$ and $A$ is a $\varphi_{1,2}$-L. set there exists a point $a$ in $A$ such that $\mathcal{F}$ $\varphi_{1,2}$-accumulates to $a$.

From Theorem 2.3 we must have $a = x$, so $x \in A$ since we have $\varphi_{1,2} \operatorname{cl} A \subset A$, and $A$ is a $\varphi_{1,2}$-closed set. ■

EXAMPLE 3.24. Let $\varphi_1 = \operatorname{int}$ and $\varphi_2 \in O(X, \tau)$ be a monotonous operation. Then:

$\varphi_1 O(X) = \tau$. $\varphi_1 O(X)$ is closed under countable intersections iff $(X, \tau)$ is a P space.

So, using $\sigma$-filters in P spaces we can obtain equivalent characterizations for a $\varphi_{1,2}$-L. set.

If $(X, \tau)$ is an $H(P)$-space, then by choosing $\varphi_1 = \operatorname{int}$ and $\varphi_2 = \operatorname{cl}$, we get equivalent conditions for a space to be $H(P)$-closed.

Proceeding in this way we can obtain many known results, some of which occur in [4,7,20,22], and also many new results.

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