МАТЕМАТИЧКИ ВЕСНИК
56 (2004), 105–114

UDK 512.552
оригинални научни рад
research paper

THE REPRESENTATIONS OF FINITE REFLECTION GROUPS

Muhittin Başer and Sait Halıcıoğlu

Abstract. The construction of all irreducible modules of the symmetric groups over an arbitrary field which reduce to Specht modules in the case of fields of characteristic zero is given by G. D. James. Halıcıoğlu and Morris describe a possible extension of James' work for Weyl groups in general, where Young tableaux are interpreted in terms of root systems. In this paper, we further develop the theory and give a possible extension of this construction for finite reflection groups which cover the Weyl groups.

1. Introduction

The representation theory of symmetric groups over fields of characteristic zero is well developed and documented with a number of books devoted to the subject. The original approach was due to G. Frobenius and I. Schur followed independently by A.Young in a long series of difficult but highly influential papers. Later, in the 1930's, W. Specht presented an alternative approach which led in an elegant way to a full set of irreducible modules, now called Specht modules. I. G. Macdonald showed [12] how to obtain irreducible modules for a Weyl groups by a construction using subsystems of the root system of the Weyl groups. Macdonald's method gives many, but in general not all, of the irreducible modules. In 1976, G. D. James in a very important paper [11], gave an easy and ingenious construction of all the irreducible modules of the symmetric groups over an arbitrary field which reduce to Specht modules in the case of fields of characteristic zero. Al-Aamily, Morris and Peel [1] showed how this construction could be extended to deal with the Weyl groups of type B_n. In [13], A. O. Morris described a possible extension of James' work for Weyl groups in general.

Later, the second author and Morris [7] gave an alternative generalisation of James' work which is an improvement and extension of the original approach suggested by Morris. In [8], L. Hawkins extended Macdonald's construction to the case where the subsystem of the roots is replaced by a parabolic subset. Although the conjugacy classes and irreducible characters are known for all finite reflection groups individually no unified approach has been obtained. We now give a possible

AMS Subject Classification: 20 C 33, 20 F 55, 22 E 45
Keywords and phrases: Specht module, tableau, tabloid, finite reflection group.

extension of James' work for finite reflection groups which is a generalization of the original approach suggested by Halıcıoğlu [6].

2. Preliminaries

In this section, we establish the notation and state some results on finite reflection groups which are required later. The basic definitions and background material required here may be found in N. Bourbaki [2], R. W. Carter [3], J. E. Humphreys [9] [10], Grove and Benson [5], Halıcıoğlu and Morris [7].

Let V be l-dimensional Euclidean space over the real field equipped with a positive definite inner product $(\ ,\ )$. For $\alpha \in V$, $\alpha \neq 0$, let τ_α be the *reflection* in the hyperplane orthogonal to α, that is, τ_α is the linear transformation on V defined by

$$\tau_\alpha \colon V \longrightarrow V, \qquad v \mapsto \tau_\alpha(v) = v - 2\,\frac{(\alpha, v)}{(\alpha, \alpha)}\,\alpha$$

for all $v \in V$. Let Φ be a root system in V and π be a simple system in Φ with corresponding positive system Φ^+ and negative system Φ^-. Then the finite reflection group

$$\mathcal{W} = \mathcal{W}(\Phi) = \langle \tau_\alpha \mid \tau_\alpha^2 \ = \ (\tau_\alpha \tau_\beta)^{m_{\alpha\beta}} = e,\ \alpha, \beta \in \pi \text{ and } \alpha \neq \beta \rangle$$

where e is the identity element of $\mathcal{W}$ and $m_{\alpha\beta}$ is the order of $\tau_\alpha \tau_\beta$. Let $l(w)$ denote the *length* of w and the *sign* of w, $s(w)$, is defined by $s(w) = (-1)^{l(w)}$, $w \in \mathcal{W}$.

2.1 To each root system Φ, there corresponds a graph Γ called the *Coxeter graph* (or *Dynkin diagram*) of $\mathcal{W}$, whose nodes are in one-to-one correspondence with the elements of π. A finite reflection group is *irreducible* if its Coxeter graph is connected. Finite irreducible reflection groups have been classified and correspond to root systems of type A_l $(l \geq 1)$, B_l $(l \geq 2)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$ $(p = 5$ or $p \geq 7)$. For example $\mathcal{W}(A_l) \cong S_{l+1}$ the symmetric group on the set $\{1, 2, \ldots, l+1\}$ and $\mathcal{W}(G_2) \cong D_6$ dihedral group of order 12.

2.2 A *subsystem* Ψ of Φ is a subset of Φ which is itself a root system in the space which it spans. The *finite reflection subgroup* $\mathcal{W}(\Psi)$ of $\mathcal{W}$ corresponding to a subsystem Ψ is the subgroup of $\mathcal{W}$ generated by the τ_α, $\alpha \in \Psi$. The subsystems Ψ_1 and Ψ_2 are *conjugate under* $\mathcal{W}$ if $\Psi_1 = w\Psi_2$ for some $w \in \mathcal{W}$.

2.3 The graphs which are Dynkin diagrams of subsystems of Φ may be obtained up to conjugacy by a standard algorithm due independently to E. B. Dynkin, A. Borel and J. de Siebenthal (see e.g. [4]).

2.4 The simple system J of Ψ can always be chosen such that $J \subset \Phi^+$ [14].

2.5 The set $D_\Psi = \{\, w \in \mathcal{W} \mid w(j) \in \Phi^+ \text{ for all } j \in J \,\}$ is a *distinguished set of coset representatives* of $\mathcal{W}(\Psi)$ in $\mathcal{W}$, that is, each element $w \in \mathcal{W}$ has unique expression of the form $d_\Psi w_\Psi$, where $d_\Psi \in D_\Psi$ and $w_\Psi \in \mathcal{W}(\Psi)$. Furthermore d_Ψ is the element of minimal length in the coset $d_\Psi \mathcal{W}(\Psi)$ [3].

2.6 If Ψ is a subsystem of Φ with simple system $J \subset \Phi^+$ and Dynkin diagram Δ then let $\Psi = \bigcup_{i=1}^{r} \Psi_i$, where Ψ_i are the indecomposable components of Ψ. Let J_i be a simple system in Ψ_i $(i = 1, 2, \ldots, r)$ and $J = \bigcup_{i=1}^{r} J_i$. Let $\Psi^{\perp}$ be the largest subsystem in Φ orthogonal to Ψ and let $J^{\perp} \subset \Phi^+$ the simple system of $\Psi^{\perp}$. Let $\Psi^{'}$ be a subsystem of Φ which is contained in $\Phi \setminus \Psi$, with simple system $J^{'} \subset \Phi^+$ and Dynkin diagram $\Delta^{'}$, $\Psi^{'} = \bigcup_{i=1}^{s} \Psi_i^{'}$, where $\Psi_i^{'}$ are the indecomposable components of $\Psi^{'}$, then let $J_i^{'}$ be a simple system in $\Psi_i^{'}$ $(i = 1, 2, \ldots, s)$ and $J^{'} = \bigcup_{i=1}^{s} J_i^{'}$. Let $\Psi^{'\perp}$ be the largest subsystem in Φ orthogonal to $\Psi^{'}$ and let $J^{'\perp} \subset \Phi^+$ the simple system of $\Psi^{'\perp}$. If $\bar{J}$ stand for the ordered set $\{J_1, J_2, \ldots, J_r; J_1^{'}, J_2^{'}, \ldots, J_s^{'}\}$, where in addition the elements in each J_i and $J_i^{'}$ are ordered, then let $\mathcal{T}_\Delta = \{w\bar{J} \mid w \in \mathcal{W}\}$. The pair $\bar{J} = \{J; J^{'}\}$ is called a *useful system* in Φ if $\mathcal{W}(J) \cap \mathcal{W}(J^{'}) = \langle e \rangle$ and $\mathcal{W}(J^{\perp}) \cap \mathcal{W}(J^{'\perp}) = \langle e \rangle$. The elements of $\mathcal{T}_\Delta$ are called Δ-*tableaux*, the J and $J^{'}$ are called the *rows* and the *columns* of $\{J; J^{'}\}$ respectively. Two Δ-tableaux $\bar{J}$ and $\bar{K}$ are $row-equivalent$, written $\bar{J} \sim \bar{K}$, if there exists $w \in \mathcal{W}(J)$ such that $\bar{K} = w\bar{J}$. The equivalence class which contains the Δ-tableau $\bar{J}$ is $\{\bar{J}\}$ and is called a Δ-*tabloid*. Let τ_Δ be the set of all Δ-tabloids. Then $\tau_\Delta = \{\{\ d\bar{J}\ \} \mid d \in D_\Psi\}$. The group $\mathcal{W}$ acts on τ_Δ as $\sigma\{\overline{w\bar{J}}\} = \{\overline{\sigma w\bar{J}}\}$ for all $\sigma \in \mathcal{W}$. Let K be arbitrary field and M^{Δ} be the K-space whose basis elements are the Δ-tabloids. Extend the action of $\mathcal{W}$ on τ_Δ linearly on M^{Δ}, then M^{Δ} becomes a $K\mathcal{W}$-module. Let

$$\kappa_{J'} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\sigma \text{ and } e_{J,J'} = \kappa_{J'}\{\bar{J}\}$$

where s is the sign function. Then $e_{J,J'}$ is called the generalized Δ-*polytabloid* associated with J. Let $S^{J,J^{'}}$ be the subspace of M^{Δ} generated by $e_{wJ,wJ'}$ where $w \in \mathcal{W}$. Then $S^{J,J^{'}}$ is called a *generalized Specht module*. A useful system $\{J; J^{'}\}$ in Φ is called a *good system* if $d\ \Psi \cap \Psi^{'} = \emptyset$ for $d \in D_\Psi$ then $\{\ \overline{d\bar{J}}\ \}$ appears with non-zero coefficient in $e_{J,J'}$. If $\{J; J^{'}\}$ is a good system, then $S^{J,J^{'}}$ is irreducible [7].

3. Specht modules for finite reflection groups

In this section we show how to construct irreducible modules for finite reflection groups.

By [7], if Ψ_1 and Ψ_2 are $\mathcal{W}$-conjugate subsystems of Φ, then the corresponding Specht modules S^{Δ_1} and S^{Δ_2} are isomorphic. Hence, it is important to choose a representative from the set of $\mathcal{W}$-conjugate subsystems. We now give a natural method to choose the representative.

Let Φ be a root system with simple system $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\alpha, \beta \in \Phi$ such that $\alpha = \sum_{i=1}^{n} a_i\alpha_i$ and $\beta = \sum_{i=1}^{n} b_i\alpha_i$,

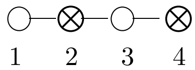
"$\alpha \prec \beta$ if and only if for some i, $a_j = b_j$ for $j < i$ and $a_i < b_i$".

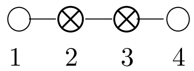
Clearly this is a total ordering on Φ. If Ψ_1 and Ψ_2 are $\mathcal{W}$-conjugate subsystems of Φ with simple systems $J_1 = \{v_1, v_2, \ldots, v_l\}$ and $J_2 = \{u_1, u_2, \ldots, u_l\}$, respectively, then

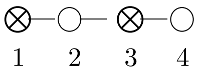
"$\Psi_1 \lhd \Psi_2$ if and only if for some i, $v_j = u_j$ for $j < i$ and $v_i \prec u_i$".

It is also easy to see that this is a total ordering on the set of $\mathcal{W}$-conjugate subsystems. From now on, we consider the maximum subsystem according to the order $\lhd$ as a representative of $\mathcal{W}$-conjugate subsystems. In the case A_l, maximum subsystems are $A_{l_1} + A_{l_2} + \cdots + A_{l_s}$, where $(l_1 + 1, l_2 + 1, \ldots, l_s + 1)$ is a partition of $l + 1$.

Example 3.1. Let $\Phi = \mathbf{A}_4$ with simple system $\pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4 - e_5\}$. In this case, we have three $\mathcal{W}(\mathbf{A}_4)$-conjugate subsystems of type $2\mathbf{A}_1$, that is, $\Psi_1 = 2\mathbf{A}_1$; $J_1 = \{\alpha_1, \alpha_3\}$, $\Psi_2 = 2\mathbf{A}_1^{'}$; $J_2 = \{\alpha_1, \alpha_4\}$ and $\Psi_3 = 2\mathbf{A}_1^{''}$; $J_3 = \{\alpha_2, \alpha_4\}$. The Dynkin diagrams for Ψ_1, Ψ_2, Ψ_3 and corresponding compositions

$\Psi_1 = 2\mathbf{A}_1;\ J_1 = \{\alpha_1, \alpha_3\}$  $\lambda_1 = (2, 2, 1)$

$\Psi_2 = 2\mathbf{A}_1';\ J_2 = \{\alpha_1, \alpha_4\}$  $\lambda_1 = (2, 1, 2)$

$\Psi_3 = 2\mathbf{A}_1'';\ J_3 = \{\alpha_2, \alpha_4\}$  $\lambda_1 = (1, 2, 2)$

Since $\Psi_3 \lhd \Psi_2 \lhd \Psi_1$, the maximum subsystem is Ψ_1 and corresponding composition is $\lambda_1 = (2, 2, 1)$ which is a partition of 5.

Let Ψ is a subsystem of Φ with simple system J and Ψ is the maximum subsystem of $\mathcal{W}$-conjugate subsystems. Let $\bar{J}$ be the ordered set

$$\{J_1, J_2, \ldots, J_r; J_1^{'}, J_2^{'}, \ldots, J_s^{'}\},$$

satisfying (2.6), $w\bar{J} = \{wJ_1, wJ_2, \ldots, wJ_r\ ;\ wJ_1^{'}, wJ_2^{'}, \ldots, wJ_s^{'}\}$ for $w \in \mathcal{W}$ and let $\mathcal{T}_\Delta = \{\, w\bar{J} \mid w \in \mathcal{W}\,\}$. Now we can give our principal definition.

Definition 3.2. Let Ψ and $\Psi^{'}$ be subsystems of Φ with simple system J and $J^{'}$ respectively. $\bar{J} = \{J, J^{'}\}$ is called *irreducible system* in Φ if

(i) $\mathcal{W}(J) \cap \mathcal{W}(J^{'}) = \langle e\rangle$ and $\mathcal{W}(J^{\perp}) \cap \mathcal{W}({J^{'}}^{\perp}) = \langle e\rangle$,

(ii) If $d\Psi \cap \Psi^{'} = \emptyset$ then there exist $\sigma \in \mathcal{W}(J^{'})$ and $\rho \in \mathcal{W}(J)$ such that $d = \sigma\rho$ for $d \in D_\Psi$.

By [7], if $\bar{J}$ is an irreducible system in Φ then $|\mathcal{T}_\Delta| = |\mathcal{W}|$.

Remark 3.3. If $\bar{J}$ is an irreducible system in Φ and $d \in D_\Psi \cap D_{\Psi'}$ then $\overline{dJ}$ is irreducible system in Φ.

Definition 3.4. Let $\bar{J}$ be an irreducible system in Φ. Then the elements of $\mathcal{T}_\Delta$ are called Δ-*tableaux*, the J and $J^{'}$ are called the *rows* and the *columns* of $\bar{J}$, respectively. The subgroups $\mathcal{W}(J)$ and $\mathcal{W}(J^{'})$ are called the *row group* and the *column group* of $\bar{J}$, respectively.

Definition 3.5. Two Δ-tableaux $\bar{J}$ and $\bar{K}$ are *row-equivalent*, written $\bar{J} \sim \bar{K}$, if there exists $w \in \mathcal{W}(J)$ such that $\bar{K} = w\bar{J}$. The equivalence class which contains the Δ-tableau $\bar{J}$ is $\{\bar{J}\}$ and is called a Δ-*tabloid*.

Let τ_Δ be the set of all Δ-tabloids. It is clear that the number of distinct elements in τ_Δ is $[\mathcal{W} : \mathcal{W}(J)]$ and by (2.5) we have

$$\tau_\Delta = \{\{\ d\bar{J}\ \} \mid d \in D_\Psi\}.$$

The group $\mathcal{W}$ acts on τ_Δ according to

$$W(\Phi) \times \tau_\Delta \longrightarrow \tau_\Delta, \qquad (\sigma\ ,\ \{\overline{wJ}\}) \mapsto \sigma\{\overline{wJ}\} = \{\overline{\sigma wJ}\}$$

This action is well defined, for if $\{\overline{w_1 J}\} = \{\overline{w_2 J}\}$, then there exists $\rho \in \mathcal{W}(w_1 J)$ such that $\overline{\rho w_1 J} = \overline{w_2 J}$. Hence since $\sigma\rho\sigma^{-1} \in \mathcal{W}(\sigma w_1 J)$ and $\overline{\sigma w_2 J} = \overline{\sigma\rho w_1 J} = (\sigma\rho\sigma^{-1})(\overline{\sigma w_1 J})$, we have $\{\overline{\sigma w_1 J}\} = \{\overline{\sigma w_2 J}\}$.

Now if K is arbitrary field, let M^Δ be the K-space whose basis elements are the Δ-tabloids. Extend the action of $\mathcal{W}$ on τ_Δ linearly on M^Δ, then M^Δ becomes a $K\mathcal{W}$-module. Then we have the following lemma.

Lemma 3.6. *The $K\mathcal{W}$-module M^Δ is a cyclic $K\mathcal{W}$-module generated by any one tabloid and $dim_K M^\Delta = [\mathcal{W} : \mathcal{W}(J)]$.*

Proof. See [7]. ∎

Now we proceed to consider the possibility of constructing a $K\mathcal{W}$-module which corresponds to the Specht module in the case of symmetric groups. In order to do this we need to define a Δ-polytabloid.

Definition 3.7. Let $\bar{J}$ be an irreducible system in Φ. Let

$$\kappa_{\bar{J}} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)\sigma \text{ and } e_{\bar{J}} = \kappa_{\bar{J}}\{\bar{J}\}$$

Then $e_{\bar{J}}$ is called the generalized Δ-*polytabloid*.

If $w \in \mathcal{W}(\Phi)$, then

$$w\kappa_{\bar{J}} = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)w\sigma = \sum_{\sigma \in \mathcal{W}(J')} s(\sigma)(w\sigma w^{-1})w = \{\sum_{\sigma \in \mathcal{W}(wJ')} s(\sigma)\sigma\}$$

Hence, for all $w \in \mathcal{W}(\Phi)$, we have

$$we_{\bar{J}} = \kappa_{\overline{wJ}}\{\overline{wJ}\} = e_{\overline{wJ}}. \tag{3.1}$$

Let $S^{\bar{J}}$ be the subspace of M^Δ generated by $e_{\overline{wJ}}$ where $w \in \mathcal{W}$. Then by (3.1) $S^{\bar{J}}$ is a $K\mathcal{W}$-submodule of M^Δ, which is called a *generalized Specht module*. Then we have the following theorem.

THEOREM 3.8. *The $K\mathcal{W}$-module $S^{\bar{J}}$ is a cyclic submodule generated by any Δ-polytabloid.*

Proof. Straightforward. ■

LEMMA 3.9. *Let $\bar{J}$ be an irreducible system in Φ and let $d \in D_{\Psi}$. If $\{\overline{dJ}\}$ appears in $e_{\bar{J}}$ then it appears only once.*

Proof. As in Lemma 3.10 [7]. ■

COROLLARY 3.10. *If $\bar{J}$ be an irreducible system in Φ, then $e_{\bar{J}} \neq 0$.*

LEMMA 3.11. *Let $\bar{J}$ be an irreducible system in Φ and let $d \in D_{\Psi}$. Then the following conditions are equivalent:*

(i) *$\{\overline{dJ}\}$ appears with non-zero coefficient in $e_{\bar{J}}$,*

(ii) *there exists $\sigma \in \mathcal{W}(J^{'})$ such that $\sigma\{\bar{J}\} = \{\overline{dJ}\}$,*

(iii) *there exists $\rho \in \mathcal{W}(J)$ and $\sigma \in \mathcal{W}(J^{'})$ such that $d = \sigma\ \rho$,*

(iv) *$d\Psi \cap \Psi^{'} = \emptyset$.*

Proof. The equivalence of (i) and (ii) follows directly from the formula

$$e_{\bar{J}} = \sum_{\sigma \in \mathcal{W}(J^{'})} s(\sigma)\sigma\{\bar{J}\}$$

(ii) $\Rightarrow$ (iii). Suppose that there exists $\sigma \in \mathcal{W}(J^{'})$ such that $\sigma\{\ \bar{J}\ \} = \{\ \overline{dJ}\ \}$. Then we have $\sigma^{-1}\ d\{\ \bar{J}\ \} = \{\bar{J}\ \}$. By the definition of equivalence there exists $\rho \in \mathcal{W}(J)$ such that $\sigma^{-1}\ d\ \bar{J} = \rho\ \bar{J}$. Then $\rho^{-1}\sigma^{-1}d \in \mathcal{W}(J^{\perp}) \cap \mathcal{W}(J^{'\perp})$. Since $\{J, J^{'}\}$ is an irreducible system in Φ then $d = \sigma\ \rho$, where $\sigma \in \mathcal{W}(J^{'})$ and $\rho \in \mathcal{W}(J)$.

(iii) $\Rightarrow$ (ii). Let $d = \sigma\rho$, where $\sigma \in \mathcal{W}(J^{'})$ and $\rho \in \mathcal{W}(J)$. Since $\rho \in \mathcal{W}(J)$, $\rho\bar{J} = \bar{J}$ then $\{\overline{dJ}\} = \{\overline{\sigma\rho J}\} = \{\overline{\sigma J}\}$.

(i) $\Rightarrow$ (iv). Let $\alpha \in d\Psi$. If $\{\overline{dJ}\}$ appears in $e_{\bar{J}}$ then by $(i) \Rightarrow$ (iii) $d = \sigma\rho$, where $\sigma \in \mathcal{W}(J^{'})$ and $\rho \in \mathcal{W}(J)$. Then $\alpha \in \sigma\rho\Psi$. Since $\rho \in \mathcal{W}(J)$, then $\alpha \in \sigma\Psi$ and $\sigma^{-1}\alpha \in \Psi$. But $\Psi \cap \Psi^{'} = \emptyset$, then $\sigma^{-1}\alpha \notin \Psi^{'}$. Since $\sigma \in \mathcal{W}(J^{'})$, $\sigma\Psi^{'} = \Psi^{'}$ then $\alpha \notin \Psi^{'}$.

(iv) $\Rightarrow$ (i). By definition of irreducible system. ■

LEMMA 3.12. *Let $\bar{J}$ be an irreducible system in Φ and let $d \in D_{\Psi}$. If $d\Psi \cap \Psi^{'} \neq \emptyset$, then $\kappa_{\bar{J}}\{\overline{dJ}\} = 0$.*

Proof. As in Lemma 3.18 [7]. ■

LEMMA 3.13. *Let $\bar{J}$ be an irreducible system in Φ and let $d \in D_{\Psi}$.*

(i) *If $\{\overline{dJ}\}$ does not appear in $e_{\bar{J}}$ then $\kappa_{\bar{J}}\{\overline{dJ}\} = 0$.*

(ii) *If $\{\overline{dJ}\}$ appears in $e_{\bar{J}}$ then there exists $\sigma \in \mathcal{W}(J^{'})$ such that $\kappa_{\bar{J}}\{\overline{dJ}\} = s(\sigma)e_{\bar{J}}$.*

Proof. See Lemma 3.20 [7]. ■

Corollary 3.14. *Let $\bar{J}$ be an irreducible system in Φ. If $m \in M^{\Delta}$ then $\kappa_{\bar{J}} m$ is a multiple of $e_{\bar{J}}$.*

We now define a bilinear form $\langle \, , \, \rangle$ on M^{Δ} by setting

$$\langle \{\bar{J}\}, \{\bar{K}\} \rangle := \begin{cases} 1, & \{\bar{J}\} = \{\bar{K}\} \\ 0, & \{\bar{J}\} \neq \{\bar{K}\}. \end{cases}$$

This is a symmetric, non-singular, $\mathcal{W}$-invariant bilinear form on M^{Δ}.

Now we can prove James' submodule theorem in this general setting.

Theorem 3.15. *Let $\bar{J}$ be an irreducible system in Φ. Let U be submodule of M^{Δ}. Then either $S^{\bar{J}} \subseteq U$ or $U \subseteq S^{\bar{J}^{\perp}}$.*

Proof. If $u \in U$ then

$$\langle u, e_{\bar{J}} \rangle = \langle u, \sum_{\sigma \in \mathcal{W}(J')} s(\sigma) \sigma \{\bar{J}\} \rangle = \sum_{\sigma \in \mathcal{W}(J')} \langle s(\sigma) \sigma^{-1} u, \{\bar{J}\} \rangle = \langle \kappa_{\bar{J}} u, \{\bar{J}\} \rangle.$$

But by Corollary 3.14 $\kappa_{\bar{J}} u = \lambda e_{\bar{J}}$, for some $\lambda \in K$. If $\lambda \neq 0$ for some $u \in U$, then $e_{\bar{J}} \in U$, that is, $S^{\bar{J}} \subseteq U$. However, if $\lambda = 0$ for all $u \in U$, then $\langle u, e_{\bar{J}} \rangle = 0$, that is, $U \subseteq S^{\bar{J}^{\perp}}$. ∎

We can now prove our principal result.

Theorem 3.16. *Let $\bar{J}$ be an irreducible system in Φ. The $K\mathcal{W}$-module*

$$S^{\bar{J}} / S^{\bar{J}} \cap S^{\bar{J}^{\perp}}$$

is zero or irreducible.

Proof. Let $S^{\bar{J}} / S^{\bar{J}} \cap S^{\bar{J}^{\perp}} \neq \{0\}$. We need to show that the $K\mathcal{W}$-module $S^{\bar{J}} / S^{\bar{J}} \cap S^{\bar{J}^{\perp}}$ is irreducible. Let $U / S^{\bar{J}} \cap S^{\bar{J}^{\perp}}$ be a submodule of $S^{\bar{J}} / S^{\bar{J}} \cap S^{\bar{J}^{\perp}}$. Then $S^{\bar{J}} \cap S^{\bar{J}^{\perp}} \subseteq U \subseteq S^{\bar{J}}$ and $U = S^{\bar{J}}$ or $U = S^{\bar{J}} \cap S^{\bar{J}^{\perp}}$ by Theorem 3.15. If $U = S^{\bar{J}}$ then $U / S^{\bar{J}} \cap S^{\bar{J}^{\perp}} = S^{\bar{J}} / S^{\bar{J}} \cap S^{\bar{J}^{\perp}}$. If $U = S^{\bar{J}} \cap S^{\bar{J}^{\perp}}$ then $U / S^{\bar{J}} \cap S^{\bar{J}^{\perp}} = \{0\}$. Thus $S^{\bar{J}} / S^{\bar{J}} \cap S^{\bar{J}^{\perp}}$ is irreducible. ∎

In the case of $K = \mathbf{Q}$ or any field of characteristic zero $\langle \, , \, \rangle$ is an inner product and $S^{\bar{J}} / S^{\bar{J}} \cap S^{\bar{J}^{\perp}} \cong S^{\bar{J}}$. Thus if for a subsystem Ψ of Φ an irreducible system $\bar{J}$ can be found, then we have a construction for irreducible $K\mathcal{W}$-modules. Hence it is essential to show for each subsystem that an irreducible system exists which satisfies Definition 3.2.

Remark 3.17. For any root system Φ with simple system π, there are two trivial irreducible system in Φ. We can describe immediately the representations arising in these cases.

If $\bar{J}_1 = \{\pi; \emptyset\}$, then $\mathcal{W}(J_1) = \mathcal{W}(\Phi)$ and $\mathcal{W}({J_1}') = \langle e \rangle$. So

$$e_{\bar{J}_1} = \sum_{\sigma \in W({J_1}')} s(\sigma) \sigma \{\bar{J}_1\} = s(e) e \{\bar{J}_1\} = \{\bar{J}_1\}.$$

We clearly have $S^{\bar{J}_1} = Sp\{e_{\bar{J}_1}\} = Sp\{\{\bar{J}_1\}\}$ and $we_{\bar{J}_1} = w\{\bar{J}_1\} = \{\bar{J}_1\} = e_{\bar{J}_1}$ for all $w \in \mathcal{W}$ and the identity representation of $\mathcal{W}$ is given.

If $\bar{J}_2 = \{\emptyset; \pi\}$, then $\mathcal{W}(J_2) = \langle e \rangle$ and $D_{\Psi_2} = \mathcal{W}(\Phi) = \mathcal{W}({J_2}')$. So

$$e_{\bar{J}_2} = \sum_{\sigma \in W({J_2}')} s(\sigma)\sigma\{\bar{J}_2\} = \sum_{\sigma \in \mathcal{W}(\Phi)} s(\sigma)\sigma\{\bar{J}_2\}.$$

Hence $S^{\bar{J}_2} = Sp\{e_{\bar{J}_2}\}$ and $we_{\bar{J}_2} = s(w)e_{\bar{J}_2}$ for all $w \in \mathcal{W}$. Thus $w \mapsto sgn(w)$ for all $w \in \mathcal{W}$, and the corresponding representation leads to the *sign character* of $\mathcal{W}$.

In the following example, we show that an irreducible system may be constructed in the case of the finite reflection group of type $\mathbf{I}_2(8)$.

Example 3.18. Let $\Phi = \mathbf{I}_2(8)$ with simple system $\pi = \{\alpha_1 = e_1, \alpha_2 = \cos\frac{7\pi}{8}e_1 + \sin\frac{7\pi}{8}e_2\}$ and $\Phi^+ = \{\alpha_1 = e_1, \alpha_2 = \cos\frac{7\pi}{8}e_1 + \sin\frac{7\pi}{8}e_2, \alpha_3 = \cos\frac{\pi}{8}e_1 + \sin\frac{\pi}{8}e_2, \alpha_4 = \cos\frac{2\pi}{8}e_1 + \sin\frac{2\pi}{8}e_2, \alpha_5 = \cos\frac{3\pi}{8}e_1 + \sin\frac{3\pi}{8}e_2, \alpha_6 = \cos\frac{4\pi}{8}e_1 + \sin\frac{4\pi}{8}e_2, \alpha_7 = \cos\frac{5\pi}{8}e_1 + \sin\frac{5\pi}{8}e_2, \alpha_8 = \cos\frac{6\pi}{8}e_1 + \sin\frac{6\pi}{8}e_2\}$. Let e, $(\tau_1\tau_2)^4$, $\tau_1\tau_2$, $(\tau_1\tau_2)^2$, $(\tau_1\tau_2)^3$, τ_1, τ_2 be representatives of conjugate classes C_1, C_2, C_3, C_4, C_5, C_6, C_7 respectively of $\mathcal{W}(\mathbf{I}_2(8))$. The character table of $\mathcal{W}(\mathbf{I}_2(8))$ is given by

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1
χ_3	1	1	-1	1	-1	1	-1
χ_4	1	1	-1	1	-1	-1	1
χ_5	2	-2	$\sqrt{2}$	0	$-\sqrt{2}$	0	0
χ_6	2	2	0	-2	0	0	0
χ_7	2	-2	$-\sqrt{2}$	0	$\sqrt{2}$	0	0

The non-conjugate subsystem of $\mathbf{I}_2(8)$ are:

1. $\Psi_1 = \mathbf{I}_2(8)$; $J_1 = \{\alpha_1, \alpha_2\}$
2. $\Psi_2 = 2\mathbf{A}_1$; $J_2 = \{\alpha_2, -\tilde{\alpha}\}$
3. $\Psi_3 = \mathbf{A}_1$; $J_3 = \{\alpha_1\}$
4. $\Psi_4 = \mathbf{A}_1'$; $J_4 = \{\alpha_2\}$
5. $\Psi_5 = \emptyset$; $J_5 = \emptyset$

where $\tilde{\alpha} = \cos\frac{3\pi}{8}e_1 + \sin\frac{3\pi}{8}e_2$ is the longest root in $\mathbf{I}_2(8)$.

(1) Let $\Psi_1 = \mathbf{I}_2(8)$ be the subsystem of Φ with simple system $J_1 = \{\alpha_1\ ,\ \alpha_2\}$. Then Coxeter graph for Ψ_1 is

$$\underset{-\tilde{\alpha}}{\otimes} \overset{8}{\text{—}} \underset{\alpha_1}{\bigcirc} \overset{8}{\text{—}} \underset{\alpha_2}{\bigcirc}$$

If $\Psi_1' = \emptyset$ with simple system $J_1' = \emptyset$, then $\{\alpha_1, \alpha_2; \emptyset\}$ is an irreducible system in Φ by Remark 3.17. Thus we have

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ψ_1	1	1	1	1	1	1	1

that is the character χ_1.

(2) Let $\Psi_2 = \mathbf{2}A_1$ be the subsystem of Φ with simple system $J_2 = \{\alpha_2\ , -\tilde{\alpha}\}$. Then Coxeter graph for Ψ_2 is

$$\underset{-\,\tilde{\alpha}}{\bigcirc} \overset{8}{\text{---}} \underset{\alpha_1}{\otimes} \overset{8}{\text{---}} \underset{\alpha_2}{\bigcirc}$$

In this case, we have not found an irreducible system in Φ.

(3) Let $\Psi_3 = \mathbf{A}_1$ be the subsystem of Φ with simple system $J_3 = \{\alpha_1\}$. Then Coxeter graph for Ψ_3 is

$$\underset{-\,\tilde{\alpha}}{\otimes} \overset{8}{\text{---}} \underset{\alpha_1}{\bigcirc} \overset{8}{\text{---}} \underset{\alpha_2}{\otimes}$$

If $\Psi_3^{'} = 2\mathbf{A}_1$ with simple system $J_3^{'} = \{\alpha_4, \alpha_8\}$, then $\{\alpha_1; \alpha_4, \alpha_8\}$ is an irreducible system in Φ. Thus we have

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ψ_2	2	2	0	-2	0	0	0

that is the character χ_6.

(4) Let $\Psi_4 = \mathbf{A}_1^{'}$ be the subsystem of Φ with simple system $J_4 = \{\alpha_2\}$. Then Coxeter graph for Ψ_4 is

$$\underset{-\,\tilde{\alpha}}{\otimes} \overset{8}{\text{---}} \underset{\alpha_1}{\otimes} \overset{8}{\text{---}} \underset{\alpha_2}{\bigcirc}$$

If $\Psi_4^{'} = 2\mathbf{A}_1$ with simple system $J_4^{'} = \{\alpha_4, \alpha_8\}$ then $\{\alpha_2; \alpha_4, \alpha_8\}$ is an irreducible system in Φ. Thus we have

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ψ_3	2	2	0	-2	0	0	0

that is the character χ_6.

(5) Let $\Psi_5 = \emptyset$ be the subsystem of Φ with simple system $J_5 = \emptyset$. Then Coxeter graph for Ψ_5 is

$$\underset{-\,\tilde{\alpha}}{\otimes} \overset{8}{\text{---}} \underset{\alpha_1}{\otimes} \overset{8}{\text{---}} \underset{\alpha_2}{\otimes}$$

If $\Psi_5^{'} = \mathbf{I}_2(8)$ with simple system $J_5^{'} = \{\alpha_1, \alpha_2\}$ then $\{\emptyset; \alpha_4, \alpha_8\}$ is an irreducible system in Φ by Remark 3.17. Thus we have

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ψ_4	1	1	1	1	1	-1	-1

that is the character χ_2.

REFERENCES

[**1**] E. Al-Aamily, A. O. Morris and M. H. Peel, *The representations of the Weyl groups of type* B_n, J. Algebra, **68** (1981), 298–305.

[**2**] N. Bourbaki, *Groupes et algèbres de Lie*, Actualites Sci. Induct 1337, Hermann, Paris, 1968.

[**3**] R. W. Carter, *Simple Groups of Lie Type*, Wiley, London, New York, Sydney, Toronto, 1989.

[**4**] R. W. Carter, *Conjugacy classes in the Weyl group*, Comp. Math., **25** (1972), 1–59.

[**5**] L. C. Grove and C. T. Benson, *Finite Reflection Groups*, Springer-Verlag, 1985.

[**6**] S. Halıcıoğlu, *Specht modules for finite reflection groups*, Glasgow Math. J. **37** (1995), 279–287.

[**7**] S. Halıcıoğlu and A. O. Morris, *Specht modules for Weyl groups*, Contrib. Alg. Geom., **34** (1993), 257–276.

[**8**] L. Hawkins, *Constructing irreducible representations of Weyl groups*, Sem. Lother. Combin., **34** (1995).

[**9**] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972.

[**10**] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990.

[**11**] G. D. James, *The irreducible representations of the symmetric group*, Bull. Lond. Math. Soc., **8** (1976), 229–232.

[**12**] I. G. Macdonald, *Some irreducible representation of Weyl groups*, Bull. Lond. Math. Soc., **4** (1972), 148–150.

[**13**] A. O. Morris, *Representations of Weyl groups over an arbitrary field*, Astèrisque, **87–88** (1981), 267–287.

[**14**] A. O. Morris and A. J. Idowu, *Some combinatorial results for Weyl groups*, Proc. Camb. Phil. Soc., **101** (1987), 405–420.

(received 16.03.2004)

M. Başer, Department of Mathematics, Faculty of Science and Arts, Kocatepe University, Afyon - Turkey

E-mail: `mbaser@aku.edu.tr`

Sait Halıcıoğlu, Department of Mathematics, Faculty of Science, Ankara University, 06100 Ankara - Turkey