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OPERATIONAL QUANTITIES DERIVED FROM THE MINIMUM MODULUS

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Abstract. The minimum modulus $\gamma(T)$ of an operator T is useful in perturbation theory because it characterizes the operators with closed range. Here we study the operational quantities derived from $\gamma(T)$. We show that the behavior of some of these quantities depends largely on whether the null space of T is finite dimensional or infinite dimensional.

1. Introduction

For every (linear bounded) non-zero operator $T \in L(X,Y)$, where X and Y are Banach spaces, the *minimum modulus* is defined by

$$\gamma(T) := \inf_{x \notin N(T)} \frac{\|Tx\|}{\text{dist }(x, N(T))}.$$

For $T = 0$ we set $\gamma(0) = 0$. It is well known that $\gamma(T) > 0$ if and only if T has closed range and $T \neq 0$ [1, Theorem IV.1.6].

Here we study the operational quantities that can be derived from the minimum modulus $\gamma(T)$.

In the preliminaries, we give a description of the procedure to derive the quantities associated to a given quantity. This procedure, applied to the norm $n(T) \equiv \|T\|$, provides three quantities in, sin and i^*n which have been studied (with a different notation) in [9], [4], [7] and [2, 3]. Applied to the injection modulus $j(T) := \inf\{\, \|Tx\| : x \in X, \|x\| = 1\,\}$, it provides three quantities s^*j, sj and isj which have been studied in [7] and [2, 3].

These operational quantities have been applied to characterize the classes of operators in Fredholm theory: see Theorem 1 in the preliminaries. For an excellent exposition of the Fredholm theory using operational quantities we refer to Chapter 14 in Schechter's book [8].

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From the minimum modulus γ we can derive eight quantities $is\gamma$, $sis\gamma$, $i^*s\gamma$, $s\gamma$, $i\gamma$, $s^*i\gamma$, $isi\gamma$ and $si\gamma$. In this paper we obtain that $i\gamma$ agrees with the injection modulus j and we derive, consequently, that the quantities $s^*i\gamma, isi\gamma$ and $si\gamma$ are the known quantities s^*j, isj and sj, respectively.

On the other hand, we prove that if the operator T has finite dimensional null space $N(T)$, then $sj(T) \leqslant s\gamma(T) \leqslant 2sj(T)$ and we obtain that $s\gamma, sis\gamma$ and $i^*s\gamma$ characterize the strictly singular operators, while $is\gamma$ characterizes the upper semi-Fredholm operators. In the case $N(T)$ infinite dimensional, we have $s\gamma(T) = n(T)$ and we obtain that the quantities $is\gamma, sis\gamma$ and $i^*s\gamma$ characterize the upper semi-Fredholm operators, the strictly singular operators and the compact operators, respectively.

Along the paper, X, Y, Z and W are infinite dimensional Banach spaces. By $L(X,Y)$ we denote the space of all (linear continuous) operators from X into Y. For a (closed) subspace M of X, J_M is the inclusion of M into X. An operator $T \in L(X,Y)$ is *upper semi-Fredholm* if its range is closed and its null space is finite dimensional; it is *strictly singular* if no restriction TJ_M of T to a closed infinite dimensional subspace M of X is an isomorphism.

2. Preliminaries

Roughly speaking, an operational quantity is a procedure a which determines a real number $a(T) \geqslant 0$ for every operator T. For two quantities a and b we write $a \leqslant b$ when

$$a(T) \leqslant b(T), \text{ for every operator } T\ .$$

We say that the operational quantities a and b are *equivalent* if $\alpha a \leqslant b \leqslant \beta a$, for some $\beta > \alpha > 0$.

Given an operational quantity a and denoting by J_M the canonical inclusion of M into X, for every operator $T \in L(X,Y)$, where X is an infinite dimensional space, we derive the following basic quantities:

$$\begin{aligned} s^*a(T) &:= \sup\{a(TJ_P) : P \text{ finite codimensional subspace of } X\}, \\ sa(T) &:= \sup\{a(TJ_M) : M \text{ infinite dimensional subspace of } X\}, \\ i^*a(T) &:= \inf\{a(TJ_P) : P \text{ finite codimensional subspace of } X\}, \\ ia(T) &:= \inf\{a(TJ_M) : M \text{ infinite dimensional subspace of } X\}. \end{aligned}$$

Repeating the procedure, we could derive new quantities like sia, $sisa$, i^*issa, $\dots$, but surprisingly we obtain only three different new quantities when a is monotone [5]:

If a is *increasing*, in the sense that $a(TJ_M) \leqslant a(T)$ for every M, then ia, sia and i^*a are the only new quantities, and they satisfy

$$ia \leqslant sia \leqslant i^*a \leqslant a.$$

If a is *decreasing*, in the sense that $a(TJ_M) \geqslant a(T)$ for every M, then sa, isa and s^*a are the only new quantities, and they satisfy

$$a \leqslant s^*a \leqslant isa \leqslant sa.$$

The norm $n(T) \equiv \|T\|$ is an increasing quantity. So we get i^*n, in and sin. The injection modulus $j(T)$ is decreasing. So we get s^*j, sj and isj. There are some relations between these quantities: $isj \leqslant in$ and $sj \leqslant sin$.

The operational quantities associated to n and j have been applied to characterize the classes of operators in Fredholm theory:

THEOREM 1. [2, 3], [7], [9]

1. $i^*n(T) = 0 \;\Leftrightarrow\; T$ *compact,*
2. $sin(T) = 0 \;\Leftrightarrow\; sj(T) = 0 \;\Leftrightarrow\; T$ *strictly singular,*
3. $in(T) > 0 \;\Leftrightarrow\; s^*j(T) > 0 \;\Leftrightarrow\; isj(T) > 0 \;\Leftrightarrow\; T$ *upper semi-Fredholm.*

The quantity isj has been introduced in [2]; moreover, it was proved in [3] that, although

$$s^*j \leqslant isj \leqslant in\ ,$$

these quantities are pairwise non-equivalent.

3. Main results

The operational quantity γ is not monotone, but $i\gamma$ is decreasing and $s\gamma$ is increasing. Hence we derive from γ the quantities

$$i\gamma \leqslant s^*i\gamma \leqslant isi\gamma \leqslant si\gamma;$$
$$is\gamma \leqslant sis\gamma \leqslant i^*s\gamma \leqslant s\gamma.$$

We begin by studying the operational quantities associated with $i\gamma$.

THEOREM 2. *For every* $T \in L(X,Y)$, $i\gamma(T) = j(T)$.

Proof. Let T in $L(X,Y)$. If $N(T) = \{0\}$ or $N(T)$ is infinite dimensional, then the statement is obvious. So we assume that $0 < \dim N(T) < \infty$, hence $j(T) = 0$.

We choose $x \in N(T)$ with $\|x\| = 1$ and $y \in X \setminus N(T)$, and we write

$$X = N(T) \oplus \langle y \rangle \oplus M,$$

where $\langle y \rangle$ is the subspace generated by y and M is a closed complement of $N(T) \oplus \langle y \rangle$. Moreover, denoting $y_n = (1/n)y$, we define

$$M_n := M \oplus \langle x + y_n \rangle.$$

Suppose that $z \in M_n \cap N(T)$ then

$$z = m + \lambda(x + y_n) = \lambda x + \lambda y_n + m,$$

for some $m \in M$ and some scalar λ. Thus $\lambda y_n = m = 0$, and we conclude $z = 0$.

Since TJ_{M_n} is injective, $\|x + y_n\| \to 1$ and $\|T(x + y_n)\| = (1/n)\|Ty\| \to 0$, we have $\gamma(TJ_{M_n}) \to 0$; hence $i\gamma(T) = 0$. ■

Corollary 1. *Let $T \in L(X,Y)$.*

1. $i\gamma(T) > 0 \Leftrightarrow T$ *isomorphism (into);*
2. $si\gamma(T) = 0 \Leftrightarrow T$ *strictly singular;*
3. $isi\gamma(T) > 0 \Leftrightarrow s^*i\gamma(T) > 0 \Leftrightarrow T$ *upper semi-Fredholm.*

Proof. From $i\gamma = j$ we derive $s^*i\gamma = s^*j$, $isi\gamma = isj$ and $si\gamma = sj$. From Theorem 1 we obtain the statement. ■

Now we study the quantities associated to $s\gamma$. We will see that the behavior of some of these quantities depends largely on whether the null space of T is finite dimensional or infinite dimensional.

Lemma 1. *Let N be a finite dimensional subspace of X, and let $0 < \varepsilon < 1$. For every infinite dimensional subspace M of X, there exists a finite codimensional subspace M_ε of M such that, for every $x \in M_\varepsilon$,*

$$\|x\| \leqslant (2+\varepsilon)\operatorname{dist}(x,N).$$

Proof. Let $\{y_1,\dots,y_k\}$ be an $(\varepsilon/2)$-net in the unit sphere of N. We choose $f_1,\dots,f_k$ in the unit sphere of the dual space X^* of X so that $f_i(y_i) = 1$ for $i = 1,2,\dots,k$, and take

$$M_\varepsilon := \{x \in M : f_1(x) = \cdots = f_n(x) = 0\}.$$

Let $x \in M_\varepsilon$. For each $y \in N$ we denote $z_i := \|y\|y_i$ $(1 \leqslant i \leqslant k)$. Then

$$\|y-x\| \geqslant \|x-z_i\| - \|y-z_i\| \geqslant f_i(z_i) - (\varepsilon/2)\|y\| = \|y\| - (\varepsilon/2)\|y\| \geqslant \frac{\|y\|}{1+\varepsilon}$$

for some i. Hence, $\|y\| \leqslant (1+\varepsilon)\|y-x\|$ for each y in N. From this we obtain

$$\|x\| \leqslant \|x-y\| + \|y\| \leqslant (2+\varepsilon)\|x-y\| ,$$

for each y in N, and this implies $\|x\| \leqslant (2+\varepsilon)\operatorname{dist}(x,N)$. ■

Theorem 3. *Let $T \in L(X,Y)$. If $N(T)$ is finite dimensional, then*

$$sj(T) \leqslant s\gamma(T) \leqslant 2sj(T).$$

Proof. Let M be an infinite dimensional subspace of X, and let $\varepsilon > 0$. By Lemma 1, there exists a finite codimensional subspace M_ε of M such that for $x \in M_\varepsilon$,

$$\|x\| \leqslant (2+\varepsilon)\operatorname{dist}(x,N(T)) \leqslant (2+\varepsilon)\operatorname{dist}(x,N(TJ_M)).$$

Then

$$\begin{aligned}\gamma(TJ_M) &= \inf_{x\in M, Tx\neq 0} \frac{\|Tx\|}{\operatorname{dist}(x,N(TJ_M))}\\ &\leqslant \inf_{x\in M_\varepsilon, Tx\neq 0} \frac{\|Tx\|}{\operatorname{dist}(x,N(TJ_M))}\\ &\leqslant \inf_{x\in M_\varepsilon, Tx\neq 0} \frac{\|Tx\|}{\|x\|}(2+\varepsilon) = (2+\varepsilon)j(TJ_{M_\varepsilon}).\end{aligned}$$

Hence, $\gamma(TJ_M) \leqslant (2+\varepsilon)sj(T)$, so $s\gamma(T) \leqslant 2sj(T)$.

The inequality $sj(T) \leqslant s\gamma(T)$ is obvious. ∎

Corollary 2. *Let $T \in L(X,Y)$. If $N(T)$ is finite dimensional, then*

1. $isj(T) \leqslant is\gamma(T) \leqslant 2isj(T)$,
2. $sj(T) \leqslant sis\gamma(T) \leqslant 2sj(T)$,
3. $sj(T) \leqslant i^*s\gamma(T) \leqslant 2sj(T)$.

Proof. Note that $N(TJ_M)$ is also finite dimensional for each M.

1. For every infinite dimensional subspace M of X we have that $sj(TJ_M) \leqslant s\gamma(TJ_M) \leqslant 2sj(TJ_M)$, hence $isj(T) \leqslant is\gamma(T) \leqslant 2isj(T)$.

2. Similar to 1., taking into account $sisj(T) = sj(T)$ [5].

3. Similar to 1., taking into account $i^*sj(T) = sj(T)$ [5]. ∎

The following result shows that the property $is\gamma(T) > 0$ is preserved by taking products.

Corollary 3. *Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If $N(T)$ and $N(S)$ are finite dimensional, then*

$$is\gamma(S)\; is\gamma(T) \leqslant 4\; is\gamma(ST)$$

Proof. It is known that $isj(S)\; isj(T) \leqslant isj(ST)$ [2], hence $is\gamma(S)\; is\gamma(T) \leqslant 2isj(S)\; 2isj(T) \leqslant 4isj(ST) \leqslant 4is\gamma(ST)$. ∎

Theorem 4. *For each operator $T \in L(X,Y)$ with $N(T)$ infinite dimensional, $s\gamma(T) = n(T)$.*

Proof. For $T = 0$ the result is obvious. Suppose $T \neq 0$. For each $x \notin N(T)$ we put $M_x := N(T) \oplus \langle x \rangle$, where $\langle x \rangle$ is the subspace generated by x.

For every $y \in M_x$, $y = \lambda x + z$, $z \in N(T)$, we obtain

$$\frac{\|Tx\|}{\|x\|} = \frac{|\lambda|\;\|Tx\|}{|\lambda|\;\|x\|} = \frac{\|Ty\|}{\|y-z\|} \leqslant \frac{\|Ty\|}{\operatorname{dist}(y, N(T))}.$$

Thus $n(T) \leqslant \gamma(TJ_{M_x}) \leqslant s\gamma(T)$. ∎

Note that for every $T \neq 0$ with $R(T)$ finite dimensional (hence $N(T)$ is infinite dimensional), $sj(T) = 0 \neq n(T)$. Thus sj and $s\gamma$ are not equivalent.

Corollary 4. *Let $T \in L(X,Y)$.*

1. *$is\gamma(T) > 0 \Leftrightarrow T$ is upper semi-Fredholm.*
2. *If $N(T)$ is finite dimensional, then*

$$s\gamma(T) = 0 \Leftrightarrow sis\gamma(T) = 0 \Leftrightarrow i^*s\gamma(T) = 0 \Leftrightarrow T \textit{ is strictly singular.}$$

3. *If $N(T)$ is infinite dimensional, then*

(a) $s\gamma(T) = 0 \Leftrightarrow T = 0$*;*

(b) $sis\gamma(T) = 0 \Leftrightarrow T$ *is strictly singular;*

(c) $i^*s\gamma(T) = 0 \Leftrightarrow T$ *is compact.*

Proof. (1) [6, Example 5.1].

(2) It is immediate from Theorem 1, Theorem 3 and Corollary 2.

(3) (a) Theorem 4.

(b) $sis\gamma(T) = 0$ is equivalent to $is\gamma(TJ_M) = 0$ for every infinite dimensional subspace M of X, which is equivalent by (1) to TJ_M is not an upper semi-Fredholm operator, and consequently to T strictly singular.

(c) Since $i^*s\gamma(T)$ is the infimum of $s\gamma(TJ_P) = n(TJ_P)$ where P runs over the finite codimensional subspaces of X, from Theorem 1, we obtain $i^*s\gamma(T) = i^*n(T) = 0$ if and only if T is compact. ■

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