МАТЕМАТИЧКИ ВЕСНИК
58 (2006), 13–17

UDK 517.521
оригинални научни рад
research paper

ON HADAMARD TYPE POLYNOMIAL CONVOLUTIONS WITH REGULARLY VARYING SEQUENCES

Slavko Simić

Abstract. For a sequence of polynomials $P_n(x) := \sum_{m \leqslant n} p_m x^m$, $n \geqslant 1$, we give a necessary and sufficient condition for the asymptotic equivalence

$$P_n^{(\alpha)}(x) := \sum_{m \leqslant n} c_m p_m x^m \sim c_n P_n(x) \qquad (n \to \infty),$$

to hold for each $x \geqslant A$ and an arbitrary regularly varying sequence $\{c_n\}$ of index $\alpha \in \mathbf{R}$.

Introduction

A sequence $\{p_n\}_{n \geqslant 1}$ of non-negative numbers generates a sequence of polynomials $\{P_n(x)\}_{n \geqslant 1}$ defined by $P_n(x) := \sum_{m \leqslant n} p_m x^m$.

A sequence $\{c_n\}_{n \geqslant 1}$ of positive numbers is regularly varying with index $\alpha \in \mathbf{R}$ if it can be represented in the form $c_n = n^\alpha \ell_n$, where $\{\ell_n\}$ is a *slowly varying* sequence, i.e. satisfying $\ell_{[\lambda n]} \sim \ell_n$ $(n \to \infty)$ for each $\lambda > 0$ ([1], [2]).

Some examples of slowly varying sequences are:

$$\log^a(n+1),\ a \in \mathbf{R};\ \log^b(\log(n+1)),\ b \in \mathbf{R};\ \exp(\log^c(n+1)),\ 0 < c < 1.$$

Our task here is to investigate asymptotic behavior of Hadamard-type convolutions $P_n^{(\alpha)}(x) := \sum_{m \leqslant n} c_m p_m x^m$ as $n \to \infty$ (cf. [2]).

In [2] we introduced an operator $Tf(x)$ in the following way.

Definition. Let $f \in C^\infty[0, \infty)$. Then

$$Tf(x) := \frac{xf'(x)}{f(x)}.$$

Under a more general framework, we obtained asymptotic behavior of $P_n^{(\alpha)}(x)$ supposing

$$T(TP_n(x)) < M, \tag{I}$$

where M does not depend on n or x.

AMS Subject Classification: 26 A 12
Keywords and phrases: Regular variation, polynomials, asymptotic behavior.

In this paper we find a necessary and sufficient condition for the given asymptotics avoiding the somewhat ambiguous condition (I).

Results

THEOREM. *Let A be a positive number. Then the asymptotic relation*

$$P_n^{(\alpha)}(x) \sim n^\alpha \ell_n P_n(x) \qquad (n \to \infty), \tag{1}$$

holds for each $x \geqslant A$, $\alpha \in \mathbf{R}$, and an arbitrary slowly varying sequence $\{\ell_n\}$, if and only if

$$TP_n(A) := \frac{AP_n'(A)}{P_n(A)} \sim n \qquad (n \to \infty). \tag{2}$$

Proof. Denote $Q_n(x) := \sum_{m \leqslant n} P_m(x)$. We can see that the condition (2) is necessary if we put in (1): $\alpha = 1$, $\ell_n = 1$, $x = A$. That it is also sufficient can be proved using the following lemmas.

LEMMA 1. *Under the condition* (2), *for each real α we have*

$$(i)\ n^\alpha Q_n(A) \to \infty; \qquad (ii)\ \sum_{m \leqslant n} m^\alpha p_m A^m \sim n^\alpha P_n(A) \qquad (n \to \infty).$$

LEMMA 2. *We have* $\sup_{m \leqslant n}(m\ell_m) \sim n\ell_n$; $\inf_{m \leqslant n}(\ell_m/m) \sim \ell_n/n$ $(n \to \infty)$.

LEMMA 3. *The function* $x \mapsto \dfrac{xP_n'(x)}{P_n(x)}$ *is non-decreasing for* $x > 0$.

LEMMA 4. (Stoltz's lemma) *If* $\sum_{m \leqslant n} b_m \to \infty$ *and* $a_n/b_n \to s$ *as* $n \to \infty$, *then*

$$\sum_{m \leqslant n} a_m \Big/ \sum_{m \leqslant n} b_m \to s \qquad (n \to \infty).$$

Proof of Lemma 1. By partial summation we get $\sum_{m \leqslant n} mp_m A^m = (n+1)P_n(A) - Q_n(A)$. Hence, the condition (2) is equivalent to

$$nP_n(A)/Q_n(A) \to \infty \qquad (n \to \infty) \tag{3}$$

Therefore, for $n > n_0$ and fixed $\alpha \in \mathbf{R}$, we deduce

$$\frac{nP_n(A)}{Q_n(A)} > |\alpha| + 1; \qquad \frac{Q_n(A) - Q_{n-1}(A)}{Q_n(A)} > \frac{|\alpha|+1}{n};$$
$$\frac{Q_{n-1}(A)}{Q_n(A)} < 1 - \frac{|\alpha|+1}{n} < \exp(-\frac{|\alpha|+1}{n}).$$

Hence

$$Q_n(A) \gg \exp((|\alpha|+1)\sum_{m \leqslant n} 1/m) \gg \exp((|\alpha|+1)\log n),$$

i.e. $n^\alpha Q_n(A) \gg n^{\alpha+|\alpha|+1}$ and the part (i) is proved.

Denoting $\Delta r_n := r_{n+1} - r_n$, by (3) we get

$$\frac{P_n(A)\Delta n^\alpha}{\Delta(n^{\alpha-1}Q_{n-1}(A))} = \frac{P_n(A)\Delta n^\alpha}{n^{\alpha-1}P_n(A) + Q_n(A)\Delta n^{\alpha-1}} \to \alpha \quad (n\to\infty).$$

Now, applying part (i), Stoltz's lemma and (3), we obtain

$$S_n(A) := \sum_{m\leqslant n} P_m(A)\Delta m^\alpha \sim \alpha n^{\alpha-1}Q_n(A) = o(n^\alpha P_n(A)) \qquad (n\to\infty).$$

Therefore, by partial summation we get

$$\sum_{m\leqslant n} m^\alpha p_m A^m = (n+1)^\alpha P_n(A) - S_n(A) = (n+1)^\alpha P_n(A) + o(n^\alpha P_n(A)) \quad (n\to\infty),$$

and the part (ii) of Lemma 1 is also proved. ■

Lemma 2. is proved in [1, p. 23].

Proof of Lemma 3. Indeed, for $x>0$ by Cauchy's inequality, we get

$$x\frac{d}{dx}\left(\frac{xP_n'(x)}{P_n(x)}\right) = \frac{\sum_{m\leqslant n} m^2 p_m x^m}{\sum_{m\leqslant n} p_m x^m} - \left(\frac{\sum_{m\leqslant n} m p_m x^m}{\sum_{m\leqslant n} p_m x^m}\right)^2 \geqslant 0.$$

Hence $TP_n(x)$ is monotone non-decreasing for $x>0$. ■

Stoltz's lemma is a classical one and is proved, for example, in [3, p. 30].

Now we can give the proof of the Theorem at the point $x=A$. By Lemmas 1 and 2, as $n\to\infty$, we get

$$P_n^\alpha(A) = \sum_{m\leqslant n} m^\alpha \ell_m p_m A^m \leqslant \sup_{m\leq n}(m\ell_m) \sum_{m\leqslant n} m^{\alpha-1} p_m A^m \sim n^\alpha \ell_n P_n(A),$$

and

$$\sum_{m\leqslant n} m^\alpha \ell_m p_m A^m \geqslant \inf_{m\leqslant n}(\ell_m/m) \sum_{m\leqslant n} m^{\alpha+1} p_m A^m \sim n^\alpha \ell_n P_n(A).$$

Hence

$$1 \leqslant \liminf_n (P_n^{(\alpha)}(A)/n^\alpha \ell_n P_n(A)) \leqslant \limsup_n (P_n^{(\alpha)}(A)/n^\alpha \ell_n P_n(A)) \leqslant 1,$$

and the proof is done. ■

For $x>A$, by Lemma 3, we obtain

$$n \sim AP_n'(A)/P_n(A) \leqslant xP_n'(x)/P_n(x) \leqslant n.$$

Hence $xP_n'(x)/P_n(x) \sim n$ $(n\to\infty)$ and we can apply the previous proof replacing A by x.

Comment. As the referee notes, the condition (2) is certainly less opaque then the former condition (I), but it still is opaque in that one has to do a calculation and some asymptotic approximations to decide if a candidate sequence satisfies it.

There is also a problem to determine the least possible A such that (2) holds.

For instance, if $p_n = a^n$ for some $a > 0$ then (2) holds for $A > 1/a$ and fails for $A \leqslant 1/a$.

Also, if $p_n = 1/n!$ then (2) never holds; but for $p_n = n!$ an easy calculation shows that (2) is valid for all $A > 0$.

Therefore we shall establish two simple criteria which can help to decide if a given sequence $\{p_n\}$ satisfies (2) or not.

PROPOSITION 1. *If A lies inside the interval of convergence of $\sum p_n x^n$ then the condition* (2) *fails.*

Proof. We have, as $n \to \infty$, $\sum_{m \leqslant n} p_m A^m \to P(A)$, and consequently,

$$\sum_{m \leqslant n} m p_m A^m \to AP'(A).$$

Hence $TP_n(A) \to 0$ $(n \to \infty)$. ■

But the divergence of $\sum p_n A^n$ does not imply that (2) is true. This can be seen from the following example.

Let $p_m = 1$ if m is in the factorial form and $p_m = 0$ otherwise. Then

$$P_{(n+1)!-1}(A) = A^{n!} + A^{(n-1)!} + \cdots .$$

For $A > 1$, we have $P_{(n+1)!-1}(A) \sim A^{n!}$, and

$$AP'_{(n+1)!-1}(A) = n!A^{n!} + (n-1)!A^{(n-1)!} + \cdots \sim n!A^{n!} \quad (n \to \infty).$$

Therefore

$$TP_{(n+1)!-1}(A) \sim \frac{n!}{(n+1)!-1} \to 0 \qquad (n \to \infty).$$

PROPOSITION 2. *If, for some $A > 0$,*

$$\lim_{n\to\infty} n\left(1 - \frac{1}{A}\frac{p_n}{p_{n+1}}\right) = +\infty, \tag{4}$$

then (2) *holds.*

Proof. Note that the condition (4) implies just a finite number of $p_n = 0$. Also, by Raabe's convergence criteria, $\sum p_n A^n$ diverges.

Now, the condition (4) is equivalent to

$$1 + (n-1)\left(1 - \frac{1}{A}\frac{p_{n-1}}{p_n}\right) \to +\infty,$$

i.e.

$$(np_n A^n - (n-1)p_{n-1}A^{n-1})/p_n A^n \to +\infty.$$

Applying Lemma 4, we get $\sum_{m\leqslant n} p_m A^m / np_n A^n \to 0$ $(n\to\infty)$. It follows that

$$\frac{np_nA^n}{np_nA^n + \sum_{m\leqslant n-1} p_m A^m} \to 1 \qquad (n\to\infty),$$

i.e.

$$\frac{np_nA^n}{n\sum_{m\leqslant n} p_m A^m - (n-1)\sum_{m\leqslant n-1} p_m A^m} \to 1.$$

Applying Lemma 4 again, we obtain the condition (2). ■

Now it is not difficult to verify the above examples using Propositions 1 and 2.

REFERENCES

[**1**] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Cambridge Univ. Press, 1987.

[**2**] S. Simić, *Orthogonal polynomials and regularly varying sequences*, Publ. Inst. Math. Belgrade (N.S) 70(84) (2001).

[**3**] G. Polya, G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Springer-Verlag, Berlin, 1964.

(received 20.10.2003, in revised form 26.10.2005)

Mathematical Institute, Kneza Mihaila 35/I, 11000 Belgrade, Serbia

E-mail: `ssimic@turing.mi.sanu.ac.yu`