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# ON PSEUDO-$BCI$ IDEALS OF PSEUDO-$BCI$ ALGEBRAS

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**Abstract.** The notions of pseudo-atoms, pseudo-$BCI$ ideals and pseudo-$BCI$ homomorphisms in pseudo-$BCI$ algebras are introduced. Characterizations of a pseudo-$BCI$ ideal are displayed, and conditions for a subset to be a pseudo-$BCI$ ideal are given. The concept of a $\diamond$-medial pseudo-$BCI$ algebra is also introduced, and its characterization is provided. We show that every pseudo-$BCI$ homomorphic image and preimage of a pseudo-$BCI$ ideal is also a pseudo-$BCI$ ideal.

## 1. Introduction

G. Georgescu and A. Iorgulescu [1] introduced the notion of a pseudo-$BCK$ algebra as an extended notion of $BCK$-algebras. In [2], Y. B. Jun, one of the present authors, gave a characterization of pseudo-$BCK$ algebra, and provided conditions for a pseudo-$BCK$ algebra to be $\wedge$-semi-lattice ordered (resp. $\cap$-semi-lattice ordered). Y. B. Jun et al. [4] introduced the notion of (positive implicative) pseudo-ideals in a pseudo-$BCK$ algebra, and then they investigated some of their properties. In [2], W. A. Dudek and Y. B. Jun introduced the notion of pseudo-$BCI$ algebras as an extension of $BCI$-algebras, and investigated some properties. In this paper, we introduce the concepts of pseudo-atoms, pseudo-$BCI$ ideals and pseudo-$BCI$ homomorphisms in pseudo-$BCI$ algebras. We display characterizations of a pseudo-$BCI$ ideal, and provide conditions for a subset to be a pseudo-$BCI$ ideal. We also introduced the notion of a $\diamond$-medial pseudo-$BCI$ algebra, and give its characterization. We prove that every pseudo-$BCI$ homomorphic image and preimage of a pseudo-$BCI$ ideal is also a pseudo-$BCI$ ideal.

## 2. Preliminaries

Recall that a *$BCI$-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$,

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- $((x*y)*(x*z))*(z*y)=0$,
- $(x*(x*y))*y=0$,
- $x*x=0$,
- $x*y=0$ and $y*x=0$ imply $x=y$.

For any $BCI$-algebra $X$, the relation $\leq$ defined by $x \leq y$ if and only if $x*y=0$ is a partial order on $X$. A nonempty subset $I$ of a $BCI$-algebra $X$ is called a $BCI$-*ideal* of $X$ if it satisfies

- $0 \in I$,
- $\forall x, y \in X$, $x*y \in I$, $y \in I \;\Rightarrow\; x \in I$.

## 3. Properties of Pseudo-$BCI$ algebras

Definition 3.1. *A pseudo-BCI algebra is a structure* $\mathfrak{X}=(X, \preceq, *, \diamond, 0)$, *where "$\preceq$" is a binary relation on a set $X$, "$*$" and "$\diamond$" are binary operations on $X$ and "0" is an element of $X$, verifying the axioms: for all $x, y, z \in X$,*

(a1) $(x*y)\diamond(x*z) \preceq z*y$, $(x\diamond y)*(x\diamond z) \preceq z\diamond y$,

(a2) $x*(x\diamond y) \preceq y$, $x\diamond(x*y) \preceq y$,

(a3) $x \preceq x$,

(a4) $x \preceq y$, $y \preceq x \Longrightarrow x=y$,

(a5) $x \preceq y \Longleftrightarrow x*y=0 \Longleftrightarrow x\diamond y=0$.

Note that every pseudo-$BCI$ algebra satisfying $x*y=x\diamond y$ for all $x, y \in X$ is a $BCI$-algebra. Every pseudo-$BCK$ algebra is a pseudo-$BCI$ algebra.

Proposition 3.2. [2] *In a pseudo-BCI algebra $\mathfrak{X}$ the following holds:*

(p1) $x \preceq 0 \;\Rightarrow\; x=0$.

(p2) $x \preceq y \;\Rightarrow\; z*y \preceq z*x,\; z\diamond y \preceq z\diamond x$.

(p3) $x \preceq y$, $y \preceq z \;\Rightarrow\; x \preceq z$.

(p4) $(x*y)\diamond z=(x\diamond z)*y$.

(p5) $x*y \preceq z \;\Leftrightarrow\; x\diamond z \preceq y$.

(p6) $(x*y)*(z*y) \preceq x*z,\; (x\diamond y)\diamond(z\diamond y) \preceq x\diamond z$.

(p7) $x \preceq y \;\Rightarrow\; x*z \preceq y*z,\; x\diamond z \preceq y\diamond z$.

(p8) $x*0=x=x\diamond 0$.

(p9) $x*\big(x\diamond(x*y)\big)=x*y$ *and* $x\diamond\big(x*(x\diamond y)\big)=x\diamond y$.

Example 3.3. Let $X=[0,\infty]$ and let $\leq$ be the usual order on $X$. Define binary operations "$*$" and "$\diamond$" on $X$ by

$$x*y:=\begin{cases} 0 & \text{if } x \leq y, \\ \frac{2x}{\pi}\arctan(\ln(\frac{x}{y})) & \text{if } y<x, \end{cases}$$

$$x\diamond y:=\begin{cases} 0 & \text{if } x \leq y, \\ xe^{-\tan(\frac{\pi y}{2x})} & \text{if } y<x, \end{cases}$$

for all $x, y \in X$. Then $\mathfrak{X} := (X, \leq, *, \diamond, 0)$ is a pseudo-$BCK$ algebra, and so a pseudo-$BCI$ algebra.

PROPOSITION 3.4 *In a pseudo-BCI algebra* $\mathfrak{X}$*, the following holds for all* $x, y \in X$*:*

(i) $0 * (x \diamond y) \preceq y \diamond x$.

(ii) $0 \diamond (x * y) \preceq y * x$.

(iii) $0 * (x * y) = (0 \diamond x) \diamond (0 * y)$.

(iv) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$.

*Proof.* (i) and (ii). We have $0 * (x \diamond y) = (x \diamond x) * (x \diamond y) \preceq y \diamond x$ and $0 \diamond (x * y) = (x * x) \diamond (x * y) \preceq y * x$ by (a1) and (a3).

(iii) and (iv). Using (a3) and (p4), we obtain

$$\begin{aligned}(0 \diamond x) \diamond (0 * y) &= (((x * y) * (x * y)) \diamond x) \diamond (0 * y)\\ &= (((x * y) \diamond x) * (x * y)) \diamond (0 * y)\\ &= (((x \diamond x) * y) * (x * y)) \diamond (0 * y)\\ &= ((0 * y) * (x * y)) \diamond (0 * y)\\ &= ((0 * y) \diamond (0 * y)) * (x * y)\\ &= 0 * (x * y)\end{aligned}$$

and

$$\begin{aligned}(0 * x) * (0 \diamond y) &= (((x \diamond y) \diamond (x \diamond y)) * x) * (0 \diamond y)\\ &= (((x \diamond y) * x) \diamond (x \diamond y)) * (0 \diamond y)\\ &= (((x * x) \diamond y) \diamond (x \diamond y)) * (0 \diamond y)\\ &= ((0 \diamond y) \diamond (x \diamond y)) * (0 \diamond y)\\ &= ((0 \diamond y) * (0 \diamond y)) \diamond (x \diamond y)\\ &= 0 \diamond (x \diamond y). \quad \blacksquare\end{aligned}$$

DEFINITION 3.5. An element $w$ of a pseudo-$BCI$ algebra $\mathfrak{X}$ is called a *pseudo-atom* if for every $x \in X$, $x \preceq w$ implies $x = w$.

Obviously, 0 is a pseudo-atom of $\mathfrak{X}$.

PROPOSIITON 3.6. *Let* $\mathfrak{X}$ *be a pseudo-BCI algebra. If an element* $w$ *of* $\mathfrak{X}$ *satisfies the identity* $y * (y \diamond (w * x)) = w * x$ *for all* $x, y \in X$*, then* $w$ *is a pseudo-atom of* $\mathfrak{X}$.

*Proof.* Let $y \in X$ be such that $y \preceq w$. Then

$$w = w * 0 = y * (y \diamond (w * 0)) = y * (y \diamond w) = y * 0 = y.$$

Hence $w$ is a pseudo-atom of $\mathfrak{X}$. $\blacksquare$

PROPOSITION 3.7. *Let* $\mathfrak{X}$ *be a pseudo-BCI algebra and let* $w$ *be a pseudo-atom of* $\mathfrak{X}$*. Then the following are true.*

(i) $w = x \diamond (x * w)$, $\forall x \in X$.
(ii) $(x * y) \diamond (x * w) = w * y$, $\forall x, y \in X$.
(iii) $w * (x \diamond y) \preceq y \diamond (x * w)$, $\forall x, y \in X$.
(iv) $(w \diamond x) * (y \diamond z) \preceq (z \diamond (y * w)) \diamond x$, $\forall x, y, z \in X$.
(v) $0 \diamond (y * w) = w * y$, $\forall y \in X$.

*Proof.* (i) Since $x \diamond (x * w) \preceq w$ by (a2), it follows that $w = x \diamond (x * w)$.

(ii) For every $x, y \in X$, we have

$$(x * y) \diamond (x * w) = (x \diamond (x * w)) * y = w * y$$

by (p4) and (i).

(iii) Using (i), (a2), (p4) and (p7), we have

$$w * (x \diamond y) = (x \diamond (x * w)) * (x \diamond y) = (x * (x \diamond y)) \diamond (x * w) \preceq y \diamond (x * w).$$

(iv) Using (p4), (p7) and (iii), we get

$$(w \diamond x) * (y \diamond z) = (w * (y \diamond z)) \diamond x \preceq (z \diamond (y * w)) \diamond x.$$

(v) For every $y \in X$, we obtain

$$\begin{aligned} w * y &= (w \diamond 0) * (y \diamond 0) && \text{by (p8)} \\ &\preceq (0 \diamond (y * w)) \diamond 0 && \text{by (iv)} \\ &= 0 \diamond (y * w) && \text{by (p8)} \\ &\preceq w * y, && \text{by Proposition 3.4(ii)} \end{aligned}$$

and so $0 \diamond (y * w) = w * y$. ■

Definition 3.8. A pseudo-$BCI$ algebra $\mathfrak{X}$ is said to be $\diamond$-*medial* if it satisfies the following identity:

(M1) $(x * y) \diamond (z * u) = (x * z) \diamond (y * u)$, $\forall x, y, z, u \in X$.

Proposition 3.9. *A pseudo-$BCI$ algebra $\mathfrak{X}$ is $\diamond$-medial if and only if it satisfies:*

(M2) $x \diamond (y * z) = (x * y) \diamond (0 * z)$, $\forall x, y, z \in X$.

*Proof.* Assume that $\mathfrak{X}$ is $\diamond$-medial. Putting $z = 0$ and $u = z$ in (M1) and using (p8), we have

$$(x * y) \diamond (0 * z) = (x * 0) \diamond (y * z) = x \diamond (y * z).$$

Suppose that $\mathfrak{X}$ satisfies the condition (M2). Then

$$\begin{aligned} (x * y) \diamond (z * u) &= (x \diamond (z * u)) * y && \text{by (p4)} \\ &= ((x * z) \diamond (0 * u)) * y && \text{by (M2)} \\ &= ((x * z) * y) \diamond (0 * u) && \text{by (p4)} \\ &= (x * z) \diamond (y * u). && \text{by (M2)} \end{aligned}$$

Therefore $\mathfrak{X}$ is $\diamond$-medial. ■

Proposition 3.10. *Every $\diamond$-medial pseudo-BCI algebra $\mathfrak{X}$ satisfies the following identities.*

(i) $x * y = 0 \diamond (y * x)$.

(ii) $0 \diamond (0 * x) = x$.

(iii) $x \diamond (x * y) = y$.

*Proof.* (i) For any $x, y \in X$, we have

$$\begin{aligned} x * y &= (x * y) \diamond 0 = (x * y) \diamond (x * x) \\ &= (x * x) \diamond (y * x) = 0 \diamond (y * x). \end{aligned}$$

(ii) If we put $y = 0$ in (i), then we have (ii).

(iii) Using (ii), (a3) and (p8), we get

$$x \diamond (x * y) = (x * 0) \diamond (x * y) = (x * x) \diamond (0 * y) = 0 \diamond (0 * y) = y. \quad \blacksquare$$

## 4. Pseudo-*BCI* ideals

Let $\mathfrak{X}$ be a pseudo-$BCI$ -algebra. For any nonempty subset $J$ of $X$ and any element $y$ of $X$, we denote

$$*(y, J) := \{x \in X \mid x * y \in J\} \text{ and } \diamond (y, J) := \{x \in X \mid x \diamond y \in J\}.$$

Note that $*(y, J) \cap \diamond(y, J) = \{x \in X \mid x * y \in J, x \diamond y \in J\}$.

Definition 4.1. A nonempty subset $J$ of a pseudo-$BCI$ algebra $\mathfrak{X}$ is called a *pseudo-BCI ideal* of $\mathfrak{X}$ if it satisfies

(I1) $0 \in J$,

(I2) $\forall y \in J$, $*(y, J) \subseteq J$ and $\diamond (y, J) \subseteq J$.

Note that if $\mathfrak{X}$ is a pseudo-$BCI$ algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notion of a pseudo-$BCI$ ideal and a $BCI$-ideal coincide.

Proposition 4.2. *Let $J$ be a pseudo-BCI ideal of a pseudo-BCI algebra $\mathfrak{X}$. If $x \in J$ and $y \preceq x$, then $y \in J$.*

*Proof* is straightforward. $\blacksquare$

Theorem 4.3. *For any element $a$ of a pseudo-BCI algebra $\mathfrak{X}$, the initial section $\downarrow a := \{x \in X \mid x \preceq a\}$ is a pseudo-BCI ideal of $\mathfrak{X}$ if and only if the following implications hold:*

(i) $\forall x, y, z \in X$, $x * y \preceq z$, $y \preceq z \Rightarrow x \preceq z$,

(ii) $\forall x, y, z \in X$, $x \diamond y \preceq z$, $y \preceq z \Rightarrow x \preceq z$.

*Proof.* Assume that for each $a \in X$, $\downarrow a$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$. Let $x, y, z \in X$ be such that $x * y \preceq z$, $x \diamond y \preceq z$, and $y \preceq z$. Then $x * y \in \downarrow z$, $x \diamond y \in \downarrow z$, and $y \in \downarrow z$, that is, $y \in \downarrow z$, $x \in *(y, \downarrow z)$ and $x \in \diamond(y, \downarrow z)$. Since $\downarrow z$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$, it follows from (I2) that $x \in \downarrow z$, i.e., $x \preceq z$. Conversely, consider $\downarrow z$ for any $z \in X$. Obviously $0 \in \downarrow z$. For every $y \in \downarrow z$, let $a \in *(y, \downarrow z)$ and $b \in \diamond(y, \downarrow z)$.

Then $a * y \in \downarrow z$ and $b \diamond y \in \downarrow z$, i.e., $a * y \preceq z$ and $b \diamond y \preceq z$. Since $y \in \downarrow z$, it follows from the hypothesis that $a \preceq z$ and $b \preceq z$, i.e., $a \in \downarrow z$ and $b \in \downarrow z$. This shows that $*(y, \downarrow z) \subseteq \downarrow z$ and $\diamond(y, \downarrow z) \subseteq \downarrow z$. Hence $\downarrow z$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$ for every $z \in X$. ■

THEOREM 4.4. *If $J$ is a pseudo-$BCI$ ideal of a pseudo-$BCI$ algebra $\mathfrak{X}$, then*

(i) $\forall x, y, z \in X,\ x, y \in J,\ z * y \preceq x \Rightarrow z \in J,$

(ii) $\forall a, b, c \in X,\ a, b \in J,\ c \diamond b \preceq a \Rightarrow c \in J.$

*Proof.* Suppose that $J$ is a pseudo-ideal of $\mathfrak{X}$ and let $x, y, z \in X$ be such that $x, y \in J$ and $z * y \preceq x$. Then $(z * y) \diamond x = 0 \in J$, and so $z * y \in \diamond(x, J) \subseteq J$. It follows that $z \in *(y, J) \subseteq J$ so that (i) is valid. Now let $a, b, c \in X$ be such that $a, b \in J$ and $c \diamond b \preceq a$. Then $(c \diamond b) * a = 0 \in J$, and thus $c \diamond b \in *(a, J) \subseteq J$. Hence $c \in \diamond(b, J) \subseteq J$, which shows (ii). ■

A *pseudo-$BCI$ subalgebra* of a pseudo-$BCI$ algebra $\mathfrak{X}$ is a subset $S$ of $\mathfrak{X}$ which satisfies $x * y \in S$ and $x \diamond y \in S$ for all $x, y \in S$. We provide conditions for a pseudo-$BCI$ subalgebra to be a pseudo-$BCI$ ideal.

THEOREM 4.5. *Let $J$ be a pseudo-$BCI$ subalgebra of a pseudo-$BCI$ algebra $\mathfrak{X}$. Then $J$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$ if and only if*

$$\forall x, y \in X,\ x \in J,\ y \in X - J \Rightarrow y * x \in X - J \text{ and } y \diamond x \in X - J.$$

*Proof.* Assume that $J$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$ and let $x, y \in X$ be such that $x \in J$ and $y \in X - J$. If $y * x \notin X - J$, then $y * x \in J$, i.e., $y \in *(x, J) \subseteq J$ which is a contradiction. Hence $y * x \in X - J$. Now if $y \diamond x \notin X - J$, then $y \diamond x \in J$ and so $y \in \diamond(x, J) \subseteq J$. This is a contradiction, and therefore $y \diamond x \in X - J$. Conversely, assume that

$$\forall x, y \in X,\ x \in J,\ y \in X - J \Rightarrow y * x \in X - J \text{ and } y \diamond x \in X - J.$$

Since $J$ is a pseudo-$BCI$ subalgebra, therefore $0 \in J$. For every $x \in J$, let $y \in *(x, J)$. Then $y * x \in J$. If $y \notin J$, then $y * x \in X - J$ by assumption. This is a contradiction, and so $y \in J$ which shows that $*(x, J) \subseteq J$. Now let $z \in \diamond(x, J)$. Then $z \diamond x \in J$. It follows from the hypothesis that $z \in J$ so that $\diamond(x, J) \subseteq J$. Consequently, $J$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$. ■

Using [2, Theorem 3.5], we know that every pseudo-$BCI$ algebra $\mathfrak{X}$ contains a maximal pseudo-$BCK$ algebra $K(\mathfrak{X}) := \{x \in X \mid 0 \preceq x\}$.

PROPOSITION 4.6. *Let $\mathfrak{X}$ be a pseudo-$BCI$ algebra. If $x \in K(\mathfrak{X})$ and $y \in X - K(\mathfrak{X})$, then $x * y \in X - K(\mathfrak{X})$ and $x \diamond y \in X - K(\mathfrak{X})$.*

*Proof.* If $x * y \in K(\mathfrak{X})$, then $x \diamond (x * y) \in K(\mathfrak{X})$ because $K(\mathfrak{X})$ is a pseudo-$BCI$ subalgebra of $\mathfrak{X}$. Hence $0 \preceq x \diamond (x * y) \preceq y$, and so $y \in K(\mathfrak{X})$. This is a contradiction. Now if $x \diamond y \in K(\mathfrak{X})$, then $x * (x \diamond y) \in K(\mathfrak{X})$ and so $0 \preceq x * (x \diamond y) \preceq y$ by (a2). Therefore $y \in K(\mathfrak{X})$, a contradiction. ■

THEOREM 4.7. *Let $\mathfrak{X}$ be a pseudo-$BCI$ algebra. Then the maximal pseudo-$BCK$ algebra $K(\mathfrak{X})$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$.*

*Proof.* Let $x, y \in X$ be such that $x \in K(\mathfrak{X})$ and $y \in X - K(\mathfrak{X})$. Using (a1) and (p8), we have

$$(y * x) \diamond y = (y * x) \diamond (y * 0) \preceq 0 * x = 0$$

and

$$(y \diamond x) * y = (y \diamond x) * (y \diamond 0) \preceq 0 \diamond x = 0$$

since $x \in K(\mathfrak{X})$. It follows from (p1) that $(y * x) \diamond y = 0$ and $(y \diamond x) * y = 0$ so that $y * x \preceq y$ and $y \diamond x \preceq y$. If $y * x \in K(\mathfrak{X})$, then $0 \preceq y * x \preceq y$, and so $y \in K(\mathfrak{X})$ which is a contradiction. Now if $y \diamond x \in K(\mathfrak{X})$, then $0 \preceq y \diamond x \preceq y$ which implies that $y \in K(\mathfrak{X})$, a contradiction. Hence $y * x \in X - K(\mathfrak{X})$ and $y \diamond x \in X - K(\mathfrak{X})$. By means of Theorem 4.5, we know that $K(\mathfrak{X})$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$. ■

THEOREM 4.8. *Let $J$ be a pseudo-$BCI$ ideal of a pseudo-$BCI$ algebra $\mathfrak{X}$. Then the following are equivalent.*

(i) *$J$ contains the maximal pseudo-$BCK$ algebra $K(\mathfrak{X})$.*

(ii) $\forall x, y \in X,\ x \preceq y,\ x \in J \Rightarrow y \in J$.

*Proof.* The sufficiency is straightforward. Assume that $K(\mathfrak{X}) \subset J$. Let $x, y \in X$ be such that $x \preceq y$ and $x \in J$. Then $x * y = 0$, and so

$$0 = 0 \diamond 0 = 0 \diamond (x * y) = (x * x) \diamond (x * y) \preceq y * x.$$

Thus $y * x \in K(\mathfrak{X}) \subset J$, which implies that $y \in *(x, J) \subseteq J$. ■

DEFINITION 4.9. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-$BCI$ algebras. A mapping $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is called a *pseudo-$BCI$ homomorphism* if $\mathfrak{f}(x*y) = \mathfrak{f}(x)*\mathfrak{f}(y)$ and $\mathfrak{f}(x\diamond y) = \mathfrak{f}(x)\diamond\mathfrak{f}(y)$ for all $x, y \in X$.

Note that if $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a pseudo-$BCI$ homomorphism, then $\mathfrak{f}(0_{\mathfrak{X}}) = 0_{\mathfrak{Y}}$ where $0_{\mathfrak{X}}$ and $0_{\mathfrak{Y}}$ are zero elements of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively.

THEOREM 4.10. *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo-$BCI$ homomorphism of pseudo-$BCI$ algebras $\mathfrak{X}$ and $\mathfrak{Y}$.* (i) *If $J$ is a pseudo-$BCI$ ideal of $\mathfrak{Y}$, then $\mathfrak{f}^{-1}(J)$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$.* (ii) *If $\mathfrak{f}$ is surjective and $I$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$, then $\mathfrak{f}(I)$ is a pseudo-$BCI$ ideal of $\mathfrak{Y}$.*

*Proof.* (i) Assume that $J$ is a pseudo-$BCI$ ideal of $\mathfrak{Y}$. Obviously $0_{\mathfrak{X}} \in \mathfrak{f}^{-1}(J)$. For every $y \in \mathfrak{f}^{-1}(J)$, let

$$a \in *(y, \mathfrak{f}^{-1}(J)) \text{ and } b \in \diamond(y, \mathfrak{f}^{-1}(J)).$$

Then $a * y \in \mathfrak{f}^{-1}(J)$ and $b \diamond y \in \mathfrak{f}^{-1}(J)$. It follows that $\mathfrak{f}(a) * \mathfrak{f}(y) = \mathfrak{f}(a * y) \in J$ and $\mathfrak{f}(b) \diamond \mathfrak{f}(y) = \mathfrak{f}(b \diamond y) \in J$ so that $\mathfrak{f}(a) \in *(\mathfrak{f}(y), J) \subseteq J$ and $\mathfrak{f}(b) \in \diamond(\mathfrak{f}(y), J) \subseteq J$ because $J$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$ and $\mathfrak{f}(y) \in J$. Hence $a \in \mathfrak{f}^{-1}(J)$ and $b \in \mathfrak{f}^{-1}(J)$, which shows that $*(y, \mathfrak{f}^{-1}(J)) \subseteq \mathfrak{f}^{-1}(J)$ and $\diamond(y, \mathfrak{f}^{-1}(J)) \subseteq \mathfrak{f}^{-1}(J)$. Hence $\mathfrak{f}^{-1}(J)$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$.

(ii) Assume that $\mathfrak{f}$ is surjective and let $I$ be a pseudo-$BCI$ ideal of $\mathfrak{X}$. Obviously, $0_{\mathfrak{Y}} \in \mathfrak{f}(I)$. For every $y \in \mathfrak{f}(I)$, let $a, b \in Y$ be such that $a \in *(y, \mathfrak{f}(I))$ and $b \in \diamond(y, \mathfrak{f}(I))$. Then $a * y \in \mathfrak{f}(I)$ and $b \diamond y \in \mathfrak{f}(I)$. It follows that there exist

$x_*, x_\diamond \in I$ such that $\mathfrak{f}(x_*) = a * y$ and $\mathfrak{f}(x_\diamond) = b \diamond y$. Since $y \in \mathfrak{f}(I)$, there exists $x_y \in I$ such that $\mathfrak{f}(x_y) = y$. Also since $\mathfrak{f}$ is surjective, there exist $x_a, x_b \in X$ such that $\mathfrak{f}(x_a) = a$ and $\mathfrak{f}(x_b) = b$. Hence

$$\mathfrak{f}(x_a * x_y) = \mathfrak{f}(x_a) * \mathfrak{f}(x_y) = a * y \in \mathfrak{f}(I)$$

and

$$\mathfrak{f}(x_b \diamond x_y) = \mathfrak{f}(x_b) \diamond \mathfrak{f}(x_y) = b \diamond y \in \mathfrak{f}(I),$$

which imply that $x_a * x_y \in I$ and $x_b \diamond x_y \in I$. Since $I$ is a pseudo-$BCI$ ideal of $\mathfrak{X}$, we get $x_a \in *(x_y, I) \subseteq I$ and $x_b \in \diamond(x_y, I) \subseteq I$, and thus $a = \mathfrak{f}(x_a) \in \mathfrak{f}(I)$ and $b = \mathfrak{f}(x_b) \in \mathfrak{f}(I)$. This shows that $*(y, \mathfrak{f}(I)) \subseteq \mathfrak{f}(I)$ and $\diamond(y, \mathfrak{f}(I)) \subseteq \mathfrak{f}(I)$. Therefore $\mathfrak{f}(I)$ is a pseudo-$BCI$ ideal of $\mathfrak{Y}$. ■

COROLLARY 4.11. *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo-BCI homomorphism of pseudo-BCI algebras $\mathfrak{X}$ and $\mathfrak{Y}$. Then the kernel*

$$\mathrm{Ker}(\mathfrak{f}) := \{x \in X \mid \mathfrak{f}(x) = 0_{\mathfrak{Y}}\}$$

*of $\mathfrak{f}$ is a pseudo-BCI ideal of $\mathfrak{X}$.*

*Proof* is straightforward. ■

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