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COMMON FIXED POINT THEOREMS FOR CONTRACTIVE MAPS

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Abstract. The aim of this paper is to obtain new common fixed point theorems under strict contractive conditions for three and four maps without continuity.

1. Introduction

With the advent of the notion of compatible maps due to Jungck [2, 3] the study of common fixed point theorems for contractive type maps has centered around the study of compatible maps and its weaker forms. However, the study of common fixed points of noncompatible maps is equally interesting and Pant [5, 6, 7, 8], Aamri and Moutawakil [1] and others have recently initiated work along these lines.

Pant [8] obtained a common fixed point theorem using the concept of R-weakly commuting maps for a pair of noncompatible maps with Lipschitz type maps without using completeness of the space and continuity of the maps involved. In a recent paper Aamri and Moutawakil [1] studied a new class of maps satisfying $(E\ A)$-property so that compatible and noncompatible maps may be studied together. We make use of this concept to obtain fixed point theorems for three and four maps without requiring continuity of maps. Further, in our formulation the completeness or compactness of the space is not needed. Our results extend and improve several known results.

We cite here the following result of Jungck [3] which was obtained for continuous maps on a compact space under very tight conditions.

Theorem J. *Let A, B, S and T be continuous self-maps of a compact metric space (X,d) with $AX \subset TX$ and $BX \subset SX$. If A,S and B,T are compatible pairs and*

$$d(Ax,By) < \max(Mxy), \quad \textit{when} \quad \max(Mxy) > 0, \tag{*}$$

where $Mxy = \{d(Sx,Ty), d(Ax,Sx), d(By,Ty), [d(By,Sx) + d(Ax,Ty)]/2\}$, then A, B, S and T have a unique common fixed point.

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For an excellent comparison of contractive maps (of the type $(*)$ with $S = T = id$ (the identity map), one may refer to Rhoades [12]. We remark that the condition $(*)$ with $A = B$ and $S = T = id$ is the condition (22) of Rhoades [op.cit.]. The condition (22) is well known to include several contractive conditions. Our results are obtained under slightly modified versions of $(*)$. Some special cases are discussed.

Fixed point theorems

Throughout this paper, let Y be an arbitrary nonempty set and (X, d) a metric space.

Definition 1. [2] Self-maps A and S of a metric space (X, d) are compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Notice that A and S will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent (see also [1], [11] and [13]).

Definition 2. [11] Self-maps A and S of a metric space (X, d) are R-weakly commuting at a point $x \in X$ if $d(ASx, SAx) \leqslant R\, d(Ax, Sx)$ for some $R > 0$. They are pointwise R-weakly commuting on X if given $x \in X$ there exists $R > 0$ such that $d(ASx, SAx) \leqslant R\, d(Ax, Sx)$.

Describing the importance of pointwise R-weakly commuting maps in fixed point considerations, Pant [5] has shown that compatible maps are necessarily pointwise R-weakly commuting but the reverse implication is not true (see also [14, p. 487]). Further, Singh and Tomar [15, p. 150] have noted that compatible maps are more general than R-weakly commuting maps. However, our formulations require only the commutativity of maps just at a coincidence point. Obviously the commutativity requirements in common fixed point considerations can not be weaker than this.

Definition 3. Let A and S be maps on Y with values in X. Then A and S are said to satisfy the $(E\,A)$-property if there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

If we take $Y = X$ then we get the definition of $(E\,A)$-property for two self-maps of X studied by Aamri and Moutawakil [1]. In such a situation, t is called a tangent point by Sastry and Murthy [13].

The following is our main result.

Theorem 1. *Let (X, d) be a metric space and $A, B, S, T \colon Y \to X$ such that*

(1.1) $d(Ax, By) < m(x, y)$, *when* $m(x, y) > 0$, *where*

$$m(x, y) = \max\{d(Sx, Ty), \alpha d(Ax, Sx), \alpha d(By, Ty), [d(By, Sx) + d(Ax, Ty)]/2\},$$

where $0 \leqslant \alpha < 1$;

(1.2) *one of the pairs* (A, S) *or* (B, T) *satisfies the* $(E\,A)$*-property;*

(1.3) $\overline{AY} \subset TY$ *and* $\overline{BY} \subset SY$.

Then:

(i) A *and* S *have a coincidence;*

(ii) B *and* T *have a coincidence.*

Further, if $Y = X$, *then*

(iii) A *and* S *have a common fixed point provided that* A *and* S *commute at their coincidence point;*

(iv) B *and* T *have a common fixed point provided that* B *and* T *commute at their coincidence point;*

(v) A, B, S *and* T *have a unique common fixed point if the pairs* (A, S) *and* (B, T) *are commuting at their coincidences.*

Proof. If the pair (B, T) satisfies the $(E\,A)$-property, then there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Since $\overline{BY} \subset SY$, for each x_n, there exists y_n in Y such that $Bx_n = Sy_n$, and $Sy_n \to t$ as well.

We show that $Ay_n \to t$. If not, there exist a subsequence $\{Ay_{n(i)}\}$ of $\{Ay_n\}$, a positive integer N, and a real number $r > 0$ such that for some positive integer $k \geqslant N$, we have $d(Ay_k, t) \geqslant r$, $d(Ay_k, Bx_k) \geqslant r$ and

$$d(Ay_k, Bx_k) < \max\{d(Sy_k, Tx_k), \alpha d(Ay_k, Sy_k), \alpha d(Bx_k, Tx_k), [d(Bx_k, Sy_k) + d(Ay_k, Tx_k)]/2\} = d(Ay_k, Bx_k) < d(Ay_k, Bx_k)$$

a contradiction, and $Ay_n \to t$.

Since $t \in \overline{BY}$ and $\overline{BY} \subset SY$, there exists an element $u \in Y$ such that $t = Su$. To show that $Au = Su$, we suppose otherwise and use the condition (1.1) to get

$$d(Au, Bx_n) < \max\{d(Su, Tx_n), \alpha d(Au, Su), \alpha d(Bx_n, Tx_n), [d(Bx_n, Su) + d(Au, Tx_n)]/2\}.$$

Making $n \to \infty$, $d(Au, Su) \leqslant \alpha d(Au, Su) < d(Au, Su)$, yielding $Au = Su$. This proves (i).

Since $\overline{AY} \subset TY$, there exists a point w in Y such that $Au = Tw$. If $Tw \neq Bw$, then by (1.1),

$$d(Au, Bw) < \max\{d(Su, Tw), \alpha d(Au, Su), \alpha d(Bw, Tw), [d(Bw, Su) + d(Au, Tw)]/2\} = d(Au, Bw) < d(Au, Bw).$$

Consequently $Tw = Au = Bw$. This proves (ii).

Now let $Y = X$. If A and S commute at their coincidence point u, then $AAu = ASu = SAu = SSu$, and by (1.1),

$$d(Au, AAu) = d(AAu, Bw) < \max\{d(SAu, Tw), \alpha d(AAu, SAu), \alpha d(Bw, Tw), [d(Bw, SAu) + d(AAu, Tw)]/2\} = d(Au, AAu).$$

This proves (iii). The proof of (iv) is analogous, and the proof of (v) is immediate. ∎

The following example shows that the pairs (A, S) and (B, T) need not have the same coincidence.

EXAMPLE 1. Let $X = [0, \infty)$ be endowed with the usual metric and $Y = [1/10, \infty)$. Define A, B, S, T from Y to X such that $Ax = x^2 + 2/9$, $Bx = x^3 + 2/9$, $Sx = 3x^2$ and $Tx = 3x^3$. Then $d(Ax, By) = |x^2 - y^3| < 3|x^2 - y^3| = d(Sx, Ty)$. So (1.1) and other hypotheses of Theorem 1 are satisfied. We see that $A(1/3) = S(1/3) = 1/3$ and $B(9^{-1/3}) = T(9^{-1/3}) = 1/3$, that is, A, S have a coincidence at $x = 1/3$ and B, T have a (different) coincidence at $x = (9^{-1/3})$.

In case $S = T$ in Theorem 1, we can obtain a slightly improved version which we state below.

THEOREM 2. *Let (X, d) be a metric space and $A, B, S\colon Y \to X$ such that (1.1) holds with $S = T$, and*

(2.1) *one of the pairs (A, S) or (B, S) satisfies the $(E\,A)$ property;*

(2.2) $\overline{AY} \cup \overline{BY} \subset SY$.

Then A, B and S have a coincidence. Further, if S commutes with each of A and B at their coincidences, then A, B and S have a unique common fixed point.

We remark that the following result is a good variant of the main result of Pant [8].

THEOREM 3. *Let A and S be noncompatible self-maps of a metric space (X, d) satisfying*

(3.1) $d(Ax, Ay) < m_a(x, y)$, *when* $m_a(x, y) > 0$, *where*

$$m_a(x, y) = \max\{d(Sx, Sy), \alpha d(Ax, Sx), \alpha d(Ay, Sy), [d(Ax, Sy) + d(Ay, Sx)]/2\},$$

where $0 \leqslant \alpha < 1$;

(3.2) $\overline{AX} \subset SX$.

Then $C(A, S)$ is nonempty. Further, A and S have a unique common fixed point provided that A and S commute at (some) $u \in C(A, S)$.

Proof. Since A and S are noncompatible, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent. Since $t \in \overline{AX}$ and $\overline{AX} \subset SX$, there exists a point $u \in X$ such that $t = Su$. Suppose $Au \neq Su$, then by (3.1),

$$d(Au, Ax_n) < \max\{d(Su, Sx_n), \alpha d(Au, Su), \alpha d(Ax_n, Sx_n), [d(Au, Sx_n) + d(Ax_n, Su)]/2\}.$$

Making $n \to \infty$ yields $d(Au, Su) \leqslant \alpha d(Au, Su) < d(Au, Su)$, and $Au = Su$. Consequently, $C(A, S)$ is nonempty.

Further, the commutativity of A and S at u implies $AAu = ASu = SAu = SSu$, and by (3.1),

$$d(Au, AAu) < \max\{d(Su, SAu), d(Au, Su), d(AAu, SAu), [d(Au, SAu) + d(AAu, Su)]/2\} = d(Au, AAu).$$

Consequently $Au = AAu = SAu$, and Au is a common fixed point of A and S. The uniqueness of the common fixed point follows easily. ∎

In view of the above proof, we have another interesting version of Theorem 3.

THEOREM 3-BIS. *Let A and S be self-maps of a metric space (X, d) satisfying the $(E\,A)$-property such that the conditions (3.1) and (3.2) hold. Then the conclusions of Theorem 3 are true.*

We remark that contractive conditions used by Aamri and Moutawakil [1, Th. 1, Cor. 1, Cor. 2 and Cor. 3] are particular cases of (3.1).

Recently Pant [8] obtained a common fixed point theorem for a pair of noncompatible pointwise R-weakly commuting Lipschitz maps A, S on a metric space X satisfying the following additional condition:

$$d(Ax, A^2x) \neq \max\{d(Ax, SAx), d(A^2x, SAx)\}. \tag{P}$$

The following example establishes the superiority of our Theorem 3 and 3-bis over Pant's theorem [op. cit.].

EXAMPLE 2. Let $X = [2, 20]$ be endowed with the usual metric. $Ax = 2$ if $x = 2$ or $x > 5$, $Ax = 6$ if $2 < x \leqslant 5$, and $S2 = 2$, $Sx = 16$ if $2 < x \leqslant 5$, $Sx = 6$ if $5 < x < 7$, $S7 = 7$, $Sx = \left(\dfrac{x+1}{4}\right)$ if $x > 7$.

We take a sequence $\{x_n = 7 + \| frac1n : n \geqslant 1\}$ to see that the maps A and S are noncompatible. It is not difficult to see that maps A and S satisfy all the hypotheses of Theorem 3 and 3-bis. Notice that the condition (P) is not satisfied (take $x = 3$).

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