МАТЕМАТИЧКИ ВЕСНИК
58 (2006), 119–124

UDK 517.547
оригинални научни рад
research paper

ON CERTAIN NEW SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In the present paper, the authors introduce a new subclass $\mathcal{K}_s^{(k)}(\alpha,\beta)$ of close-to-convex functions. The subordination and inclusion relationship, and some coefficient inequalities for this class are provided. The results presented here would provide extensions of those given in earlier works.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbf{C} : |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{K}$ denote the usual subclasses of $\mathcal{A}$ whose members are univalent, starlike and close-to-convex in $\mathcal{U}$, respectively. Also let $\mathcal{S}^*(\alpha)$ denote the class of starlike functions of order α, $0 \leq \alpha < 1$.

Sakaguchi [4] introduced a class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points, consisting of functions $f(z) \in \mathcal{S}$ satisfying

$$\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0 \quad (z \in \mathcal{U}).$$

In a later paper, Gao and Zhou [1] discussed a class $\mathcal{K}_s$ of analytic functions related to the starlike functions, that is the subclass of $f(z) \in \mathcal{S}$ satisfying the following inequality

$$\Re\left\{\frac{z^2 f'(z)}{g(z)g(-z)}\right\} < 0 \quad (z \in \mathcal{U}),$$

where $g(z) \in \mathcal{S}^*(\frac{1}{2})$.

AMS Subject Classification: 30C45.

Keywords and phrases: Starlike functions, close-to-convex functions, differential subordination.

This work was supported by the Scientific Research Fund of Hunan Provincial Education Department and the Hunan Provincial Natural Science Foundation (No. 05JJ30013) of People's Republic of China.

More recently, Wang, Gao and Yuan [6] discussed a subclass $\mathcal{K}_s(\alpha,\beta)$ of the class $\mathcal{K}_s$, that is the subclass of $f(z)\in\mathcal{S}$ satisfying the following inequality

$$\left|\frac{z^2f'(z)}{g(z)g(-z)}+1\right|<\beta\left|\frac{\alpha z^2f'(z)}{g(z)g(-z)}-1\right| \quad (z\in\mathcal{U}),$$

where $0\le\alpha\le1,\ 0<\beta\le1$ and $g(z)\in\mathcal{S}^*(\frac{1}{2})$. Note that $\mathcal{K}_s(1,1)=\mathcal{K}_s$, so $\mathcal{K}_s(\alpha,\beta)$ is a generalization of $\mathcal{K}_s$.

Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $|\omega(z)|\le|z|$ and $f(z)=F(\omega(z))$, denoted by $f\prec F$ or $f(z)\prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0)=F(0)$ and $f(\mathcal{U})\subset F(\mathcal{U})$ (see [3]).

In the present paper, we introduce and investigate the following class of analytic functions, and obtain some interesting results.

Definition 1. Let $\mathcal{K}_s^{(k)}(\alpha,\beta)$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\left|\frac{z^kf'(z)}{g_k(z)}-1\right|<\beta\left|\frac{\alpha z^kf'(z)}{g_k(z)}+1\right| \quad (z\in\mathcal{U}), \tag{1.1}$$

where $0\le\alpha\le1,\ 0<\beta\le1,\ g(z)\in\mathcal{S}^*(\frac{k-1}{k}),\ k\ge1$ is a fixed positive integer and $g_k(z)$ is defined by the following equality

$$g_k(z)=\prod_{\nu=0}^{k-1}\varepsilon^{-\nu}g(\varepsilon^{\nu}z) \quad (\varepsilon^k=1). \tag{1.2}$$

Note that $\mathcal{K}_s^{(2)}(\alpha,\beta)=\mathcal{K}_s(\alpha,\beta)$, so $\mathcal{K}_s^{(k)}(\alpha,\beta)$ is a generalization of $\mathcal{K}_s(\alpha,\beta)$.

In the present paper, we shall provide the subordination and inclusion relationships, and some coefficient inequalities for the class $\mathcal{K}_s^{(k)}(\alpha,\beta)$. The results presented here would provide extensions of those given in earlier works.

2. Coefficient estimate

We first give two meaningful conclusions about the class $\mathcal{K}_s^{(k)}(\alpha,\beta)$. The proof of Theorem 1 below is much akin to that of Theorem 1 in [6], here we omit the details.

Theorem 1. *A function $f(z)\in\mathcal{K}_s^{(k)}(\alpha,\beta)$ if and only if*

$$\frac{z^kf'(z)}{g_k(z)}\prec\frac{1+\beta z}{1-\alpha\beta z} \quad (z\in\mathcal{U}). \tag{2.1}$$

Remark 1. From Theorem 1, we know that

$$\Re\left\{\frac{zf'(z)}{g_k(z)/z^{k-1}}\right\}>0 \quad (z\in\mathcal{U}), \tag{2.2}$$

because of $\Re\left\{\dfrac{1+\beta z}{1-\alpha\beta z}\right\}>0\ (z\in\mathcal{U})$.

In order to prove our next theorem, we shall require the following lemma.

Lemma 1. *Let* $\psi_i(z) \in \mathcal{S}^*(\alpha_i)$, *where* $0 \leq \alpha_i < 1$ $(i = 0, 1, \dots, k-1)$. *Then for* $k - 1 \leq \sum\limits_{i=0}^{k-1} \alpha_i < k$, *we have*

$$\frac{\prod_{i=0}^{k-1} \psi_i(z)}{z^{k-1}} \in \mathcal{S}^* \left(\sum_{i=0}^{k-1} \alpha_i - (k-1) \right).$$

Proof. Since $\psi_i(z) \in \mathcal{S}^*(\alpha_i)$ $(i = 0, 1, \dots, k-1)$, we have

$$\Re \left\{ \frac{z\psi_0'(z)}{\psi_0(z)} \right\} > \alpha_0, \quad \Re \left\{ \frac{z\psi_1'(z)}{\psi_1(z)} \right\} > \alpha_1, \ \dots, \ \Re \left\{ \frac{z\psi_{k-1}'(z)}{\psi_{k-1}(z)} \right\} > \alpha_{k-1}. \tag{2.3}$$

We now let

$$F(z) = \frac{\psi_0(z)\psi_1(z)\cdots\psi_{k-1}(z)}{z^{k-1}}. \tag{2.4}$$

Differentiating (2.4) logarithmically, we have

$$\frac{zF'(z)}{F(z)} = \frac{z\psi_0'(z)}{\psi_0(z)} + \frac{z\psi_1'(z)}{\psi_1(z)} + \cdots + \frac{z\psi_{k-1}'(z)}{\psi_{k-1}(z)} - (k-1), \tag{2.5}$$

from (2.5) together with (2.3), we can get

$$\begin{aligned} \Re \left\{ \frac{zF'(z)}{F(z)} \right\} &= \Re \left\{ \frac{z\psi_0'(z)}{\psi_0(z)} \right\} + \Re \left\{ \frac{z\psi_1'(z)}{\psi_1(z)} \right\} + \cdots + \Re \left\{ \frac{z\psi_{k-1}'(z)}{\psi_{k-1}(z)} \right\} - (k-1) \\ &> \sum_{i=0}^{k-1} \alpha_i - (k-1). \end{aligned}$$

Thus, if $0 \leq \sum\limits_{i=0}^{k-1} \alpha_i - (k-1) < 1$, we know that

$$F(z) = \frac{\prod_{i=0}^{k-1} \psi_i(z)}{z^{k-1}} \in \mathcal{S}^* \left(\sum_{i=0}^{k-1} \alpha_i - (k-1) \right). \quad \blacksquare$$

Theorem 2. *Let* $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \left(\frac{k-1}{k} \right)$, *then* $g_k(z)/z^{k-1} \in \mathcal{S}^* \subset \mathcal{S}$.

Proof. From (1.2), we know

$$\begin{aligned} \frac{g_k(z)}{z^{k-1}} &= \frac{\prod_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^\nu z)}{z^{k-1}} = \frac{\prod_{\nu=0}^{k-1} \varepsilon^{-\nu} \left[\varepsilon^\nu z + \sum_{n=2}^{\infty} b_n (\varepsilon^\nu z)^n \right]}{z^{k-1}} \\ &= \frac{\prod_{\nu=0}^{k-1} \left[z + \sum_{n=2}^{\infty} b_n \varepsilon^{(n-1)\nu} z^n \right]}{z^{k-1}}. \end{aligned} \tag{2.6}$$

Now, suppose that

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \left(\frac{k-1}{k} \right).$$

Then, by Lemma 1 and equality (2.6), we can get the assertion of Theorem 2 easily. ■

Remark 2. From Theorem 2 and inequality (2.2), we know that if $f(z) \in \mathcal{K}_s^{(k)}(\alpha,\beta)$, then $f(z)$ is a close-to-convex function. So $\mathcal{K}_s^{(k)}(\alpha,\beta)$ is a subclass of the class of close-to-convex functions.

In order to give the coefficient estimate of $f(z) \in \mathcal{K}_s^{(k)}(\alpha,\beta)$, we shall require the following lemma.

Lemma 2 [5] *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, and satisfy the inequality*

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < \beta \left|\frac{\alpha z f'(z)}{g(z)} + 1\right| \quad (z \in \mathcal{U}),$$

where $0 \le \alpha \le 1$, $0 < \beta \le 1$. Then for $n \ge 2$, we have

$$|na_n - b_n|^2 \le 2(1+\alpha\beta^2) \sum_{k=1}^{n-1} k\,|a_k|\,|b_k| \quad (|a_1| = |b_1| = 1). \tag{2.7}$$

We now give the following theorem.

Theorem 3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, and satisfy the inequality (1.1). Then for $n \ge 2$, we have*

$$|na_n - B_n|^2 \le 2(1+\alpha\beta^2) \sum_{k=1}^{n-1} k\,|a_k|\,|B_k| \quad (|a_1| = |B_1| = 1), \tag{2.8}$$

where B_n is given by (2.11).

Proof. Note that inequality (1.1) can be written as

$$\left|\frac{zf'(z)}{g_k(z)/z^{k-1}} - 1\right| < \beta \left|\frac{\alpha z f'(z)}{g_k(z)/z^{k-1}} + 1\right|. \tag{2.9}$$

At the same time, we know that equality (2.6) can be written as

$$\frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} B_n z^n \in \mathcal{S}^* \subset \mathcal{S}. \tag{2.10}$$

Now, suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ and satisfy (2.9). So $f(z)$ and $g_k(z)/z^{k-1}$ satisfy the condition of Lemma 2. Thus, from (2.7), we can get (2.8) easily. ■

3. Inclusion relationship

In order to give the inclusion relationship for the class $\mathcal{K}_s(\lambda,\alpha,\beta)$, we shall require the following lemma.

Lemma 3. [2] *Let* $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$. *Then*

$$\frac{1+A_1z}{1+B_1z} \prec \frac{1+A_2z}{1+B_2z}.$$

Theorem 4. *Let* $0 \le \alpha_1 \le \alpha_2 \le 1$ *and* $0 < \beta_1 \le \beta_2 \le 1$. *Then we have*

$$\mathcal{K}_s^{(k)}(\alpha_1,\beta_1) \subset \mathcal{K}_s^{(k)}(\alpha_2,\beta_2).$$

Proof. Suppose that $f(z) \in \mathcal{K}_s^{(k)}(\alpha_1,\beta_1)$. By Theorem 1 we have

$$\frac{z^kf'(z)}{g_k(z)} \prec \frac{1+\beta_1z}{1-\alpha_1\beta_1z}.$$

Since $0 \le \alpha_1 \le \alpha_2 \le 1$ and $0 < \beta_1 \le \beta_2 \le 1$, we have

$$-1 \le -\alpha_2\beta_2 \le -\alpha_1\beta_1 < \beta_1 \le \beta_2 \le 1.$$

Thus, by Lemma 3, we have

$$\frac{z^kf'(z)}{g_k(z)} \prec \frac{1+\beta_2z}{1-\alpha_2\beta_2z},$$

that is $f(z) \in \mathcal{K}_s^{(k)}(\alpha_2,\beta_2)$. This means that $\mathcal{K}_s^{(k)}(\alpha_1,\beta_1) \subset \mathcal{K}_s^{(k)}(\alpha_2,\beta_2)$. Hence the proof is complete. ∎

Taking $k = 2$ in Theorem 4, we have

Corollary 1. *Let* $0 \le \alpha_1 \le \alpha_2 \le 1$ *and* $0 < \beta_1 \le \beta_2 \le 1$. *Then we have*

$$\mathcal{K}_s(\alpha_1,\beta_1) \subset \mathcal{K}_s(\alpha_2,\beta_2).$$

4. Sufficient condition

At last, we give the sufficient condition for functions belonging to the class $\mathcal{K}_s^{(k)}(\alpha,\beta)$.

Theorem 5. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$, $g(z) = z + \sum_{n=2}^{\infty} b_nz^n$ *be analytic in* $\mathcal{U}$. *If for* $0 \le \alpha \le 1$, $0 < \beta \le 1$, *we have*

$$\sum_{n=2}^{\infty} n(1+\alpha\beta)\,|a_n| + \sum_{n=2}^{\infty}(1+\beta)\,|B_n| \le (1+\alpha)\beta, \tag{4.1}$$

where B_n *is given by (2.10), then* $f(z) \in \mathcal{K}_s^{(k)}(\alpha,\beta)$.

Proof. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$. In view of (2.9) and (2.10), let M be denoted by

$$\begin{aligned} M &= \left|zf'(z) - \frac{g_k(z)}{z^{k-1}}\right| - \beta\left|\alpha zf'(z) + \frac{g_k(z)}{z^{k-1}}\right| \\ &= \left|z + \sum_{n=2}^{\infty} na_nz^n - z - \sum_{n=2}^{\infty} B_nz^n\right| - \beta\left|\alpha z + \sum_{n=2}^{\infty} n\alpha a_nz^n + z + \sum_{n=2}^{\infty} B_nz^n\right|. \end{aligned}$$

Thus, for $|z| = r < 1$, we have

$$M \leq \sum_{n=2}^{\infty} n\,|a_n|\,r^n + \sum_{n=2}^{\infty} |B_n|\,r^n - \beta\left[(1+\alpha)r - \sum_{n=2}^{\infty} n\alpha\,|a_n|\,r^n - \sum_{n=2}^{\infty} |B_n|\,r^n\right]$$
$$< \left[-(1+\alpha)\beta + \sum_{n=2}^{\infty} n(1+\alpha\beta)\,|a_n| + \sum_{n=2}^{\infty} (1+\beta)\,|B_n|\right] r.$$

From inequality (4.1), we know that $M < 0$. Thus, we have

$$\left|\frac{z^k f'(z)}{g_k(z)} - 1\right| < \beta\left|\frac{\alpha z^k f'(z)}{g_k(z)} + 1\right| \quad (z \in \mathcal{U}),$$

that is $f(z) \in \mathcal{K}_s^{(k)}(\alpha,\beta)$. This completes the proof of Theorem 5. ■

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(received 01.06.2005)

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