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SPACES RELATED TO γ-SETS

Filippo Cammaroto[1] **and Ljubiša D.R. Kočinac**[2]

Abstract. We characterize Ramsey theoretically two classes of spaces which are related to γ-sets.

1. Introduction

The notation and terminology are mainly as in [2]. X will denote an infinite Hausdorff topological space.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose members are families of subsets of an infinite set X. Then (see [7], [4]):

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection principle:

For each sequence $(A_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbf{N})$ such that for each $n \in \mathbf{N}$, $b_n \in A_n$ and $\{b_n : n \in \mathbf{N}\} \in \mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:

For each sequence $(A_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbf{N})$ of finite (not necessarily non-empty) sets such that for each $n \in \mathbf{N}$, $B_n \subset A_n$ and $\bigcup_{n\in\mathbf{N}} B_n$ is an element of $\mathcal{B}$.

The symbol $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ [7] denotes an infinitely long game for two players, ONE and TWO, which play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots ; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbf{N}\} \in \mathcal{B}$; otherwise, ONE wins.

If ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, then the selection hypothesis $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is true, but the converse need not be always true. In many cases the game characterizes the corresponding selection principle.

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For positive integers n and m the symbol $\mathcal{A} \to (\mathcal{B})^n_m$ denotes the statement:

For each $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1, \cdots, m\}$ there are a set $B \subset A$, $B \in \mathcal{B}$, and an $i \in \{1, \cdots, m\}$ such that for each $Y \in [B]^n$, $f(Y) = i$.

Here $[A]^n$ denotes the set of n-element subsets of A. We call f a "coloring" and say that "B is homogeneous of color i for f".

This symbol is called the *ordinary partition symbol* [7]. Several selection principles of the form $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ have been characterized by the ordinary partition relation (see [7], [4], [6]).

An open cover $\mathcal{U}$ of a space X is an *ω-cover* (resp. *k-cover*) if X does not belong to $\mathcal{U}$ and every finite (resp. compact) subset of X is contained in a member of $\mathcal{U}$. Because we deal with k-covers, we assume that spaces we consider are (infinite) non-compact. An open cover $\mathcal{U}$ of X is called a *γ-cover* [3] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Notice that it is equivalent to the assertion: Each finite subset of X belongs to all but finitely many members of $\mathcal{U}$. An open cover of a space X is called a *γ_k-cover* of X if each compact subset of X is contained in all but finitely many elements of $\mathcal{U}$ and X is not a member of the cover ([5]).

We suppose that all covers are *countable*. Recall that spaces in which every open k-cover contains a countable k-subcover are called *k-Lindelöf*.

For a topological space X we denote:

1. Ω – the family of ω-covers of X;
2. $\mathcal{K}$ – the family of k-covers of X;
3. Γ – the family of γ-covers of X;
4. Γ_k – the family of γ_k-covers of X.

Let us observe that we have

$$\Gamma_k \subset \Gamma \subset \Omega, \quad \Gamma_k \subset \mathcal{K} \subset \Omega.$$

In [3], Gerlits and Nagy introduced the following notion: a space X is a *γ-space* (or a *γ-set*) if each ω-cover $\mathcal{U}$ of X contains a countable family $\{U_n : n \in \mathbf{N}\}$ which is a γ-cover of X. They have also proved that the γ-set property of a space X is equivalent to the statement that X satisfies the selection property $\mathsf{S}_1(\Omega, \Gamma)$. It was shown in [4] that the γ-set property is equivalent also to the selection hypothesis $\mathsf{S}_{fin}(\Omega, \Gamma)$.

In [7], it was proved:

Theorem 1. *For a space X the following statements are equivalent:*

(a) *X is a γ-set;*

(b) *ONE has no winning strategy in the game $\mathsf{G}_1(\Omega, \Gamma)$ on X;*

(c) *For all $n, m \in \mathbf{N}$, X satisfies $\Omega \to (\Gamma)^n_m$.*

We shall prove here that similar results are true for two recently introduced classes of spaces which are similar to γ-sets. In fact, we give Ramsey theoretical characterizations of those classes of spaces.

2. k-γ-sets

In [1], the class of *k-γ-sets* was introduced as the class of $\mathsf{S}_1(\mathcal{K},\Gamma)$-sets and the following result regarding that class of spaces was shown.

THEOREM 2. *For a space X the following are equivalent:*

(1) *X is a k-γ-set;*

(2) *X satisfies $\mathsf{S}_{fin}(\mathcal{K},\Gamma)$;*

(3) *ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{K},\Gamma)$ on X.*

We show here that this class of spaces also can be described Ramsey-theoretically.

THEOREM 3. *For a k-Lindelöf space X the following are equivalent:*

(1) *X is a k-γ-set;*

(2) *For positive integers n and m, X satisfies $\mathcal{K} \to (\Gamma)^n_m$.*

Proof. We consider the case $n = m = 2$, because the general case can be easily obtained from it by standard induction arguments.

(1) $\Rightarrow$ (2): Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a countable k-cover of X and let $f : [\mathcal{U}]^2 \to \{1,2\}$ be a coloring. For $j \in \{1,2\}$ let $\mathcal{H}_j = \{V \in \mathcal{U} : f(\{U_1, V\}) = j\}$. Then at least one of the sets $\mathcal{H}_1$ and $\mathcal{H}_2$ is a k-cover of X. Denote such a set by $\mathcal{U}_1$ and the corresponding j by i_1. In a similar way define inductively sets $\mathcal{U}_n$ of k-covers of X and elements i_n from $\{1,2\}$ such that

$$\mathcal{U}_n = \{V \in \mathcal{U}_{n-1} : f(\{U_n, V\}) = i_n\}.$$

Apply now the $\mathsf{S}_1(\mathcal{K},\Gamma)$ property of X to the sequence $(\mathcal{U}_n : n \in \mathbf{N})$ to choose for each $n \in \mathbf{N}$ a $V_n \in \mathcal{U}_n$ such that $\mathcal{V} = \{V_n : n \in \mathbf{N}\} \in \Gamma$. Consider now the sets $\mathcal{V}_1 := \{V_m \in \mathcal{V} : i_m = 1\}$ and $\mathcal{V}_2 := \{V_m \in \mathcal{V} : i_m = 2\}$. At least one of them is infinite and so is a γ-cover of X, as each infinite subset of a γ-cover is also a γ-cover. So, one may suppose that there is an $i \in \{1,2\}$ satisfying: for each $U_m \in \mathcal{V}$, $i_m = i$. We have that $f(\{A,B\}) = i$ for each $\{A,B\} \in [\mathcal{V}]^2$.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of countable k-covers of X and let us suppose that for each $n \in \mathbf{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbf{N}\}$. Define now $\mathcal{V}$ to be the family of all nonempty sets of the form $U_{1,n} \cap U_{n,m}$, $n, m \in \mathbf{N}$. Clearly, $\mathcal{V}$ is a k-cover of X. Let $f : [\mathcal{V}]^2 \to \{1,2\}$ be define by

$$f(U_{1,n_1} \cap U_{n_1,m}, U_{1,n_2} \cap U_{n_2,l}) = \begin{cases} 1, & \text{if } n_1 = n_2, \\ 2, & \text{otherwise.} \end{cases}$$

Apply $\mathcal{K} \to (\Gamma)^2_2$ to find a γ-cover $\mathcal{W} \subset \mathcal{V}$ and an $i \in \{1,2\}$ such that whenever U and V are from $\mathcal{W}$, then $f(\{U,V\}) = i$. Consider two possibilities:

(i) $i = 1$: Then there is some $n \in \mathbf{N}$ such that for each $W \in \mathcal{W}$ we have $W \subset U_{1,n}$. However, this implies that $\mathcal{W}$ is not a (γ-) cover of X and this contradiction shows that this case is impossible.

(ii) $i = 2$: Whenever $W \in \mathcal{W}$ is of the form $U_{1,n} \cap U_{n,m}$ choose (one element) $H_n = U_{n,m} \in \mathcal{U}_n$. Otherwise, let $H_n = \emptyset$. Then the set $\mathcal{H} := \{H_n : n \in \mathbf{N}\}$ is a γ-cover of X (because $\mathcal{W}$ refines $\mathcal{H}$), and the sequence $(H_n : n \in \mathbf{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbf{N})$ that X satisfies $\mathsf{S}_{fin}(\mathcal{K}, \Gamma)$. By Theorem 2 it is equivalent to $\mathsf{S}_1(\mathcal{K}, \Gamma)$, i.e. (1) holds. ■

3. γ_k'-sets

The following class of spaces was introduced in [5]. A space X is said to be a γ_k'-*set* if it satisfies the selection hypothesis $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$.

A characterization of γ_k'-sets from [5] is given in the next theorem.

THEOREM 4. *For a space X the following are equivalent:*

(1) *X is a γ_k'-set;*

(2) *X satisfies $\mathsf{S}_{fin}(\mathcal{K}, \Gamma_k)$;*

(3) *ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{K}, \Gamma_k)$ played on X.*

We give now a Ramsey-theoretical characterization of γ_k'-sets.

THEOREM 5. *For a k-Lindelöf space X the following are equivalent:*

(1) *X is a γ_k'-set;*

(2) *For all $n, m \in \mathbf{N}$, X satisfies $\mathcal{K} \to (\Gamma_k)^n_m$.*

Proof. We again consider only the case $n = m = 2$.

(1) ⇒ (2): We shall use that (1) is equivalent to the fact that ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{K}, \Gamma_k)$ played on X (Theorem 4).

Suppose $\mathcal{U} = \{U_1, U_2, \cdots\}$ is a k-cover of X and let $f : [\mathcal{U}]^2 \to \{1, 2\}$ be a coloring. Let us define a strategy σ for ONE in the game $\mathsf{G}_1(\mathcal{K}, \Gamma_k)$.

In the first round ONE plays $\sigma(\emptyset) = \mathcal{U}$. Then choose $i_n \in \{1, 2\}$, $n \in \mathbf{N}$, such that $\sigma(U_n) = \{V \in \mathcal{U} : \sigma(\{U_n, V\}) = i_n\}$ is a k-cover of X (see the proof of Theorem 3). Let us write $\sigma(U_n) = \{U_{n,m} : m \in \mathbf{N}\}$. Suppose for each finite sequence $(n_1, \ldots, n_p)$ of natural numbers we have defined sets $U_{n_1,\ldots,n_p}$ and $i_{n_1,\ldots,n_{p-1}} \in \{1, 2\}$ satisfying the condition $\{U_{n_1,\ldots,n_p,m} : m \in \mathbf{N}\}$ is a k-cover of X which is equal to the set

$$\{V \in \sigma(U_{n_1}, U_{n_1,n_2}, \ldots, U_{n_1,n_2,\ldots,n_p}) : f(\{U_{n_1,n_2,\ldots,n_p}, V\}) = i_{n_1,n_2,\ldots,n_p}\}.$$

In this way one defines a strategy σ for ONE in $\mathsf{G}_1(\mathcal{K}, \Gamma_k)$. As ONE has no winning strategy, there is a play (for TWO)

$$U_{n_1}, U_{n_1,n_2}, \ldots, U_{n_1,n_2,\ldots,n_m}, \ldots$$

which defeats this strategy. The set $\{U_{n_1}, \ldots, U_{n_1,n_2,\ldots,n_m}, \ldots\}$ is a γ_k-cover of X. Besides, if $p < q$, then

$$f(\{U_{n_1,n_2,\ldots,n_p}, U_{n_1,n_2,\ldots,n_q}\}) = i_{n_1,n_2,\ldots,n_p}.$$

We may choose $i \in \{1,2\}$ such that for infinitely many m we have $i_{n_1,n_2,\dots,n_m} = i$. Then define

$$\mathcal{V} = \{U_{n_1,n_2,\dots,n_m} : i_{n_1,n_2,\dots,n_m} = i\} \subset \mathcal{U}.$$

This set is a γ_k-cover of X (because an infinite subset of a γ_k-cover is also a γ_k-cover) and, by construction, is homogeneous for f of color i.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of countable k-covers of X and suppose that for each n, $\mathcal{U}_n = \{U_{n;m} : m \in \mathbf{N}\}$. Consider now the set $\mathcal{V}$ of all nonempty sets of the form $U_{1;m} \cap U_{m;k}$, $n,k \in \mathbf{N}$. Clearly, $\mathcal{V}$ is a k-cover of X. Define $f : [\mathcal{V}]^2 \to \{1,2\}$ by

$$f(U_{1;n_1} \cap U_{n_1;k}, U_{1;n_2} \cap U_{n_2;m}) = \begin{cases} 1, & \text{if } n_1 = n_2, \\ 2, & \text{otherwise.} \end{cases}$$

Since $\mathcal{K} \to (\Gamma_k)^2_2$ holds there are $j \in \{1,2\}$ and a homogeneous for f of color j collection $\mathcal{W} \subset \mathcal{V}$ such that $\mathcal{W} \in \Gamma_k$. Consider two possibilities:

(i) $j = 1$: Then there is some n such that for each $W \in \mathcal{W}$ we have $W \subset U_{1,n}$. However, this means that $\mathcal{W}$ is not a γ_k-cover of X and we have a contradiction which shows that this case is impossible.

(ii) $j = 2$: For each $W \in \mathcal{W}$ choose, when it is possible, $U_{n;k_n}$ to be the second term in the chosen representation of W; otherwise let $U_{n;k_n} = \emptyset$. Let $\mathcal{V}'$ be the set of all U_{n,k_n}'s chosen in this way. Then $\mathcal{V}'$ is a γ_k-cover of X witnessing for $(\mathcal{U}_n : n \in \mathbf{N})$ that X satisfies $\mathsf{S}_{fin}(\mathcal{K}, \Gamma_k)$. Apply now Theorem 4

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Filippo Cammaroto, Dipartimento di Matematica, Università di Messina, 98166 Messina, Italia

E-mail: camfil@unime.it

Ljubiša D.R. Kočinac, Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia

E-mail: lkocinac@ptt.yu