МАТЕМАТИЧКИ ВЕСНИК
58 (2006), 131–138

UDK 515.122
оригинални научни рад
research paper

METRIZABLE GROUPS AND STRICT o-BOUNDEDNESS

Liljana Babinkostova

Abstract. We show that for metrizable topological groups being a strictly o-bounded group is equivalent to being a Hurewicz group. In [5] Hernandez, Robbie and Tkachenko ask if there are strictly o-bounded groups G and H for which $G \times H$ is not strictly o-bounded. We show that for metrizable strictly o-bounded groups the answer is no. In the same paper the authors also ask if the product of an o-bounded group with a strictly o-bounded group is again an o-bounded group. We show that if the strictly o-bounded group is metrizable, then the answer is yes.

1. Definitions and notation

Let H and G be topological spaces with G a subspace of H. We shall use the notations:

- $\mathcal{O}_H$: The collection of open covers of H.
- $\mathcal{O}_{HG}$: The collection of covers of G by sets open in H.

An open cover $\mathcal{U}$ of a topological space H is said to be

- an ω-cover if H is not a member of $\mathcal{U}$, but for each finite subset F of H there is a $U \in \mathcal{U}$ such that $F \subset U$ [3]. The symbol ω denotes the collection of ω-covers of H.
- *groupable* if there is a partition $\mathcal{U} = \bigcup_{n\in\mathbf{N}} \mathcal{U}_n$, where each $\mathcal{U}_n$ is finite, and for each $x \in H$ the set $\{n : x \notin \bigcup \mathcal{U}_n\}$ is finite [9]. The symbol $\mathcal{O}^{gp}$ denotes the collection of groupable open covers of the space.
- γ-cover if it is infinite, and each infinite subset of it is still an open cover of the space. The symbol γ denotes the collection of γ-covers of the space.
- *large* if each element of the space is contained in infinitely many elements of the cover. The symbol Λ denotes the collection of large covers of the space.

Now let $(G, *)$ be a topological group with identity element e. We will assume that G is not compact. For a neighborhood U of e, and for a finite subset F of G the set $F * U$ is a neighborhood of the finite set F. Thus, the set $\{F * U : F \subset G \text{ finite}\}$ is an ω-cover of G, which is denoted by the symbol $\Omega(U)$. The set

$$\Omega_{nbd} = \{\Omega(U) : U \text{ a neighborhood of } e\}$$

is the set of all such ω-covers of G.

AMS Subject Classification: 54H11

Keywords and phrases: Metrizable topological group, strict o-boundedness, product groups.

The set $\mathcal{O}(U) = \{x * U : x \in G\}$ is an open cover of G. The symbol

$$\mathcal{O}_{nbd} = \{\mathcal{O}(U) : U \text{ a neighborhood of } e\}$$

denotes the collection of all such open covers of G. Now we describe the selection principles relevant to this topic. Let S be an infinite set, and let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of S.

The symbol $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the statement that there is for each sequence $(O_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ a sequence $(T_n : n \in \mathbf{N})$ such that for each n $T_n \in O_n$, and $\{T_n : n \in \mathbf{N}\} \in \mathcal{B}$. The earliest example of this sort of selection principle was introduced in [14] by Rothberger, and is $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ in our notation.

The symbol $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ is defined as follows: For each sequence $(O_n : n \in \mathbf{N})$ from $\mathcal{A}$ there is a sequence $(T_n : n \in \mathbf{N})$ of finite sets such that for each n $T_n \subset O_n$, and $\bigcup_{n \in \mathbf{N}} T_n \in \mathcal{B}$. The earliest example of this selection principle was introduced by Hurewicz in [7], and in our notation is $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$. Because of its equivalence in metric spaces with a basis property of Menger, $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ is also called the Menger property. In [7] Hurewicz introduced also a second selection principle, now called the Hurewicz property, and characterized in [9] as the statement $\mathsf{S}_{fin}(\Lambda, \mathcal{O}^{gp})$.

Each selection principle has a corresponding game. The game that we will use in this paper is the game corresponding to $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$. The game is denoted $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ and is played as follows: In the n-th inning ONE chooses an $O_n \in \mathcal{A}$, and TWO responds with a $T_n \in O_n$. They play an inning for each natural number n. A play

$$O_1, T_1, \cdots, O_n, T_n, \cdots$$

is won by TWO if $\{T_n : n \in \mathbf{N}\}$ is in $\mathcal{B}$. Otherwise, ONE wins.

2. Hurewicz-bounded groups

The notion of Hurewicz-bounded group was introduced by Kočinac in 1998 in one of his unpublished notes. A topological group $(G, *)$ is said to be a *Hurewicz-bounded* group if it satisfies the selection principle $\mathsf{S}_1(\Omega_{nbd}(G), \mathcal{O}_G^{gp})$. Let $(G, *)$ be a subgroup of the group $(H, *)$. Then G is *Hurewicz-bounded H* if the selection principle $\mathsf{S}_1(\Omega_{nbd}(H), \mathcal{O}_{HG}^{gp})$ holds.

THEOREM 1. *For a subgroup $(G, *)$ of an infinite topological group $(H, *)$ the following are equivalent:*

1. $\mathsf{S}_1(\Omega_{nbd}(H), \mathcal{O}_{HG}^{gp})$.
2. $\mathsf{S}_1(\Omega_{nbd}(H), \Gamma_{HG})$.

Proof. We need to prove only that $1 \Rightarrow 2$. For each n let $\mathcal{U}_n$ be an element of $\Omega_{\mathrm{nbd}}(H)$, and choose for each n an open neighborhood U_n of e such that $\mathcal{U}_n = \Omega(U_n)$. For each n put $V_n = \cap_{j \leq n} U_j$. For each n put $\mathcal{V}_n = \Omega(V_n)$. Apply $\mathsf{S}_1(\Omega_{\mathrm{nbd}}, \mathcal{O}^{gp})$ to $(\mathcal{V}_n : n \in \mathbf{N})$. For each n choose $W_n \in \mathcal{V}_n$ such that $\{W_n : n \in \mathbf{N}\}$ is a groupable open cover of G. Choose a sequence $m_1 < m_2 < \cdots < m_n < \cdots$ such that for each $x \in G$, for all but finitely many n, $x \in \bigcup_{m_n \leq j < m_{n+1}} W_j$. For each n also choose a finite set $F_n \subset H$ with $W_n = F_n * V_n$. Now define, for each k, the finite set G_k by

$$G_k = \begin{cases} \bigcup_{i\le m_1} F_i, & \text{if } k \le m_1 \\ \bigcup_{m_n<i\le m_{n+1}} F_i, & \text{if } m_n < k \le m_{n+1} \end{cases}$$

For each n put $A_n = G_n * U_n$, an element of $\Omega(U_n)$. We claim that $\{A_n : n \in \mathbf{N}\}$ is a γ-cover of G. For consider $g \in G$. Choose $M \in \mathbf{N}$ so that for all $n \ge M$ we have $g \in \bigcup_{m_n<i\le m_{n+1}} W_i$. But for $m_n < i \le m_{n+1}$ we have $W_i = F_i * V_i \subset A_k = G_k * U_k$ for $m_n < k \le m_{n+1}$. Thus, for all $k > m_M$ we have $g \in A_k$. It follows that $\{A_k : k \in \mathbf{N}\}$ is a γ-cover of G. ∎

THEOREM 2. *Let $(G, *)$ be a subgroup of the topological group $(H, *)$. Then the following are equivalent:*

1. $\mathsf{S}_1(\Omega_{\mathrm{nbd}}(H), \mathcal{O}^{gp}_{HG})$.
2. $\mathsf{S}_1(\Omega_{\mathrm{nbd}}(G), \mathcal{O}^{gp}_{G})$.

Proof. The implication (2) ⇒ (1) is evidently true. We must show that (1) ⇒ (2). Thus, let $(\Omega(U_n) : n \in \mathbf{N})$ be a sequence in $\Omega_{\mathrm{nbd}}(G)$. Since each U_n is a neighborhood (in G) of the group identity we may choose for each n a neighborhood T_n in H of the identity, such that $U_n = T_n \cap G$. Next, choose for each n a neighborhood S_n in H of the identity, such that $S_n^{-1} * S_n \subseteq T_n$.

Apply (2) to the sequence $(\Omega(S_n) : n \in \mathbf{N})$ we get for each n a finite set $F_n \subset H$ such that for each element x of G there is an N such that for all $n \ge N$ we have $x \in F_n * S_n$. For each n, and for each $f \in F_n$, choose a $g_f \in G$ as follows:

$$g_f \begin{cases} \in G \cap f * S_n, & \text{if nonempty} \\ = e, \text{ the group identity}, & \text{otherwise} \end{cases}$$

Then put $G_n = \{g_f : f \in F_n\}$, a finite subset of G. For each n we have $G_n * U_n \in \Omega(U_n) \in \Omega_{\mathrm{nbd}}(G)$. We must show that for each $x \in G$ there is an N such that for each $n \ge N$, $x \in G_n * U_n$.

Let $x \in G$ be given. Choose N so that for all $n \ge N$ we have $x \in F_n * S_n$. Fix $n \ge N$ and choose $f_x \in F_n$ so that $x \in f_x * S_n$. Then evidently $G \cap f_x * S_n$ is nonempty, and so $g_{f_x} \in G$ is defined as an element of this intersection. Since $g_{f_x} \in f_x * S_n$, we have $f_x \in g_{f_x} * S_n^{-1}$, and so $x \in g_{f_x} * S_n^{-1} * S_n \subseteq g_{f_x} * T_n$. Now $g_{f_x}^{-1} * x \in G \cap T_n = U_n$, and so we have $x \in g_{f_x} * U_n \subset G_n * U_n$. This completes the proof. ∎

This result implies the following:

COROLLARY 3. *If $(H, *)$ has property $\mathsf{S}_1(\Omega_{\mathrm{nbd}}(H), \Gamma_H)$, then for each infinite subgroup G of H, $\mathsf{S}_1(\Omega_{\mathrm{nbd}}(G), \Gamma_G)$ holds.*

3. A characterization of strict *o*-boundedness for metrizable groups

According to Hausdorff [4, 25] a metric space (X, d) is *totally bounded* if there is for each $\delta > 0$ a partition of X into finitely many sets, each of diameter less than δ. The metric space is σ-totally bounded if it is a union of countably many sets, each totally bounded.

Measure-like properties of metrizable spaces can be equivalent to selection properties. Similarly, for metrizable groups measure-like properties with respect to left-invariant metrics can be equivalent to selection properties. Let $\mathcal{A}$ be a collection of sets, each a set of subsets of H. As in [1] we say: Metric space (H, d) has $\mathcal{A}$-*measure zero* if for each sequence $(\epsilon_n : n \in \mathbf{N})$ of positive real numbers, there is a sequence $(\mathcal{F}_n : n \in \mathbf{N})$ where:

1. For each n, $\mathcal{F}_n$ is a finite set of subsets of H,
2. Each element of $\mathcal{F}_n$ has diameter less than ϵ_n, and
3. $\bigcup_{n\in\mathbf{N}} \mathcal{F}_n \in \mathcal{A}$.

The following theorem of Kakutani will be used below:

THEOREM 4. [6] *Let $(U_k : k < \infty)$ be a sequence of subsets of the topological group $(H, *)$ where $\{U_k : k < \infty\}$ is a neighborhood basis of the identity element e and each U_k is symmetric*[1], *and for each k also $U_{k+1}^2 \subseteq U_k$. Then there is a left-invariant metric d on H such that*

1. *d is uniformly continuous in the left uniform structure on $H \times H$.*
2. *If $y^{-1} * x \in U_k$ then $d(x, y) \leq (\frac{1}{2})^{k-2}$.*
3. *If $d(x, y) < (\frac{1}{2})^k$ then $y^{-1} * x \in U_k$.*

The equivalence of the first two statements of the following theorem apparently has been independently obtained also by H. Michalewski [13].

THEOREM 5. *For a subgroup $(G, *)$ of a metrizable group $(H, *)$ the following are equivalent:*

1. *TWO has a winning strategy in the game $\mathsf{G}_1(\Omega_{nbd}(H), \mathcal{O}_{HG})$.*
2. *$(G, *)$ is σ-totally bounded in all left-invariant metrics generating the topology of H.*
3. *$\mathsf{S}_1(\Omega_{\mathrm{nbd}}(H), \Gamma_{HG})$ holds.*
4. *H has the $\mathcal{O}_{HG}^{gp}$-measure zero property in all left invariant metrics on H.*

Proof. $1 \Rightarrow 2$: Since $(H, *)$is a metrizable group, it is first countable. Let d be a left-invariant metric of H and let $(U_n : n \in \mathbf{N})$ be a neighborhood basis of the identity element e of H such that for each n, $U_n \supset U_{n+1}$ and $diam_d(U_n) < \frac{1}{2^n}$. Let σ be TWO's winning strategy. Define:

$$G_\emptyset = \bigcap_{n\in\mathbf{N}} \sigma(\Omega(U_n)).$$

[1] U_k is symmetric if $U_k = U_k^{-1}$

Then, for $n_1, \cdots, n_k < \infty$ given, define

$$G_{n_1,\cdots,n_k} = \bigcap_{n\in\mathbf{N}} \sigma(\Omega(U_{n_1}), \cdots, \Omega(U_{n_k}), \Omega(U_n)).$$

First, we show that

$$G \subseteq \bigcup_{\tau\in{}^{<\omega}\mathbf{N}} G_\tau. \tag{1}$$

For suppose on the contrary that (1) is false. Choose an $x \in G \setminus \bigcup_{\tau\in{}^{<\omega}\mathbf{N}} G_\tau$. Since x is not in $G_\emptyset$, choose n_1 so that $x \notin \sigma(\Omega(U_{n_1}))$. Since x is not in G_{n_1}, choose n_2 so that $x \notin \sigma(\Omega(U_{n_1}), \Omega(U_{n_2})$, and so on. In this way we obtain a σ-play

$$\Omega(U_{n_1}), \sigma(\Omega(U_{n_1})), \Omega(U_{n_2}), \sigma(\Omega(U_{n_1}), \Omega(U_{n_2})), \cdots$$

which is lost by TWO, since x is never covered by TWO. This contradicts the fact that σ is a winning strategy for TWO.

Next, we observe that each G_τ is totally bounded. Fix $\tau = (n_1 \cdots, n_k)$. Let an $\epsilon > 0$ be given. Choose n so large that $\frac{1}{2^n} < \epsilon$. Now G_τ is a subset of $\sigma(\Omega(U_{n_1}), \cdots, \Omega(U_{n_k}), \Omega(U_n))$, which is of the form $F * U_n$ for some finite subset F of G. But then $\{f * U_n : f \in F\}$ is a finite family of sets, each of diameter less than $\frac{1}{2^n} < \epsilon$, and covers G_τ.

This completes the proof that G is σ-totally bounded.

$2 \Rightarrow 1$: Fix a left-invariant metric d of H and assume that G is σ-totally bounded. Write $G = \bigcup_{n\in\mathbf{N}} G_n$, where each G_n is totally bounded. We define a winning strategy σ for TWO as follows:

When player ONE plays $\Omega(U_1)$, put $\epsilon_1 = diam(U_1)$, and $\delta_1 = \frac{\epsilon_1}{4}$. Since G_1 is totally bounded, choose a finite family $\mathcal{F}$ of open sets, each of diameter less than δ_1, so that $G_1 \subseteq \bigcup\mathcal{F}$. For each $F \in \mathcal{F}$, choose a point $x_F \in F$. Then $F \subseteq x_F * U$ (by diameter considerations), and so, setting $S_1 = \{x_F : F \in \mathcal{F}\}$, TWO responds with $\sigma(\Omega(U_1)) = S_1 * U_1 \in \Omega(U_1)$.

Suppose it is the n-th inning, and ONE has played $\Omega(U_1), \cdots, \Omega(U_n)$ so far. Put $\epsilon_n = \min\{diam(U_j) : j \leq n\}$, and put $\delta_n = \frac{\epsilon_n}{4}$. Since $G_1 \cup \cdots \cup G_n$ is totally bounded, choose a finite set $\mathcal{F}$ of open subsets of G, each of diameter less than δ_n, such that $G_1 \cup \cdots \cup G_n \subseteq \cup\mathcal{F}$. For each $F \in \mathcal{F}$ choose an $x_F \in F$ and put $S_n = \{x_F : F \in \mathcal{F}\}$. Then by diameter considerations, $\bigcup\mathcal{F} \subset S_n * U_n$. Now TWO plays $\sigma(\Omega(U_1), \cdots, \Omega(U_n)) = S_n * U_n \in \Omega(U_n)$.

It is evident that σ is a winning strategy for TWO.

$2 \Rightarrow 3$: The winning strategy σ described for TWO above has the effect of choosing for a sequence $(\Omega(U_n) : n \in \mathbf{N})$ a corresponding sequence $(T_n : n \in \mathbf{N})$ such that for each n we have $T_n \in \Omega(U_n)$, and $G_1 \cup \cdots \cup G_n \subseteq T_n$. This witnesses $\mathsf{S}_1(\Omega_{\mathrm{nbd}}(H), \Gamma_{HG})$.

$3 \Rightarrow 2$: Since H is metrizable it is first countable and has a left-invariant metric. Choose a left invariant metric d for H and a sequence $(U_n : n \in \mathbf{N})$ of neighborhoods of the identity e, which forms a neighborhood basis at e, and such that $\lim_{n\to\infty} diam_d(U_n) = 0$. Apply $\mathsf{S}_1(\Omega_{\rm nbd}(H), \Gamma_{HG})$ to the sequence $(\Omega(U_n) : n \in \mathbf{N})$: For each n we find a $T_n = S_n * U_n \in \Omega(U_n)$ where S_n is a finite subset of G, such that $\{T_n : n \in \mathbf{N}\}$ is a γ-cover of G. For each $n \in \mathbf{N}$ define $G_n = \cap_{k\geq n} T_k$. Then $G \subseteq \bigcup_{n\in\mathbf{N}} G_n$. Each G_n is totally bounded: For let an n be given and let $\epsilon > 0$ be given. Choose an $N > n$ so large that $diam(U_N) < \epsilon$. Then T_N is a union of finitely many sets of the form $x * U_N$, and $G_n \subset T_N$.

3 $\Rightarrow$ 4: Let d be a left-invariant metric on H and let $(\epsilon_n : n \in \mathbf{N})$ be a sequence of positive real numbers. Choose for each n a neighborhood U_n of the identity element e of H such that $diam_d(U_n) < \epsilon_n$. Then apply $\mathsf{S}_1(\Omega_{\rm nbd}(H), \mathcal{O}^{gp}_{HG})$ to the sequence $(\Omega_H(U_n) : n \in \mathbf{N})$. For each n we find F_n of H such that each element of G is in all but finitely many of the $F_n * U_n$'s. Since d is left-invariant, each element of $\mathcal{V}_n = \{x * U_n : x \in F_n\}$ has diameter less than ϵ_n.

4 $\Rightarrow$ 3: Let $(U_n : n \in \mathbf{N})$ be a sequence of neighborhoods of e, the identity element of H. Choose for each n a symmetric neighborhood V_n such that $V^2_{n+1} \subseteq V_n \subset U_n$. By Kakutani's theorem choose a left-invariant metric d as in Theorem 5 corresponding to the sequence$(V_n : n \in \mathbf{N})$. For each n put $\epsilon_n = \frac{1}{2^n}$. We may assume that G is not totally bounded in d. Choose $\delta \geq 0$ witnessing this. Choose N large enough such that $\epsilon_N \geq \delta$, and apply 4 to $(\epsilon_n : n \geq N)$. For each $n \geq N$ choose a finite set $\mathcal{H}_n$ of subsets of H such that each set in H_n has d-diameter less that ϵ. For each $S \in \mathcal{H}_n$ choose $T_s \in \mathcal{O}_H(V_n)$ with $S \in T_s$. Choose $W_s \in \mathcal{O}_H(U_n)$ with $S \subseteq W_s$. For each n, put $\mathcal{G} = \{W_s : S \in \mathcal{H}_n\}$, and put $G_n = \bigcup \mathcal{G}_n \in \Omega(U_n)$. Then, $\{G_n : n \geq N\}$ is a γ cover of G. ■

Problem 4.1 of [5] asks if there are strictly o-bounded groups G and H for which $G \times H$ is not strictly o-bounded. We show that for metrizable strictly o-bounded groups the answer is no. Problem 4.2 asks if the product of an o-bounded group with a strictly o-bounded group is again an o-bounded group. We show that if the strictly o-bounded group is metrizable, then the answer is yes. Both of these answers are results of the following theorem.

Theorem 6. *Let $(G, *)$ be a group satisfying $\mathsf{S}_1(\Omega_{\rm nbd}, \Gamma)$. Let $\mathcal{A}$ be one of $\mathcal{O}$, Ω, Γ. If $(H, *)$ is a group with property $\mathsf{S}_1(\Omega_{\rm nbd}, \mathcal{A})$, then $(G \times H, *)$ also has this property.*

Proof. Consider a sequence $(\Omega(U_n) : n \in \mathbf{N})$ in $\Omega_{\rm nbd}(G \times H)$. For each n choose neighborhoods V_n of e_G and W_n of e_H such that $V_n \times W_n \subset U_n$. Consider the sequences $(\Omega(V_n) : n \in \mathbf{N})$ and $(\Omega(W_n) : n \in \mathbf{N})$. For each n choose a finite set $G_n \subset G$ and a finite set $H_n \subset H$ such that $(G_n * V_n : n \in \mathbf{N})$ is in Γ, and $(H_n * W_n : n \in \mathbf{N})$ is in $\mathcal{A}$. For each n put $F_n = G_n \times H_n$. We claim that $(F_n * U_n : n \in \mathbf{N})$ has the required properties:

1. $\mathcal{A} = \mathcal{O}$: By the Remark following Theorem 1 of [2], we may assume that there is for each $y \in H$ infinitely many n so that $y \in H_n * W_n$. Consider any $(x, y) \in G \times H$. Fix an N so that for all $n \geq N$ we have $x \in G_n * V_n$. Then choose an $n > N$ so that also $y \in H_n * W_n$. Then $(x, y) \in F_n * U_n$.

2. $\mathcal{A} = \Omega$: Observe that for each finite subset K of G there is an N such that for all $n \geq N$, $K \subseteq G_n * V_n$. Since for each finite subset L of H there are infinitely many n with $L \subseteq H_n * W_n$, apply the argument from before.
3. $\mathcal{A} = \Gamma$: The argument is similar for this case. ■

COROLLARY 7. *If $(G_j, *_j)$, $j \leq n$ are metrizable strictly o-bounded groups, then so is their product.*

Proof. By Theorem 5, metrizable strictly o-bounded groups are characterized by $\mathsf{S}_1(\Omega_{\rm nbd}, \Gamma)$. Apply Theorem 6. ■

COROLLARY 8. *If $(G, *)$ is an o-bounded group and $(H, *)$ is a metrizable strictly o-bounded group, then $G \times H$ is an o-bounded group.*

Proof. Apply Theorems 5 and 6. ■

We characterize for metrizable groups when TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\rm nbd}(H), \mathcal{O}_{HG})$:

THEOREM 9. *Let $(H, *)$ be a metrizable group with a subgroup $(G, *)$. The following are equivalent:*

1. *TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}_{\rm nbd}(H), \mathcal{O}_{HG})$.*
2. *G is a countable set.*

Proof. We only need to prove $1 \Rightarrow 2$: Let σ be a winning strategy for TWO. Since $(H, *)$ is metrizable, let d be a left-invariant metric for it, and let $(U_n : n \in \mathbf{N})$ be a neighborhood basis for e in H such that $\lim_{n\to\infty} diam_d(U_n) = 0$. Define $G_\emptyset = \bigcap_{n\in\mathbf{N}} \sigma(\mathcal{O}(U_n))$. For $(n_1, \cdots, n_k)$ a given finite sequence, define $G_{(n_1,\cdots,n_k)} = \bigcap_{n\in\mathbf{N}} \sigma(\mathcal{O}(U_{n_1}), \cdots, \mathcal{O}(U_{n_k}), \mathcal{O}(U_n))$. We claim that

$$G \subseteq \bigcup_{\tau \in {}^{<\omega}\mathbf{N}} G_\tau. \tag{2}$$

For suppose on the contrary x is not in $\bigcup_{\tau\in{}^{<\omega}\mathbf{N}} G_\tau$. Choose an n_1 with $x \notin \sigma(\mathcal{O}(U_{n_1}))$, and then choose n_2 so that $x \notin \sigma(\mathcal{O}(U_{n_1}), \mathcal{O}(U_{n_2}))$, and so on. In this way we obtain a sequence $n_1, n_2, \cdots, n_k, \cdots$ so that for each k, $x \notin \sigma(\mathcal{O}(U_{n_1}), \cdots, \mathcal{O}(U_{n_k}))$. But then we obtain a σ-play lost by TWO, a contradiction.

Next, we observe that each G_τ has at most one element: This is because d is left-invariant, so that $diam(\sigma(\mathcal{O}(U_{n_1}), \cdots, \mathcal{O}(U_{n_k}), \mathcal{O}(U_n)) = diam(U_n)$, and since these U_n's form a neighborhood base for e, the diameter of $H_{n_1,\cdots,n_k}$ is 0. ■

REFERENCES

[1] L. Babinkostova, *Selektivni principi vo topologijata*, Ph.D. thesis, Skopje (2001).

[2] L. Babinkostova, Lj.D.R. Kočinac, M. Scheepers, *Combinatorics of open covers (XI): Menger- and Rothberger-bounded groups*, Topology Appl., to appear.

[3] J. Gerlits and Zs. Nagy, *Some properties of C(X), I*, Topology Appl. **14** (1982), 151–161.

[**4**] F. Hausdorff, *Set Theory*, Chelsea Publishing Company (1978).

[**5**] C. Hernández, D. Robbie and M. Tkachenko, *Some properties of o-bounded groups and strictly o-bounded groups*, Applied General Topology **1** (2000), 29–43.

[**6**] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, 1963.

[**7**] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Math. Zeitschrift **24** (1925), 401–421.

[**8**] Lj.D.R. Kočinac, *Selection principles in uniform spaces*, Note di Matematica **22** : 2 (2003/2004), 127–139.

[**9**] Lj.D.R. Kočinac and M. Scheepers, *The Combinatorics of open covers (VII): Groupability*, Fund. Math. **179** (2003), 131–155.

[**10**] A. Krawczyk and H. Michalewski, *Linear metric spaces close to being σ-compact*, Technical Report **46** (2001), Institute of Mathematics, Warsaw University.

[**11**] A. Krawczyk and H. Michalewski, *An example of a topological group*, Topology Appl. **127** (2003), 325–330.

[**12**] M. Machura and B. Tsaban, *The combinatorics of the Baire group*, preprint.

[**13**] H. Michalewski, *Function spaces with topology of pointwise convergence*, Ph.D. dissertation, Faculty of Mathematics, Informatics and Mechanics, Warsaw University (2003).

[**14**] F. Rothberger, *Eine Verschärfung der Eigenschaft* C, Fund. Math. (1938), 50–55.

[**15**] B. Tsaban, *o-Bounded groups and other topological groups with strong combinatorial properties*, Proc. AMS **134** (2006), 881–891.

(received 12.11.2005, in revised form 22.01.2006)