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# RELATIONS BETWEEN SOME TOPOLOGIES

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**Abstract.** Generalizations of openness, such as semi-open, preopen, semi-pre-open, $\alpha$-open, etc. are important in topological spaces and in particular in topological spaces on which ideals are defined. $\alpha$-equivalent topologies and $*$-equivalent topologies with respect to an ideal have some common properties. Relations between these aforementioned notions of openness are investigated within the framework of $\alpha$-equivalence and $*$-equivalence.

## 1. Introduction

The subject of ideals in general topological spaces was introduced by Kuratowski [8] and Vaidyanathaswamy [18]. An ideal $\mathcal{I}$ on a set $X$ is a nonempty collection of subsets of $X$ which satisfies

(i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$,

(ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

By $(X, \tau, \mathcal{I})$ we will denote a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$. No separation properties are assumed on $X$. For a space $(X, \tau, \mathcal{I})$ and a subset $A \subset X$,

$$A^*(\tau, \mathcal{I}) = \{\, x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x) \,\}$$

(where $\tau(x) = \{U \in \tau : x \in U\}$) is called the *local function of $A$ with respect to $\mathcal{I}$ and $\tau$* [8]. Note that $\mathrm{cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$ [15] on $X$. If there is no chance of confusion, we simply write $A^*$ or $A^*(\mathcal{I})$ instead of $A^*(\tau, \mathcal{I})$, and $\tau^*$ instead of $\tau^*(\mathcal{I})$.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on $X$, then $\mathcal{I} \vee \mathcal{J} = \{I \cup J : I \in \mathcal{I} \text{ and } J \in \mathcal{J}\}$ is also an ideal on $X$ [6].

In a topological space $(X, \tau)$, for any subset $A$, $A^o$, $\mathrm{int}\, A$ or $\tau\, \mathrm{int}\, A$ will stand for the interior of $A$ and $\overline{A}$, $\mathrm{cl}\, A$, or $\tau\, \mathrm{cl}\, A$ will stand for the closure of $A$. A subset $A$ of a space $(X, \tau)$ is said to be *semi-open* (*pre-open*, *$\alpha$-open*, *semi-pre-open*, *regular open*, *nowhere dense*, *codense*) if $A \subset \overline{A^o}$ ($A \subset \overline{A}^{o}$, $A \subset \overline{A^o}^{o}$, $A \subset \overline{\overline{A}^{o}}$, $A = \overline{A}^{o}$, $\overline{A}^{o} = \emptyset$, $A^o = \emptyset$), respectively.

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A point $x$ of $X$ is called a *$\theta$-interior point of* $A$ if there exists an open set $U$ such that $x \in U \subset \bar{U} \subset A$. Also, $\theta$-$\operatorname{int} A$ will stand for the set of $\theta$-interior points of $A$. $A$ is $\theta$-open iff $A \subset \theta$-$\operatorname{int} A$ [10]. The family of all $\alpha$-open sets in $(X, \tau)$ is a topology on $X$ which is finer than $\tau$ and it is denoted by $\tau^\alpha$. Topologies $\tau$ and $\sigma$ on $X$ are called *$\alpha$-equivalent* if they have the same $\alpha$-open sets [14].

A supratopology $\mathcal{A}$ on $X$ is a nonempty collection of subsets of $X$ which satisfies

(i) $\emptyset \in \mathcal{A}$, $X \in \mathcal{A}$,

(ii) $\mathcal{A}$ is closed under arbitrary unions [10].

The $\mathcal{A}$-*interior* (shortly, $\mathcal{A}$-int) of a subset $A$ of $X$ is defined as

$$\mathcal{A}\text{-}\operatorname{int} A = \bigcup\{U : U \subset A, U \in \mathcal{A}\}$$

[9]. It is well known that the family of semi-open (pre-open, semi-pre-open) sets of a topological space is a supratopology on this space. If $\mathcal{A}$ is a supratopology on $X$, then

$$\mathcal{T}_{\mathcal{A}} = \{A \subset X : A \cap B \in \mathcal{A} \text{ for each } B \in \mathcal{A}\}$$

is a topology and $\mathcal{T}_{\mathcal{A}} \subset \mathcal{A}$ [19].

We will use the following notational conventions:

$\tau \in Top\,(X) \iff \tau$ is a topology on $X$,

$\mathcal{I} \in Id(X) \iff \mathcal{I}$ is an ideal on $X$,

$A \in D(X) \iff A$ is dense in $X$,

$A \in CD(X) \iff A^o = \emptyset$ (i.e. $A$ is codense),

$A \in NO(X) \iff A^{\underline{o}} = \emptyset$ (i.e. $A$ is nowhere dense),

$A \in SO(X) \iff A \subset A^{\overline{o}}$,

$A \in PO(X) \iff A \subset A^{\underline{o}}$,

$A \in \alpha O(X) \iff A \subset A^{\frac{o}{o}}$,

$A \in SPO(X) \iff A \subset A^{\overline{\underline{o}}}$,

$A \in RO(X) \iff A = A^{\underline{o}}$,

$A \in \theta O(X) \iff A$ is $\theta$-open,

$A \in SC(X) \iff X - A$ is semi-open (i.e. $A$ is semi-closed),

$A \in SR(X) \iff A$ is semi-open and semi-closed (i.e. $A$ is semi-regular),

$\sigma \in [\tau]^\alpha \iff \sigma \in Top\,(X),$ and $\sigma^\alpha = \tau^\alpha$ (i.e. $\tau$ and $\sigma$ are $\alpha$-equivalent).

$I_n$ (or $I_n(\tau)$) and $I_n(\sigma)$ will stand for the family of nowhere dense sets in $X$ with respect to $\tau$ and $\sigma$, respectively. From now on, $A^o$ and $\bar{A}$ will be reserved for the interior and closure of $A$ with respect to topology $\tau$, respectively.

In a topological space scl, sint, pcl, spcl, etc. will stand for the operations semi-closure, semi-interior, pre-closure, semi-pre-closure, respectively. Where it is necessary to indicate the topology, we will write, for example, $\tau$-scl or $\sigma$-scl.

In the following theorem we recall some known results in the literature which will be used in this paper. They appear in [1, 2, 5, 6, 13–18].

THEOREM 1.1. *In any topological space* $(X,\tau)$ *we have the following results:*
(1) *For any subset* $A$ *of* $X$,

$$\operatorname{scl} A = A \cup A^{\overset{o}{-}}, \qquad \operatorname{sint} A = A \cap A^{\overline{o}},$$

$$\operatorname{pcl} A = A \cup A^{\overline{o}}, \qquad \operatorname{pint} A = A \cap A^{\overset{o}{-}},$$

$$\operatorname{spcl} A = A \cup A^{\overset{o}{\underset{o}{-}}}, \qquad \operatorname{spint} A = A \cap A^{\overline{\overset{o}{-}}},$$

$$\alpha\text{-}\operatorname{cl} A = \tau^{\alpha} \operatorname{cl} A = A \cup A^{\overline{\overset{o}{-}}}, \qquad \alpha\text{-}\operatorname{int} A = \tau^{\alpha} \operatorname{int} A = A \cap A^{\overset{o}{\underset{o}{-}}} \ [1,2].$$

(2) $\tau \subset \mathcal{T}_{SO(X)} \cap \mathcal{T}_{PO(X)} \cap \mathcal{T}_{SPO(X)}$ [1,2].
*In the remaining results given below,* $\sigma \in Top(X)$ *and* $\mathcal{I}, \mathcal{J} \in Id(X)$.
(3) *If* $\tau \subset \sigma \subset \tau^{\alpha}$, *then* $\sigma \in [\tau]^{\alpha}$ [14].
(4) $\tau \subset \tau^*(\mathcal{I})$ *and* $(\tau^*(\mathcal{I}))^*(\mathcal{I}) = \tau^*(\mathcal{I})$.
(5) $A^*(\tau,\mathcal{I})$ *is* $\tau$*-closed and* $A^*(\tau,\mathcal{I}) = A^*(\tau^*(\mathcal{I}),\mathcal{I})$ *for each* $A \subset X$,
(6) $I \in \mathcal{I} \implies I$ *is* $\tau^*(\mathcal{I})$*-closed and* $I^* = \emptyset$.
(7) $\mathcal{I} \subset \mathcal{J} \implies A^*(\tau,\mathcal{J}) \subset A^*(\tau,\mathcal{I})$ *for each* $A \subset X$ *and* $\tau^*(\mathcal{I}) \subset \tau^*(\mathcal{J})$.
(8) $\tau \subset \sigma \implies A^*(\sigma,\mathcal{I}) \subset A^*(\tau,\mathcal{I})$ *for each* $A \subset X$ *and* $\tau^*(\mathcal{I}) \subset \sigma^*(\mathcal{I})$.
(8) $\tau \cap \mathcal{I} = \{\emptyset\} \iff \tau^*(\mathcal{I}) \cap \mathcal{I} = \{\emptyset\} \iff X = X^* \iff U \subset U^*$ *for each* $U \in \tau$.
(9) $\mathcal{B}(\tau^*(\mathcal{I})) = \{U - I : U \in \tau, I \in \mathcal{I}\}$ *is a base for the topology* $\tau^*(\mathcal{I})$.
(10) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then for each* $U \in \tau$ *and for each* $I \in \mathcal{I}$, *we have* $\bar{U} = U^* = (U - I)^* = (U - I)^- = \tau^*\text{-}\operatorname{cl}(U - I) = \tau^*\text{-}\operatorname{cl} U$.
(11) $\tau^*(\mathcal{I} \vee \mathcal{J}) = (\tau^*(\mathcal{I}))^*(\mathcal{J})$ [6].
(12) $\tau^*(\mathcal{I}_n) = \tau^{\alpha}$, $\tau \cap \mathcal{I}_n = \{\emptyset\}$, $PO(X,\tau) \cap \mathcal{I}_n = \{\emptyset\}$.
(13) $A \subset B \implies A^*(\mathcal{I}) \subset B^*(\mathcal{I})$.
(14) $(A - I)^* = A^*$ *for each* $A \subset X$ *and each* $I \in \mathcal{I}$.
(15) $A^*(\mathcal{I}_n) = A^{\overline{\overset{o}{-}}}$ *for each* $A \subset X$.
(16) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then for each* $U \in \tau^*(\mathcal{I})$ *we have* $\tau\text{-}\operatorname{cl} U = \tau^*(\mathcal{I})\text{-}\operatorname{cl} U$.

## 2. Relations between topologies and some special sets

Firstly, some relations between families such as $SO(X), PO(X), SPO(X)$, etc. on a set $X$ with two topologies are investigated. Then, these relations will be carried over to topological spaces on which ideals are defined. Some known results will be obtained by a different method.

THEOREM 2.1. *Let* $\tau, \sigma, \omega \in Top(X)$. *Then we have the following results.*
(1) *If* $\tau \subset \sigma \subset SPO(X,\tau)$, *then:* (a) $SPO(X,\sigma) \subset SPO(X,\tau)$, (b) $\mathcal{I}_n(\tau) \subset \mathcal{I}_n(\sigma)$.
(2) *If* $\tau \subset \sigma \subset PO(X,\tau)$, *then* $PO(X,\sigma) \subset PO(X,\tau)$, *and the relations* (*a*) *and* (*b*) *in* (1) *are valid.*
(3) I. *If* $\tau \subset \sigma$, *and* $\tau \operatorname{cl} U = \sigma \operatorname{cl} U$ *for each* $U \in \tau$, *then*

(a) *For each* $A \subset X$ *we have*

$$\begin{aligned}
&\tau \operatorname{cl}(\tau \operatorname{int} A) = \sigma \operatorname{cl}(\tau \operatorname{int} A) \subset \sigma \operatorname{cl}(\sigma \operatorname{int} A),\\
&\sigma \operatorname{int}(\sigma \operatorname{cl} A) \subset \sigma \operatorname{int}(\tau \operatorname{cl} A) = \tau \operatorname{int}(\tau \operatorname{cl} A),\\
&\tau \operatorname{int}(\tau \operatorname{cl}(\tau \operatorname{int} A)) \subset \tau \operatorname{int}(\sigma \operatorname{cl}(\sigma \operatorname{int} A)) \subset \sigma \operatorname{int}(\sigma \operatorname{cl}(\sigma \operatorname{int} A)),\\
&\sigma \operatorname{cl}(\sigma \operatorname{int}(\sigma \operatorname{cl} A)) \subset \sigma \operatorname{cl}(\tau \operatorname{int}(\tau \operatorname{cl} A)) = \tau \operatorname{cl}(\tau \operatorname{int}(\tau \operatorname{cl} A)).
\end{aligned}$$

(b) *For each* $U \in SO(X,\tau)$, *we have* $\tau \operatorname{cl} U = \sigma \operatorname{cl} U = \tau \operatorname{cl}(\tau \operatorname{int} U) = \sigma \operatorname{cl}(\sigma \operatorname{int} U)$,

(c) $SO(X,\tau) \subset SO(X,\sigma)$,

(d) $PO(X,\sigma) \subset PO(X,\tau)$,

(e) $SPO(X,\sigma) \subset SPO(X,\tau)$,

(f) $\alpha O(X,\tau) \subset \alpha O(X,\sigma)$,

(g) $\mathcal{I}_n(\tau) \subset \mathcal{I}_n(\sigma)$,

(h) $RO(X,\tau) \subset RO(X,\sigma)$.

II. *If* $\tau \subset \sigma$, *and* $\tau \operatorname{cl} U = \sigma \operatorname{cl} U$ *for each* $U \in \sigma$, *then in addition to the results given in (3.I) above, we have* $RO(X,\tau) = RO(X,\sigma)$, $SR(X,\tau) = SR(X,\sigma)$, *and* $\theta O(X,\tau) = \theta O(X,\sigma)$,

(4) *If* $\tau \subset \sigma \subset SO(X,\tau)$ *and* $\tau \operatorname{cl} U = \sigma \operatorname{cl} U$ *for each* $U \in \tau$, *then* $\tau \operatorname{cl} U = \sigma \operatorname{cl} U$ *for each* $U \in \sigma$. *So, the results in* (3) *are valid.*

(5) $\tau \subset \omega \subset \sigma$, *and* $\tau clU = \sigma \operatorname{cl} U$ *for each* $U \in \sigma$, *then* $\tau \operatorname{cl} U = \sigma \operatorname{cl} U = \omega \operatorname{cl} U$ *for each* $U \in \sigma$ (*hence for each* $U \in \omega$).
*So, results similar to those given in* (3) *are valid for* $\tau, \omega$ *and* $\sigma$.

*Proof.* (1a) Let $\tau \subset \sigma \subset SPO(X,\tau)$ and $U \in SPO(X,\sigma)$. We have:

$$\begin{aligned}
U &\subset \sigma \operatorname{cl}(\sigma \operatorname{int}(\sigma \operatorname{cl} U)) \subset \tau \operatorname{cl}(\sigma \operatorname{int}(\tau \operatorname{cl} U)) \subset \tau \operatorname{cl}(\tau \operatorname{cl}(\tau \operatorname{int}(\tau \operatorname{cl}(\sigma \operatorname{int}(\tau clU)))))\\
&\subset \tau \operatorname{cl}(\tau \operatorname{int}(\tau clU)).
\end{aligned}$$

(1b) The proof is clear from Corollary 2.13 below.

2. Let $\tau \subset \sigma \subset PO(X,\tau)$ and $U \in PO(X,\sigma)$. We have:

$$\begin{aligned}
U &\subset \sigma \operatorname{int}(\sigma \operatorname{cl} U) \subset \sigma \operatorname{int}(\tau \operatorname{cl} U) \subset \tau \operatorname{int}(\tau \operatorname{cl}(\sigma \operatorname{int}(\tau \operatorname{cl} U))) \subset \tau \operatorname{int}(\tau \operatorname{cl}(\tau \operatorname{cl} U))\\
&= \tau \operatorname{int}(\tau \operatorname{cl} U).
\end{aligned}$$

So, $U \in PO(X,\tau)$. Since $PO(X,\tau) \subset SPO(X,\tau)$, the results in (1) are valid.

(3Ib) Let $U \in SO(X,\tau)$. Since $U \subset \tau \operatorname{cl}(\tau \operatorname{int} U)$ and $\sigma clU \subset \tau \operatorname{cl} U$, we have

$$\sigma \operatorname{cl} U \subset \tau \operatorname{cl} U = \tau \operatorname{cl}(\tau \operatorname{int} U) = \sigma \operatorname{cl}(\tau \operatorname{int} U) \subset \sigma \operatorname{cl}(\sigma \operatorname{int} U) \subset \sigma \operatorname{cl} U.$$

Hence we have $\sigma \operatorname{cl} U = \tau \operatorname{cl} U = \tau \operatorname{cl}(\tau \operatorname{int} U) = \sigma \operatorname{cl}(\sigma \operatorname{int} U)$.

(3Ic–f). These are clear from (3Ia).

(3Ig) This is clear from (1a) since $\tau \subset \sigma \subset SPO(X,\tau)$.

(3Ih) Let $U \in RO(X,\tau)$. We have $U = \tau \operatorname{int}(\tau \operatorname{cl} U)$ and $U \in SO(X,\tau)$. Now if we use (3Ib), we obtain that $\sigma \operatorname{int}(\sigma \operatorname{cl} U) = \sigma \operatorname{int}(\tau \operatorname{cl} U) = \tau \operatorname{int}(\tau \operatorname{cl} U) = U$.

(3II) Under the hypothesis, since $RO(X,\tau) = RO(X,\sigma)$[11], it is clear that $SR(X,\tau) = SR(X,\sigma)$ [7], and $\theta O(X,\tau) = \theta O(X,\sigma)$[12].

(4). The proof is clear from (3Ib). ∎

COROLLARY 2.2. *If* $\tau \cap \mathcal{I} = \{\emptyset\}$ *for* $\tau \in Top(X)$ *and* $\mathcal{I} \in Id(X)$*, then*

(a) *The results* (3) *in Theorem 2.1. are valid by taking* $\tau^*(\mathcal{I})$ *instead of* $\sigma$*.*

(b) *For* $\mathcal{J} \in Id(X)$ *and* $\omega \in Top(X)$*, if* $\omega \cap \mathcal{J} = \{\emptyset\}$ *and* $\omega^*(\mathcal{J}) = \tau^*(\mathcal{I})$*, then* $(X,\tau),(X,\omega),(X,\omega^*(\mathcal{J}))$ *and* $(X,\tau^*(\mathcal{I}))$ *have the same* $RO(X)$*,* $SR(X)$ *and* $\theta O(X)$ *sets.*

(c) *If* $\tau^*(\mathcal{I}) = \sigma^*(\mathcal{I})$ *for* $\sigma \in Top(X)$*, then the results* (3) *in Theorem 2.1. are valid by taking* $\sigma^*(\mathcal{I})$ *instead of* $\sigma$*, and then* $(X,\tau),(X,\tau^*(\mathcal{I}),(X,\sigma)$ *and* $(X,\sigma^*(\mathcal{I}))$ *have the same* $RO(X)$*,* $SR(X)$ *and* $\theta O(X)$ *sets.*

The following theorem and corollaries can be obtained by using Lemma 2.7. below and the results of Andrijević given in [1,2,3]. We note that Corollary 2.6.(1) was given by Rose and Hamlett using a different method [16]. We will obtain these results by using the results given here.

THEOREM 2.3. *Let* $\tau,\sigma \in Top(X)$*. If* $\tau \subset \sigma \subset SO(X,\tau)$*, and* $\tau\,\mathrm{cl}\,U = \sigma\,\mathrm{cl}\,U$ *for each* $U \in \sigma$*, then we have the following results.*

(1) *For each* $A \subset X$*, we have*

(a) $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)$*,*

(b) $\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A) = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,A)$*,*

(c) $\tau\,\mathrm{cl}(\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)) = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A))$*,*

(d) $\tau\,\mathrm{int}(\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A)) = \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}(\sigma int A))$*.*

(2) $(X,\tau)$ *and* $(X,\sigma)$ *have the same* $SO(X)$*,* $PO(X)$*,* $SPO(X)$*,* $RO(X)$*,* $SR(X)$*,* $NO(X)$*,* $D(X)$*,* $\alpha O(X)$*,* $CD(X)$ *and* $\theta O(X)$ *sets.*

*Proof.* (1a) Let $A \subset X$. Then $\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A) \subset \sigma\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)$. Since $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) \in \sigma$ and $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) \subset A \cup \tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \tau\,\mathrm{scl}\,A \subset \sigma\,\mathrm{cl}\,A$, we have $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \sigma\,\mathrm{int}(\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)) \subset \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)$. Hence $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)$.

The remaining proofs are clear. ∎

COROLLARY 2.4. *Let* $\tau \in Top(X), \mathcal{I} \in Id(X)$*. If* $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $\tau^*(\mathcal{I}) \subset SO(X,\tau)$*, then the results of Theorem 2.3 are satisfied by taking* $\tau^*(\mathcal{I})$ *instead of* $\sigma$*.*

COROLLARY 2.5. *Since* $\tau \cap \mathcal{I}_n = \{\emptyset\}$ *and* $\tau^*(\mathcal{I}_n) = \tau^\alpha \subset SO(X,\tau)$*, the results of the above theorem are satisfied by taking* $\tau^\alpha$ *instead of* $\sigma$*.*

COROLLARY 2.6. *If* $\sigma^\alpha = \tau^\alpha$ *(i.e. if* $\sigma \in [\tau]^\alpha$ *in the sense of Njåstad* [14]*), then we have the following results.*

(1) *For each* $A \subset X$*, we have*

(a) $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \tau^\alpha\,\mathrm{int}(\tau^\alpha\,\mathrm{cl}\,A) = \sigma^\alpha\,\mathrm{int}(\sigma^\alpha\,\mathrm{cl}\,A) = \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)$*,*

(b) $\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A) = \tau^\alpha\,\mathrm{cl}(\tau^\alpha\,\mathrm{int}\,A) = \sigma^\alpha\,\mathrm{cl}(\sigma^\alpha\,\mathrm{int}\,A = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,A)$*,*

(c) $\tau\,\mathrm{int}(\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A)) = \tau^\alpha\,\mathrm{int}(\tau^\alpha\,\mathrm{cl}(\tau^\alpha\,\mathrm{int}\,A)) = \sigma^\alpha\,\mathrm{int}(\sigma^\alpha\,\mathrm{cl}(\sigma^\alpha\,\mathrm{int}\,A))$*.*

(d) $\tau\,\mathrm{cl}(\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)) = \tau^\alpha\,\mathrm{cl}(\tau^\alpha\,\mathrm{int}\,A) = (\sigma^\alpha\,\mathrm{cl}\,A(\sigma^\alpha\,\mathrm{int}(\sigma^\alpha\,\mathrm{cl}\,A) = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A))$.

(2) *$(X,\tau),(X,\tau^\alpha),(X,\sigma^\alpha)$ and $(X,\sigma)$ have the same $SO(X)$, $PO(X)$, $SPO(X)$, $NO(X)$, $D(X)$, $CD(X)$, $\alpha O(X)$, $RO(X)$, $SR(X)$ and $\theta O(X)$ sets* [3].

LEMMA 2.7. *Let $\tau,\sigma \in Top(X)$ and $\tau \subset \sigma$. Then, $\sigma \subset SO(X,\tau)$ and $\tau\,\mathrm{cl}\,U = \sigma\,\mathrm{cl}\,U$ for each $U \in \sigma$ iff $\sigma \in [\tau]^\alpha$.*

Njåstad defined $\alpha$-equivalent topologies and $*$-equivalent topologies in [14], [15], respectively. Njåstad showed that if $\tau \subset \sigma \subset \tau^\alpha$ for $\tau,\sigma \in Top(X)$, then $\tau$ and $\sigma$ are $\alpha$-equivalent. For $\tau,\sigma \in Top(X)$, $\mathcal{I} \in Id(X)$, if $\tau^*(\mathcal{I}) = \sigma^*(\mathcal{I})$, then we say that $\sigma$ and $\tau$ are $*\mathcal{I}$-equivalent.

The $\alpha$-equivalence or $*\mathcal{I}$-equivalence of topologies on a set on which ideals are defined is important.

For any ideal $\mathcal{I}$ on $(X,\tau)$, $\tau^*(\mathcal{I})$ and $\tau^*(\mathcal{I})^\alpha$ are $\alpha$-equivalent. We know that $\tau^*(\mathcal{I})^\alpha = (\tau^*(\mathcal{I}))^*(\mathcal{I}_n(\tau^*(\mathcal{I}))) = \tau^*(\mathcal{I} \text{ V } \mathcal{I}_n(\tau^*(\mathcal{I})))$. Hence, for each ideal $\mathcal{J}$ such that $\mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$, $\tau^*(\mathcal{I})$, $\tau^*(\mathcal{I} \vee \mathcal{J})$ and $\tau^*(\mathcal{I} \vee \mathcal{I}_n(\tau^*(\mathcal{I})))$ are $\alpha$-equivalent.

At the same time, for any ideal $\mathcal{I}$, since $\tau^*(\mathcal{I}) \cap \mathcal{I}_n(\tau^*(\mathcal{I}))) = \{\emptyset\}$ and $\tau \subset \tau^*(\mathcal{I})$, we have that that $\tau \cap \mathcal{I}_n(\tau^*(\mathcal{I})) = \{\emptyset\}$. Hence for any ideal $\mathcal{J}$ such that $\mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$ we have $\tau \cap \mathcal{J} = \{\emptyset\}$. And, if $\tau \cap \mathcal{I} = \{\emptyset\}$, then we know that $\mathcal{I}_n(\tau) \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$. Now, we can give the following result.

THEOREM 2.8. *Let $\mathcal{I}$ be an ideal on $(X,\tau)$ and $\mathcal{J}$ any ideal such that $\mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$. Then the following are equivalent.*

(a) $\tau \cap \mathcal{I} = \{\emptyset\}$
(b) $\mathcal{I} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$
(c) $\mathcal{I} \vee \mathcal{I}_n \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$
(d) $\mathcal{I} \vee \mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$
(e) $\mathcal{I} \vee \mathcal{I}_n \vee \mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$
(f) $\tau \cap (\mathcal{I} \vee \mathcal{J}) = \{\emptyset\}$
(g) $\tau \cap (\mathcal{I} \vee \mathcal{J} \vee \mathcal{I}_n) = \{\emptyset\}$
(h) $\tau \cap (\mathcal{I} \vee \mathcal{I}_n) = \{\emptyset\}$.

*Proof.* (a) $\Longrightarrow$ (b) Let $I \in \mathcal{I}$. Then $I$ is $\tau^*(\mathcal{I})$-closed and since $\tau^*(\mathcal{I}) \cap \mathcal{I} = \{\emptyset\}$ we have that $\tau^*(\mathcal{I})$-$\mathrm{int}\,I = \emptyset$. So, $\tau^*(\mathcal{I})$-$\mathrm{int}(\tau^*(\mathcal{I})$-$\mathrm{cl}\,I) = \emptyset$ and $I \in \mathcal{I}_n(\tau^*(\mathcal{I}))$.

The remaining proofs are clear. ■

We deduce that if $\tau \cap \mathcal{I} = \{\emptyset\}$ for an ideal $\mathcal{I}$, then $\mathcal{I} \vee \mathcal{I}_n(\tau^*(\mathcal{I})) = \mathcal{I}_n(\tau^*(\mathcal{I}))$, and $(\tau^*(\mathcal{I}))^\alpha = \tau^*(\mathcal{I}_n(\tau^*(\mathcal{I})))$.

COROLLARY 2.9. *If $\tau \cap \mathcal{I} = \{\emptyset\}$, then for any ideal $\mathcal{J}$ satisfying $\mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$ we have that $\tau^*(\mathcal{I})$, $\tau^*(\mathcal{I} \vee \mathcal{J})$, $\tau^*(\mathcal{I} \vee \mathcal{J} \vee \mathcal{I}_n(\tau^*(\mathcal{I})))$ and $\tau^*(\mathcal{I}_n(\tau^*(\mathcal{I})))$ are all $\alpha$-equivalent.*

Several statements equivalent to $\mathcal{I} \subset \mathcal{I}_n$ have been given in the literature. Since $\tau$ and $\tau^*(\mathcal{I})$ are $\alpha$-equivalent when $\mathcal{I} \subset \mathcal{I}_n$, we give some further conditions for $\mathcal{I} \subset \mathcal{I}_n$, in the following theorem.

THEOREM 2.10. *Let $\mathcal{I}$ be an ideal on $(X,\tau)$ and $\mathcal{I}_n$ the ideal of nowhere dense sets in $(X,\tau)$. Then the following are equivalent.*

(1) $\mathcal{I} \subset \mathcal{I}_n$,

(2) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $\tau$ *and* $\tau^*(\mathcal{I})$ *are* $\alpha$*-equivalent,*

(3) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $\tau^*(\mathcal{I}) \subset \tau^\alpha$,

(4) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $\tau^*(\mathcal{I}) \subset SO(X,\tau)$,

(5) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $SO(X,\tau^*(\mathcal{I})) \subset SO(X,\tau)$,

(6) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $PO(X,\tau) \subset PO(X,\tau^*(\mathcal{I}))$,

(7) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $SPO(X,\tau) \subset SPO(X,\tau^*(\mathcal{I}))$,

(8) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $D(X,\tau) \subset D(X,\tau^*(\mathcal{I}))$,

(9) $A^*(\mathcal{I}_n) \subset A^*(\mathcal{I})$ *for each* $A \subset X$,

(10) $A^{\underline{\circ}} \subset A^*(\mathcal{I})$ *for each* $A \in D(X,\tau)$,

(11) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and*$A^{\underline{\circ}} \subset \tau^*(\mathcal{I})\text{-cl}\,A$ *for each* $A \subset X$,

(12) $\tau \cap \mathcal{I} = \{\emptyset\}$ *and* $\mathcal{I} \subset SC(X,\tau)$.

*Proof.* (1) $\Longrightarrow$ (2) Let $\mathcal{I} \subset \mathcal{I}_n$. We have, $\tau \subset \tau^*(\mathcal{I}) \subset \tau^*(\mathcal{I}_n) = \tau^\alpha$, so $\tau$ and $\tau^*(\mathcal{I})$ are $\alpha$-equivalent from Theorem 1.1.(3).

(2) $\Longrightarrow$ (1) If $\tau^*(\mathcal{I}) \in [\tau]^\alpha$, then we have $\mathcal{I}_n(\tau) = \mathcal{I}_n(\tau^*(\mathcal{I}))$. If $\tau \cap \mathcal{I} = \{\emptyset\}$, then $\mathcal{I} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$. Hence, $\mathcal{I} \subset \mathcal{I}_n$, if $\tau \cap \mathcal{I} = \{\emptyset\}$ and $\tau^*(\mathcal{I}) \in [\tau]^\alpha$.

(2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) Clear.

(2) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (8) Clear from [3, Theorem 1] and Corollary 2.2

(1) $\Longleftrightarrow$ (9) Known from the literature.

(9) $\Longrightarrow$ (10) Clear.

(10) $\Longrightarrow$ (8) Since $X \in D(X,\tau)$, we have $X^{\underline{\circ}} = X \subset X^*(\mathcal{I})$, $X = X^*(\mathcal{I})$ and hence $\tau \cap \mathcal{I} = \{\emptyset\}$. For $A \in D(X,\tau)$ we deduce $\bar{A} = X$, $A^{\underline{\circ}} = X$, $A^*(\mathcal{I}) = X$ and $\tau^*(\mathcal{I})\text{-cl}\,A = A \cup A^*(\mathcal{I}) = X$. Hence we have $D(X,\tau) \subset D(X,\tau^*(\mathcal{I}))$.

(4) $\Longrightarrow$ (11) We have $\text{scl}\,A \subset \tau^*(\mathcal{I})\text{-cl}\,A$ for each $A \subset X$. Since $\text{scl}\,A = A \cup A^{\underline{\circ}}$, the result is clear.

(11) $\Longrightarrow$ (3) Under the hypothesis of (11) we obtain $A^{\overline{\underline{\circ}}} = \tau^*(\mathcal{I})\text{-cl}\,A^{\underline{\circ}} \subset \tau^*(\mathcal{I})\text{-cl}\,A$ and $\tau^\alpha\text{-cl}\,A = A \cup A^{\overline{\underline{\circ}}} \subset \tau^*(\mathcal{I})\text{-cl}\,A$ for each subset $A$. Hence we have $\tau^*(\mathcal{I}) \subset \tau^\alpha$.

(4) $\Longrightarrow$ (12) We know that each $I \in \mathcal{I}$ is $\tau^*(\mathcal{I})$-closed. Since $\tau^*(\mathcal{I}) \subset SO(X,\tau)$, it follows that each $I \in \mathcal{I}$ is $\tau$-semiclosed.

(12) $\Longrightarrow$ (4) If $\mathcal{I} \subset SC(X,\tau)$, then $U - I \in SO(X,\tau)$ for any $U \in \tau$ and any $I \in \mathcal{I}$. So, $SO(X,\tau)$ contains a base of $\tau^*(\mathcal{I})$ (from Theorem 1.1.(10)). Hence $\tau^*(\mathcal{I}) \subset SO(X,\tau)$. ∎

COROLLARY 2.11. *If $\mathcal{I} \subset \mathcal{I}_n$, then we have the following results.*

(a) $\mathcal{I}_n(\tau^*(\mathcal{I})) = \mathcal{I}_n(\tau)$,

(b) $\sigma \in [\tau]^\alpha$ *iff* $\sigma \in [\tau^*(\mathcal{I})]^\alpha$,

(c) *If* $\sigma^*(\mathcal{I}) = \tau^*(\mathcal{I})$, *then* $\tau$, $\tau^*(\mathcal{I})$ *and* $\sigma^*(\mathcal{I})$ *are all* $\alpha$*-equivalent.*

Some other statements equivalent to $\mathcal{I} \subset \mathcal{I}_n$ can be seen from Corollary 2.13. and Corollary2.15.

If $\mathcal{A}$ is a supratopology and $\mathcal{I}$ an ideal on $X$, then it is clear that $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ iff $\mathcal{A}$-int $I = \emptyset$ for each $I \in \mathcal{I}$. In the following theorem, the results are clear and almost all of them are known.

Theorem 2.12. *Let $(X, \tau)$ be a topological space. Then we have the following results for any $A \subset X$.*

(1) $A^{\overset{o}{-}} = \emptyset \iff \text{pre-}\operatorname{int} A = \emptyset \iff A^{\overline{\overset{o}{-}}} = \emptyset \iff \text{semi-pre-}\operatorname{int} A = \emptyset$,

(2) $A^{\overset{o}{\underset{o}{-}}} = \emptyset \iff \alpha\text{-}\operatorname{int} A = \emptyset \iff A^o = \emptyset \iff A^{\overline{o}} = \emptyset \iff \text{semi-}\operatorname{int} A = \emptyset$,

(3) $A^{\overline{o}} = X \iff \text{pre-}\operatorname{cl} A = X \iff A^{\overset{o}{\underset{o}{-}}} = X \iff \text{semi-pre-}\operatorname{cl} A = X$,

(4) $A^{\overline{\overset{o}{-}}} = X \iff \alpha\text{-}\operatorname{cl} A = X \iff \bar{A} = X \iff A^{\overset{o}{-}} = X \iff \operatorname{scl} A = X$.

Clearly, in a topological space $(X, \tau)$, for any $x \in X$, $\{x\} \notin \mathcal{I}_n$ iff pre-int $\{x\} \neq \emptyset$ iff semi-pre-int $\{x\} \neq \emptyset$ iff $\{x\}$ is pre-open iff $\{x\}$ is semi-pre-open.

In the following corollary we assume that the necessary ideals are defined on the topological space $(X, \tau)$.

Corollary 2.13. *We have the following results.*

(1) $\mathcal{I}_n = \{A : A^{\overset{o}{-}} = \emptyset\} = \{A : \text{pre-}\operatorname{int} A = \emptyset\} = \{A : \text{semi-pre-}\operatorname{int} A = \emptyset\} = \{A : A^{\overline{\overset{o}{-}}} = \emptyset\} = CD(X, \tau) \cap SC(X, \tau)$.

(2) $\mathcal{I}_n \cap PO(X, \tau) = \{\emptyset\}$ *and* $\mathcal{I}_n \cap SPO(X, \tau) = \{\emptyset\}$.

(3) $\mathcal{I} \subset \mathcal{I}_n$ *iff* $\mathcal{I} \cap PO(X, \tau) = \{\emptyset\}$ [4] *iff* $\mathcal{I} \cap SPO(X, \tau) = \{\emptyset\}$.

(4) *For* $\sigma \in Top\,(X)$, *if* $PO(X, \sigma) \subset PO(X, \tau)$ *or* $SPO(X, \sigma) \subset SPO(X, \tau)$, *then* $\mathcal{I}_n(\tau) \subset \mathcal{I}_n(\sigma)$.

(5) $CD(X) = \{A : A^o = \emptyset\} = \{A : A^{\overset{o}{\underset{o}{-}}} = \emptyset\} = \{A : \alpha\text{-}\operatorname{int} A = \emptyset\} = \{A : \text{semi-}\operatorname{int} A = \emptyset\} = \{A : A^{\overline{o}} = \emptyset\}$. (6) *For* $\sigma \in Top\,(X)$, *if* $SO(X, \tau) \subset SO(X, \sigma)$ *or* $\tau^\alpha \subset \sigma^\alpha$, *then* $D(X, \sigma) \subset D(X, \tau)$ *and* $CD(X, \sigma) \subset CD(X, \tau)$.

(7)(a) $\tau \cap \mathcal{I} = \{\emptyset\}$ *iff* $SO(X, \tau) \cap \mathcal{I} = \{\emptyset\}$ [4] *iff* $\tau^\alpha \cap \mathcal{I} = \{\emptyset\}$ *iff* $\tau^*(\mathcal{I}) \cap \mathcal{I} = \{\emptyset\}$ *iff* $SO(X, \tau^*(\mathcal{I})) \cap \mathcal{I} = \{\emptyset\}$ *iff* $\tau^*(\mathcal{I})^\alpha \cap \mathcal{I} = \{\emptyset\}$.

(b) *Let* $\sigma \in Top\,(X)$ *and* $\sigma^*(\mathcal{I}) = \tau^*(\mathcal{I})$, *then we have that* $\tau \cap \mathcal{I} = \{\emptyset\}$ *iff* $\sigma \cap \mathcal{I} = \{\emptyset\}$.

(c) *Let* $\sigma \in Top\,(X)$ *and* $\sigma \in [\tau^*(\mathcal{I})]^\alpha$, *then we have that* $\tau \cap \mathcal{I} = \{\emptyset\}$ *iff* $\sigma \cap \mathcal{I} = \{\emptyset\}$.

Now, by combining the results given above with the following facts,

(i) If $\tau \subset \sigma \subset \tau^*(\mathcal{I})$ for $\sigma \in Top\,(X)$, then $\sigma^*(\mathcal{I}) = \tau^*(\mathcal{I})$,

(ii) If $\tau^*(\mathcal{I}) \subset \sigma \subset (\tau^*(\mathcal{I}))^\alpha = (\tau^*(\mathcal{I}))^*(\mathcal{I}_n(\tau^*(\mathcal{I})))$ then $\sigma \in [\tau^*(\mathcal{I}))]^\alpha$,

we can obtain several conditions equivalent to $\tau \cap \mathcal{I} = \{\emptyset\}$.

For a supratopology $\mathcal{A}$ on $X$, $\mathcal{T}_\mathcal{A}$ will stand for the topology

$$\{U : A \in \mathcal{A} \implies U \cap A \in \mathcal{A}\} \text{ [19].}$$

We know that $\tau \subset \mathcal{T}_{PO(X,\tau)}$, $\tau \subset \mathcal{T}_{SO(X,\tau)}$, $\tau \subset \mathcal{T}_{SPO(X,\tau)}$ and $\tau \subset \tau^\alpha = \mathcal{T}_{\tau^\alpha}$ [1–3].

Theorem 2.14. *Let $(X,\tau)$ be a topological space, $\mathcal{I} \in Id(X)$ and $\mathcal{A}$ a supratopology on $X$. Then we have the following results.*

(1) *If $\tau \cap \mathcal{I} = \{\emptyset\}$ and $A^o \neq \emptyset$ for each $A \in \mathcal{A} - \{\emptyset\}$, then $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$.*

(2) *If $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ and $\tau \subset \mathcal{T}_{\mathcal{A}}$, then*

(a) *$A \subset A^*$ for each $A \in \mathcal{A}$,*

(b) *$\bar{A} = A^* = \tau^*$-cl $A$ for each $A \in \mathcal{A}$.*

(3)(a) *If $A^*(\mathcal{I}) \neq \emptyset$ for each $A \in \mathcal{A} - \{\emptyset\}$, then $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$,*

(b) *If $\tau \subset \mathcal{T}_{\mathcal{A}}$, then $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ iff $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.*

(4) *$\mathcal{A} \cap \mathcal{I}_n = \{\emptyset\}$ iff $A^{\underline{o}} \neq \emptyset$ for each $A \in \mathcal{A} - \{\emptyset\}$.*

(5) *If $\mathcal{A} \cap \mathcal{I}_n = \{\emptyset\}$ and $\tau \subset \mathcal{T}_{\mathcal{A}}$, then $\mathcal{A} \subset SPO(X,\tau)$.*

(6) *If $PO(X,\tau) \subset \mathcal{A} \subset SPO(X,\tau)$, then $\mathcal{A} \cap \mathcal{I}_n = \{\emptyset\}$ iff $\mathcal{I} \subset \mathcal{I}_n$ iff $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.*

(7) *If $\tau \subset \mathcal{A} \subset SO(X,\tau^*(\mathcal{I}))$, then $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ iff $\tau \cap \mathcal{I} = \{\emptyset\}$ iff $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.*

*Proof.* (2a) Let $A \in \mathcal{A}$ and $x \in A$. If $x \in U \in \tau$, then since $\tau \subset \mathcal{T}_{\mathcal{A}}$, we have $\emptyset \neq U \cap A \in \mathcal{A}$. Hence, $U \cap A \notin \mathcal{I}$. So, we have $x \in A^*$.

(2b) We know that $A^*$ is $\tau$-closed, $A^* \subset \bar{A}$ and $\tau^* clA = A \cup A^*$. Result is clear from (a).

(3a) It is known that $I^* = \emptyset$ for any ideal $\mathcal{I}$ and for each $I \in \mathcal{I}$. Now, result is clear.

(3b) Clear from (3a) and (2a)

(4) Clear from Corollary 2.13.

(5) Let $A \in \mathcal{A}$. From (3)(b) we have $A \subset A^*(\mathcal{I}_n) = A^{\overline{\underline{o}}}$. Hence, $A \in SPO(X,\tau)$.

(6) Let $PO(X,\tau) \subset \mathcal{A} \subset SPO(X,\tau)$. If $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ then $PO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ and hence $\mathcal{I} \subset \mathcal{I}_n$. If $\mathcal{I} \subset \mathcal{I}_n$, then $SPO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ and hence $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$.

If $\mathcal{I} \subset \mathcal{I}_n$, then $SPO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ . Now, from Corollary 2.15.(1) below, and since $\mathcal{A} \subset SPO(X,\tau)$, we have $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.

If $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$, then we have $A \subset A^*(\mathcal{I})$ for each $A \in PO(X,\tau)$. So, from Corollary 2.15(1), we have $\mathcal{I} \subset \mathcal{I}_n$.

(7) Let $\tau \subset \mathcal{A} \subset SO(X,\tau^*(\mathcal{I}))$. If $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$, then $\tau \cap \mathcal{I} = \{\emptyset\}$. If $\tau \cap \mathcal{I} = \{\emptyset\}$, then from Corollary 2.13(7) we have $SO(X,\tau^*(\mathcal{I})) \cap \mathcal{I} = \{\emptyset\}$. So, $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$.

If $\tau \cap \mathcal{I} = \{\emptyset\}$, then from Corollary 2.15(2) below we have $A \subset A^*(\mathcal{I})$ for each $A \in SO(X,\tau^*(\mathcal{I}))$. So, $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.

If $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$, then from Theorem 1.1.(9) we have $U \subset U^*$ for each $U \in \tau$, and $\tau \cap \mathcal{I} = \{\emptyset\}$. ∎

Corollary 2.15. *We have the following results.*

(1) *$PO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ iff $A \subset A^*(\mathcal{I})$ for each $A \in PO(X,\tau)$ [4] iff $A \subset A^*(\mathcal{I})$ for each $A \in SPO(X,\tau)$.*

(2) *$\tau \cap \mathcal{I} = \{\emptyset\}$ iff $A \subset A^*(\mathcal{I})$ for each $A \in SO(X,\tau)$. [4] iff $A \subset A^*(\mathcal{I})$ for*

*each* $A \in \tau^\alpha$ *iff* $A \subset A^*(\mathcal{I})$ *for each* $A \in SO(X, \tau^*(\mathcal{I}))$ *iff* $A \subset A^*(\mathcal{I})$ *for each* $A \in \tau^*(\mathcal{I})^\alpha$.

*Proof.* The proofs are clear from Theorem 1.1.(2), Theorem 2.14(3b) and Theorem 1.1(5). ∎

COROLLARY 2.16. *We have the following results.*

(1) *If* $PO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$, *then* $\tau\text{-cl}\, A = A^* = \tau^*\text{-cl}\, A$ *for each* $A \in SPO(X,\tau)$.

(2) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then* $\tau\text{-cl}\, A = A^* = \tau^*\text{-cl}\, A$ *for each* $A \in SO(X, \tau^*(\mathcal{I}))$.

LEMMA 2.17. *Let* $(X,\tau)$ *be topological space,* $\mathcal{A}$ *a supratopology on* $X$ *such that* $\tau \subset \mathcal{T}_{\mathcal{A}}$. *If* $\mathcal{A}\text{-}\operatorname{int} B = \emptyset$ *for a subset* $B$, *then* $(A \cap B)^- \subset (A - B)^-$ *and* $\bar{A} = (A - B)^-$ *for each* $A \in \mathcal{A}$.

*Proof.* Let $B \subset X$, $\mathcal{A}\text{-}\operatorname{int} B = \emptyset$, $A \in \mathcal{A}$, $x \in (A \cap B)^-$ and $x \in U \in \tau$. Then $U \cap A \cap B \neq \emptyset$ and $U \cap A \neq \emptyset$. Since $\tau \subset \mathcal{T}_{\mathcal{A}}$, we have $\emptyset \neq U \cap A \in \mathcal{A}$. So $U \cap A \not\subset B$, and $(A - B) \cap U = A \cap U - B \neq \emptyset$. Hence $x \in (A - B)^-$.

This now gives $\bar{A} = (A - B)^- \cup (A \cap B)^- = (A - B)^-$. ∎

COROLLARY 2.18. *Let* $\mathcal{I} \in Id(X)$ *and* $\mathcal{A}$ *a supratopology on* $X$. *If* $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$, *then we have the following results.*

(a) *If* $\tau \subset \mathcal{T}_{\mathcal{A}}$, *then* $\bar{A} = (A - I)^-$ *for each* $A \in \mathcal{A}$ *and for each* $I \in \mathcal{I}$.

(b) *If* $\tau^*(\mathcal{I}) \subset \mathcal{T}_{\mathcal{A}}$, *then* $\bar{A} = (A - I)^-$ *and* $\tau^*\text{-}clA = \tau^*\text{-cl}(A - I)$ *for each* $A \in A$ *and for each* $I \in \mathcal{I}$.

*Proof.* (a) If $I \in \mathcal{I}$ we obtain $\mathcal{A}\text{-}\operatorname{int} I = \emptyset$. The proof is now clear from Lemma 2.17.

(b) If $\tau^* \subset \mathcal{T}_{\mathcal{A}}$ then we obtain $\tau \subset \tau^* \subset \mathcal{T}_{\mathcal{A}}$. The proof is now clear from Lemma 2.17. ∎

COROLLARY 2.19. *Let* $(X,\tau)$ *be a topological space and* $\mathcal{I} \in Id(X)$. *Then we have the following results.*

(1) *If* $PO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$, *then* $\bar{A} = (A - I)^- = A^* = \tau^*\text{-}clA = (A - I)^*$ *for each* $I \in \mathcal{I}$ *and for each* $A \in SPO(X,\tau)$.

(2) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then* $\bar{A} = (A - I)^- = A^* = \tau^*\text{-cl}\, A = \tau^*\text{-cl}(A - I) = (A - I)^*$ *for each* $I \in \mathcal{I}$ *and for each* $A \in SO(X, \tau^*(\mathcal{I}))$.

(3) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then we have* $\tau\text{-scl}\, A = \tau^*\text{-scl}\, A$ *for each* $I \in \mathcal{I}$ *and for each* $A \in SO(X, \tau^*(\mathcal{I}))$.

*Proof.* (1),(2) Clear from Corollory 2.16, Corollary 2.18 and Theorem 1.1.(15).

(3) From (2) we will have $A^{\underline{o}} = \tau\text{-int}(\tau^*\text{-cl}\, A) = \tau^*\text{-int}(\tau^*\text{-cl}\, A)$ and $\tau\text{-scl}\, A = A \cup A^{\underline{o}} = A \cup \tau^*\text{-int}(\tau^*\text{-cl}\, A) = \tau^*\text{-scl}\, A$. ∎

REFERENCES

[1] D. Andrijević, *Some properties of the topology of* $\alpha$*-sets*, Mat. Vesnik, 36 (1984), 1–10.

[**2**] D. Andrijević, *A note on preopen sets*, Supplemento ai Rendiconti del Circolo Matematica di Palermo, Serie II, 18, (1988), 195–201.

[**3**] D. Andrijević, *A note on $\alpha$-equivalent topologies*, Mat. Vesnik, 45 (1993), 65–69.

[**4**] R. Devi, D. Sivaraj and T. Chelvam, *Codense and completely codense ideals*, Acta Math. Hungar., 108, 3, (2005),197–205.

[**5**] E. Hayashi, *Topologies defined by local properties*, Math. Ann., 156 (1964), 205–215.

[**6**] D. Janković and T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, 97 (1990), 295–310,

[**7**] D. Janković and I.L. Reilly, *On semi seperation properties*, Indian J. Math., 6, (1985), 957–967.

[**8**] K. Kuratowski, *Topologie I*, Warszawa, 1933.

[**9**] P.E. Long and L.L. Herrington, *The $\tau_\theta$-topology and faintly continuous functions*, Kyungpook Math. J., 22 (1982), 7–14.

[**10**] A.S. Mashhour, T.H. Khedr and S.Abd El-Bakkey, *On supra-$R_o$ and supra-$R_1$ spaces*, Indian J. Pure Appl. Math., 16 (1985), 1300–1306.

[**11**] J. Mioduszewkski and L. Rudolf, *H-closed and externally disconnected Hausdorff spaces*, Dissertationes Math., 66 (1969), 1–55.

[**12**] M.N. Mukherjee and C.K. Basu, *$\theta$-equivalent spaces- A new approach to RO-equivalent spaces and semi-reguler properties*, Indian J. pure appl. Math., 22, 9(1991), 745–750.

[**13**] L. Newcomb, *Topologies which are compact mudulo an ideal*, Ph.D. Dissertation, Univ. of Cal. at. Santa Barbara (1967).

[**14**] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. 15, 3 (1965), 961–970.

[**15**] O. Njåstad, *Remarks on topologies defined by local properties*, Avh. Norske vid.-Akad. Oslo I(N.S), 8(1966), 1–16.

[**16**] D. Rose and T.R. Hamlett, *Ideally equivalent topologies and semitopological properties*, Math. Choronicle 20 (1991), 149–156.

[**17**] P. Samuels, *A topology formed from a given topology and ideal*, J. London Math. Soc., 2,10 (1975), 409–416.

[**18**] R. Vaidyanathaswamy, *The localization theory in set-topology*, Proc. Indian Acad. Sci., 20 (1945), 51–61.

[**19**] T.H. Yalvac, *Relations between new topologies obtained from old ones*, Acta Math. Hungar., 64, 3 (1994), 231–235.

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