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INVERTIBLE COMPOSITION OPERATORS ON BANACH FUNCTION SPACES

Rajeev Kumar

Abstract. In this paper, we relate composition operators with multiplication operators on the general Banach function spaces on a σ-finite measure space. We use this relation to study the invertibility and Fredholmness properties of composition operators on the Banach function spaces.

1. Introduction

Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite measure space. By $L(\mu)$, we denote the linear space of all equivalence classes of Σ-measurable functions on Ω, where we identify any two functions that are equal μ-a.e. on Ω. Let $T\colon \Omega \to \Omega$ be a non-singular measurable transformation, that is, $T^{-1}(A) \in \Sigma$, for each $A \in \Sigma$ and $\mu(T^{-1}(A)) = 0$ for each $A \in \Sigma$ whenever $\mu(A) = 0$. The Radon-Nikodym Theorem ensures the existence of a non-negative locally integrable function f_T on Ω so that the measure $\mu \circ T^{-1}$ can be represented as

$$\mu \circ T^{-1}(A) = \int_A f_T(x)\, d\mu(x), \text{ for each } A \in \Sigma.$$

Then T defines a well-defined composition transformation C_T from $L(\mu)$ into itself defined by

$$C_T f(x) = f(T(x)),\ x \in \Omega,\ f \in L(\mu).$$

In case C_T maps a Banach function space X into itself, then it is called a composition operator on X induced by T.

The study of composition operators on Banach function spaces has been initiated in [16]. For the study of composition operators on L^p-spaces and Orlicz spaces (see [7], [18], [25] and the references therein).

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For $u \in L(\mu)$, we define the multiplication transformation M_u on $L(\mu)$ as

$$M_u f = uf, \text{ for each } f \in L(\mu).$$

In case M_u is continuous on X, we call it a multiplication operator on X. Note that a multiplication operator M_u on a Banach function space X is continuous if and only if $u \in L^\infty(\mu)$, the Banach space of all essentially bounded measurable functions on Ω.

The study of multiplication operators on the general Banach function spaces has been initiated in [1] and [11]. In [11], the multiplication operators find their applications in semigroup theory in solving the abstract Cauchy problem.

A Banach function space X is a Banach space defined as

$$X = \{f \in L(\mu) : \|f\|_X < \infty\},$$

where the function norm $\|.\|_X$ on X has the following properties: for each $f, g, f_n \in L(\mu)$, $n \geq 1$, we have

(i) $0 \leq | g(x) | \leq | f(x) |$ μ-a.e. $x \in \Omega \Rightarrow \|g\|_X \leq \|f\|_X$,

(ii) $0 \leq | f_n(x) | \nearrow | f(x) |$ μ-a.e. $x \in \Omega \Rightarrow \|f_n\|_X \nearrow \|f\|_X$, and

(iii) $E \in \Sigma$ with $\mu(E) < \infty$ implies that $\chi_E \in X$, and

$$\int_E |f| \, d\mu \leq c_E \|f\|_X,$$

for some constant $0 < c_E < \infty$ depending on E and the norm $\|.\|_X$ but independent of f.

A function f in a Banach function space X is said to have an *absolutely continuous norm* if $\|f\chi_{E_n}\|_X \to 0$ for each sequence $\{E_n\}_{n=1}^\infty$ in Σ satisfying $E_n \to \emptyset$ μ-a.e., that is, $\chi_{E_n} \to 0$ μ-a.e.

Let X_a be the set of all functions in X having absolutely continuous norm. If $X = X_a$, we say that X is a Banach function space having absolutely continuous norm.

Let X_b denotes the closure of the set of all μ-simple functions in X. Note that $X_a \subseteq X_b \subseteq X$. Throughout this paper, we assume that $X = X_b$, that is, the μ-simple functions are dense in X. In case X has absolutely continuous norm, we have

$$X_a = X_b = X$$

and then its Banach space dual X^* and its associate space X' coincide, where X' is also a Banach function space defined as

$$X' = \{g \in L(\mu) : \|g\|_{X'} < \infty\},$$

and

$$\|g\|_{X'} = \sup\{\int_\Omega |f(x)g(x)| \, d\mu(x) : f \in X, \ \|f\|_X \leq 1\}.$$

The monotone convergence theorem holds in every Banach function space X in the form of the weak Fatou property, by axiom (ii). Note that X_a is the largest subspace of X in which the suitable dominated convergence theorem holds (cf., [3, Proposition 3.6]. For details on Banach function spaces, we refer to [3], [19], [20] and [21].

Note that $X = X(\mathbf{N}, P(\mathbf{N}), \mu)$ indicates a Banach sequence space considered on $\mathbf{N}$ with a weight function $\mu\colon \mathbf{N} \to (0, \infty)$, where $P(\mathbf{N})$ denotes the power set of $\mathbf{N}$. If $\mu(n) = 1$, for each $n \in \mathbf{N}$, we write $X = X(\mathbf{N})$.

The separable Banach function spaces form a proper subclass of the absolutely continuous ones. The examples of Banach function spaces having absolutely continuous norm are L^p-spaces, Orlicz spaces with Δ_2-conditions [23], Lorentz spaces [3], the Orlicz-Lorentz spaces as defined in [10], etc.

By $\langle f, g\rangle$, we denote the duality pairing between the Banach spaces X and X^*, that is, $\langle f, g\rangle = g(f)$, for each $f \in X$ and $g \in X^*$. In case X is a Banach function space, then for each $f \in X$ and $g \in X'$, we have

$$\langle f, g\rangle = \int_\Omega |f(x)g(x)|\, d\mu(x).$$

We recall from [16, Theorem 2.4] that $C_T\colon X \to X$ and $C_T\colon X' \to X'$ are bounded composition operators on rearrangement invariant spaces X and X' on a resonant measure space (Ω, Σ, μ) if and only if

$$\mu(T^{-1}(A)) \leq \mu(A), \text{ for each } A \in \Sigma.$$

This result forms the base for our assumption that C_T maps a Banach function $X = X_b$ into itself and also maps the corresponding associate space X' into itself to prove our main result which is used to prove many important results in the following sections.

Note that the adjoint operator of the composition operator C_T on a Banach function space X is $C_T^*\colon X^* \to X^*$ such that

$$\langle f, C_T^* g\rangle = \langle C_T f, g\rangle,$$

for each $f \in X$ and $g \in X^*$

In section 2, we study composition operators with closed range. In section 3, we discuss the invertibility of composition operators on Banach function spaces and we also give a necessary and sufficient condition for the injectiveness of composition operators on these spaces. We discuss invertible composition operators induced by invertible transformations in section 4. In section 5, we characterise Fredholm composition operators on X and we also give conditions so that the operator C_T^* on X^* becomes a composition operator on X'.

2. Composition operators with closed range

In this section, we prove our main result with the help of the following lemma. Note that an operator between Banach spaces has closed range if and only if its adjoint does and the range of the adjoint is closed if and only if it is wk$*$-closed.

Lemma 2.1. *Suppose $u \in L(\mu)$ such that $M_u\colon X \to X$. Then M_u is one-to-one if and only if $u(x) \neq 0$, μ-a.e. $x \in \Omega$.*

Proof. Suppose that $u(x) \neq 0$, μ-a.e. $x \in \Omega$. Let $g \in \ker(M_u)$. Then $\|M_u g\|_X = 0$ and by definition, for each $E \in \Sigma$ with $\mu(E) < \infty$, we have

$$\int_E |M_u g(x)|\, d\mu(x) \leq c_E \|M_u g\|_X = 0$$

which clearly implies that $g(x) = 0$, μ-a.e. $x \in \Omega$, since μ is σ-finite. This proves that $g = 0$, that is M_u is one-to-one.

Let $A = \{x \in \Omega : u(x) = 0\}$ have a positive measure. Then, there is a measurable set $F \subseteq A$ with $0 < \mu(F) < \infty$ so that $\chi_F \in X$. Clearly, $M_u \chi_F = 0$ which implies that M_u is not injective. This proves the lemma. ∎

Theorem 2.2. (Main result) *Let $X = X(\Omega, \Sigma, \mu)$ be a Banach function space. Let $T\colon \Omega \to \Omega$ be a non-singular measurable transformation such that $C_T\colon X \to X$, $C_T\colon X' \to X'$ and $C_T^*\colon X' \to X'$ are continuous, $\|f_T\|_\infty < \infty$, and $T(\Omega) = \Omega$. Then we have*

(i) *$C_T^* C_T = M_{f_T}$ on X'.*

(ii) *$C_T C_T^* P = M_{f_T \circ T} P$ on X', where $P\colon X' \to \overline{\mathrm{range}(C_T)} \subseteq X'$ is the projection operator.*

(iii) *If C_T has dense range in X', then $C_T C_T^* = M_{f_T \circ T}$ on X'. Further, this equality implies that C_T has dense range in X.*

Proof. (i) For each $f \in X$ and $g \in X'$, we have

$$\begin{aligned}\langle f, C_T^* C_T g\rangle &= \langle C_T f, C_T g\rangle = \int_\Omega |f(T(x))g(T(x))|\, d\mu(x)\\ &= \int_{T(\Omega)} |f(y)g(y)| f_T(y)\, d\mu(y) = \int_\Omega |f(y)(M_{f_T} g)(y)|\, d\mu(y)\\ &= \langle f, M_{f_T} g\rangle\end{aligned}$$

which implies that $C_T^* C_T = M_{f_T}$ on X'.

(ii) Let $P(g) \in \overline{\mathrm{range}(C_T)} \subseteq X'$, so there exists a sequence $\{g_n\}_{n\geq 1}$ in X' such that $C_T g_n \to Pg$ in norm, that is, $\|C_T g_n - Pg\|_{X'} \to 0$. Now, we have

$$C_T C_T^* Pg = \lim_n C_T C_T^*(C_T g_n) = \lim_n C_T(M_{f_T} g_n) = M_{f_T\circ T} \lim_n (C_T g_n) = M_{f_T \circ T} Pg.$$

Therefore, we have

$$C_T C_T^* P = M_{f_T \circ T} P \text{ on } X'.$$

(iii) Suppose C_T has dense range in X', that is, $C_T C_T^* = M_{f_T \circ T}$, since then $P = I$ on X'. Now, $C_T C_T^* = M_{f_T \circ T}$ and since $f_T \circ T \neq 0$ a.e., we see that $M_{f_T\circ T}$ is injective so that $C_T C_T^*$ is also injective. Also, we see that $\ker(C_T^*) \subseteq \ker(C_T C_T^*) = \{0\}$. Therefore, C_T^* is also injective.

Thus, we have

$$\overline{\mathrm{range}_X(C_T)} =^{\perp} (\ker(C_T^*)) = X.$$

Hence C_T has dense range in X. ∎

Corollary 2.3. *In case $C_T^*\colon X \to X$, then the assertions of the above theorem hold true on X.*

Corollary 2.4. *If X is a Banach function space such that $X = Y'$, for some Banach function space Y. Let T be a non-singular measurable transformation such that $C_T\colon X \to X$, $C_T^*\colon X \to X$, then the assertions of the above theorem hold true on X.*

The next theorem characterises composition operators with closed range.

Theorem 2.5. *Let X be a Banach function space with the associate space X' and let T be a non-singular measurable transformation such that $C_T\colon X' \to X'$, $C_T^*\colon X' \to X'$ and C_T and $C_T^* C_T$ have closed range together in X'. Then C_T has closed range in $X' \subseteq X^*$ if and only if f_T is bounded away from 0 on its support.*

Proof. By Theorem 2.2, we have $C_T^* C_T = M_{f_T}$ on X' and so C_T has closed range in X' if and only if M_{f_T} has so in X' if and only if f_T is bounded away from 0 a.e. on its support, by [11, Theorem 2.3]. ■

Theorem 2.6. *Let $T\colon \Omega \to \Omega$ be a non-singular measurable transformation satisfying the hypothesis of Theorem 2.5 with $\Omega = (\mathbf{N}, \mu)$. If $a = \inf \mu(n) > 0$ and $b = \sup \mu(n) < \infty$, then C_T has closed range in X'.*

Proof. Follows from the above theorem since in this case the Radon-Nikodym function f_T is bounded away from 0. ■

Corollary 2.7. *If $a > 0$, $b < \infty$ and T is a non-singular transformation on $\mathbf{N}$ satisfying the hypothesis of Theorem 2.5 with $\Omega = (\mathbf{N}, \mu)$, then C_T has closed range in $X = X(\mathbf{N}, \mu)$.*

Definition 1. A Banach function space $X(\mathbf{N}, \mu)$ is said to admit composition operators with non-closed range if there exists a measurable transformation $T\colon \mathbf{N} \to \mathbf{N}$ such that range(C_T) is not closed in $X(\mathbf{N}, \mu)$.

Theorem 2.8. *If $a > 0$ and T is a non-singular transformation on $\mathbf{N}$ satisfying the hypothesis of Theorem 2.5 with $\Omega = (\mathbf{N}, \mu)$, then $X'(\mathbf{N}, \mu)$ admits composition operators with non-closed range if and only if $b = \infty$.*

Proof. Suppose that $b < \infty$. Then by Theorem 2.6, for every non-singular transformation T such that $C_T \in B(X'(\mathbf{N}, \mu))$, we see that C_T has closed range which proves that $b = \infty$.

Conversely, let $b = \infty$. Then for each $n \in \mathbf{N}$, there exists some $m \in \mathbf{N}$ such that $\mu(n)/\mu(m) < 1/n$. Define a measurable transformation T as $T(n) = m$. Then, we can choose m such that T is one-to-one and T induces a composition operator on $X'(\mathbf{N}, \mu)$. Since f_T either takes the value $\mu(n)/\mu(m)$ or 0, for each $n \in \mathbf{N}$, so that f_T is not bounded below on its support(f_T). Hence C_T does not have closed range in $X'(\mathbf{N}, \mu)$. ■

Theorem 2.9. *Let $b < \infty$ and T be a non-singular transformation on $\mathbf{N}$ satisfying the hypothesis of Theorem 2.5 with $\Omega = (\mathbf{N}, \mu)$. Then $B(X'(\mathbf{N}, \mu))$ admits composition operators with closed range if and only if $a = 0$.*

Proof. If $a > 0$, then again by Theorem 2.6, we see that C_T has closed range in X' so that $a = 0$.

Conversely, if $a = 0$, choose a subsequence $\{\mu(n_m)\}_{m\geq 1}$ of the sequence $\{\mu(n)\}_{n\geq 1}$ such that $\sum_{m\geq 1} \mu(n_m) < \infty$ and choose another subsequence $\{\mu(n_{m_i})\}_{i\geq 1}$ in $\{\mu(n_m)\}_{m\geq 1}$ such that

$$\frac{\mu(n_{m_i})}{\mu(n_i)} \to 0.$$

Define T as

$$T(n) = \begin{cases} n_i & \text{if } n = m_i, \\ 1 & \text{if } n = n_j, \text{ but } n \neq n_{m_i} \\ n & \text{if } n \neq n_j. \end{cases}$$

Then T induces a composition operator which does not have closed range. This proves the theorem. ■

3. Invertible composition operators

The next theorem characterizes one-to-one composition operators. For each $f \in L(\mu)$, we define the essential range of f as

$$\text{ess ran}\, f = \{\lambda \in \mathbf{C} : \mu(f^{-1}(F)) \neq 0, \text{ for each neighbourhood } F \text{ of } \lambda\}.$$

Definition 2. A measurable transformation $T\colon \Omega \to \Omega$ is said to be one-to-one (or left invertible) if there exists a measurable transformation $S\colon \Omega \to \Omega$ such that

$$(S \circ T)(x) = x \ \mu\text{-a.e. } x \in \Omega.$$

T is said to be onto (or right invertible) if there exists a measurable transformation $w\colon \Omega \to \Omega$ such that

$$(T \circ w)(x) = x \ \mu\text{-a.e. } x \in \Omega.$$

T is said to be invertible if there exists a measurable transformation $S\colon \Omega \to \Omega$ such that

$$(T \circ S)(x) = (S \circ T)(x) = x \ \mu\text{-a.e. } x \in \Omega.$$

Theorem 3.1. *Let X be a Banach function space. Let T be a non-singular measurable transformation such that $C_T\colon X \to X$. Then the following are equivalent.*

(i) *C_T is one-to-one on X.*

(ii) *$\text{ess ran}\, f = \text{ess ran}\, C_T f$, for each $f \in X$.*

(iii) *$\mu \ll \mu \circ T^{-1}$.*

(iv) *$f_T \neq 0$ a.e.*

Proof. The proof of the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is on similar lines as given in [25, Theorem 2.2.2] for L^p-spaces.

(iv) $\Rightarrow$ (i) Since the simple functions are dense in X, and the simple functions are nothing but the linear combinations of characteristic functions, so it suffices to prove the result for characteristic functions.

Let $E \in \Sigma$ be such that $C_T\chi_E = 0 \Rightarrow \chi_{T^{-1}}(E) = 0$ a.e. so that $\mu(T^{-1}(E)\,) = 0$. Then $f_T(x) = 0$ on E. Therefore, $\mu(E) = 0$ so that $\chi_E = 0$, then using the continuity of C_T we see that C_T is one-to-one. This proves our result. ∎

Corollary 3.2. *Let $T\colon \Omega \to \Omega$ be a non-singular measurable transformation satisfying the hypothesis of Theorem 2.2 on a Banach function space X, then the following are equivalent.*

(i) *C_T is one-to-one on X'.*

(ii) ess ran $f =$ ess ran $C_T f$, *for each $f \in X'$.*

(iii) $\mu \ll \mu \circ T^{-1}$.

(iv) $f_T \neq 0$ *a.e.*

Proof. We prove only (iv) $\Rightarrow$ (i), since rest of the implications follow from the above theorem. By Theorem 2.2, we have $C_T^*C_T = M_{f_T}$ on X'. Now if $f_T \neq 0$, then by Lemma 2.1, we see that M_{f_T} is one-to-one on X' which implies that $\ker C_T^*C_T = \{0\}$, and $\ker C_T \subseteq \ker(C_T^*C_T)$. Hence C_T is one-to-one on X'. ∎

Corollary 3.3. *If T is a non-singular measurable transformation on $\Omega = (\mathbf{N}, P(\mathbf{N}), \mu)$, then C_T is one-to-one on $X = X(\mathbf{N})$ if and only if T is onto.*

Corollary 3.4. *Let T be a non-singular measurable transformation on Ω such that ess ran $f =$ ess ran $C_T f$, for each $f \in X$. Then $C_T f$ is a characteristic function if and only if f is so.*

Theorem 3.5. *Let T be a non-singular measurable transformation on Ω such that $C_T : X \to X$ is continuous. Let T be right invertible (onto) such that the right inverse of T is non-singular. Then C_T is one-to-one on X.*

Proof. In view of Theorem 2.5, it suffices to prove that $\mu \ll \mu \circ T^{-1}$. If T is onto, there is a measurable transformation w on Ω such that $T \circ w(x) = x$, μ-a.e. $x \in \Omega$. Let $E \in \Sigma$. Then

$$(T \circ w)^{-1}(E) = w^{-1}(T^{-1}(E)) = E.$$

Suppose $\mu(T^{-1}(E)) = 0$, since w is non-singular, we have

$$\mu(w^{-1}(T^{-1}(E))) = 0 \Rightarrow \mu(E) = 0.$$

This proves that C_T is injective on X. ∎

Remark 1. The converse of the above theorem need not be true. For example, let $\Omega = \mathbf{R}$, $\Sigma =$ Borel σ-algebra on $\mathbf{R}$ with Lebesgue measure μ. Let $T(x) = x^2$, for each $x \in \mathbf{R}$, then T is not onto. But C_T is injective on any Banach function space $X = X(\mathbf{R})$, since $\mu \ll \mu \circ T^{-1} \ll \mu$.

The next theorem characterizes the surjective composition operators. First we prove the next lemma.

Lemma 3.6. *Let T be a non-singular measurable transformation on Ω such that $C_T : X \to X$ is continuous. Then*

$$\overline{\text{range } (C_T)} = X(\Omega, T^{-1}(\Sigma), \mu).$$

Proof. Proof is on the similar lines as in [25, Theorem 2.2.6]. ■

Theorem 3.7. *Let $X = X(\Omega, \Sigma, \mu)$ be a Banach function space with the associate space X' and let T be a non-singular measurable transformation on Ω satisfying the hypothesis of Theorem 2.5. Then C_T is onto on X' if and only if there is some $c \in \mathbf{R}$ such that $f_T \geq c > 0$ on support(f_T) and $T^{-1}(\Sigma) = \Sigma$, where $T^{-1}(\Sigma) = \{T^{-1}(E) : E \in \Sigma\}$.*

Proof. Suppose $T^{-1}(\Sigma) = \Sigma$. Then by the above lemma, range(C_T) is dense in X. If $f_T \geq c > 0$ on support(f_T), by Theorem 2.5, C_T has closed range in X', which implies that C_T is onto on X'.

Conversely, suppose that C_T is onto on X'. Then C_T has closed range and so by Theorem 2.5, f_T is bounded away from 0 on support$(f_T) \subseteq \Omega$. The family $T^{-1}(\Sigma)$ is always a subfamily of Σ. Let $E \in \Sigma$ and $\mu(E) < \infty$, since C_T is onto, there is some $g \in X'$ such that $C_T g = \chi_E$. Let $F = \{x \in \Omega : g(x) = 1\}$. Then

$$C_T \chi_F = \chi_E \Rightarrow T^{-1}(F) = E \Rightarrow E \in T^{-1}(\Sigma).$$

Therefore $T^{-1}(\Sigma) = \Sigma$. This proves the result. ■

Theorem 3.8. *Let X be a Banach function space with the associate space X'. Let T be a non-singular measurable transformation on Ω such that $C_T : X' \to X'$ is continuous. Then C_T has dense range in X' if and only if $T^{-1}(\Sigma) = (\Sigma)$.*

Proof. Let $T^{-1}(\Sigma) = (\Sigma)$, then Lemma 3.6 implies that C_T has dense range in X'.

Conversely, by Theorem 2.5, if C_T has dense range in X', then f_T is bounded way from 0. Also, $\overline{C_T(X')} = X' \Rightarrow X'(\Omega, T^{-1}(\Sigma), \mu) = X'(\Omega, \Sigma, \mu) \Rightarrow T^{-1}(\Sigma) = \Sigma$. This proves the result. ■

Corollary 3.9. *Under the same hypothesis as in above theorem, if T is one-to-one, then $\overline{\text{range}(C_T)} = X'$.*

Remark 2. The converse of the above theorem is not true in general. For example, let $\Omega = \{a, b, c, d\}$, $\Sigma = \{\emptyset, \Omega, \{a, b\}, \{c, d\}\}$ and define $T : \Omega \to \Omega$ as

$$T(a) = a = T(b) \text{ and } T(c) = c = T(d).$$

Then, $T^{-1}(\Sigma) = \Sigma$. Thus C_T has dense range in $X(= l^2(\mathbf{N}), \text{say})$ since $T^{-1}(\Sigma) = \Sigma$ but T is not one-to-one. Note that here $\mu \ll \mu \circ T^{-1} \ll \mu =$ counting measure.

THEOREM 3.10. *Let $X = X(\mathbf{N}, \mu)$ be a Banach function space and let T be a non-singular measurable transformation on Ω such that $C_T\colon X \to X$ is continuous. Then C_T has dense range in X' if and only if T is one-to-one.*

Proof. The sufficiency follows from Corollary 3.9. To prove the necessary part, suppose T is not one-to-one. Then there exists a set N_1 containing $p\,(p \geq 1)$ elements $n_1, n_2, \ldots, n_p$ such that $T(n_i) = m$ for $1 \leq i \leq p$. We claim that $\{n_i\} \notin T^{-1}(\Sigma)$. If $\{n_i\} \in T^{-1}(\Sigma)$, then

$$\{n_i\} = T^{-1}(\{n_i'\}) \text{ for some } \{n_i'\} \in \Sigma.$$

Since $T(\{n_i\}) = \{m\}$, it follows that $\{n_i'\} = \{m\}$. Hence $T^{-1}(\{n_i'\}) = T^{-1}(\{m\})$ or in other words

$$\{n_i\} = \{n_1, n_2, \ldots, n_p\},$$

which is a contradiction. Thus $T^{-1}(\Sigma) \neq \Sigma$. Hence C_T dose not have dense range. This completes the proof of the theorem. ■

COROLLARY 3.11. *Let $X = X(\mathbf{N}, \mu)$ be a Banach function space. If $a > 0$ and $b < \infty$ and T is a non-singular measurable transformation on $\mathbf{N}$ such that C_T is continuous on X. Then C_T is surjective on X' if and only if T is one-to-one.*

The next theorem characterizes invertible composition operators on the associate space of a Banach function space X.

THEOREM 3.12. *Let X be a Banach function space with the associate space X' and T is a non-singular measurable transformation on Ω such that C_T is continuous on X and X' and $T(\Omega) = \Omega$. Then C_T is invertible on X' if and only if $f_T \geq c > 0$ a.e. on Ω and $T^{-1}(\Sigma) = \Sigma$.*

Proof. Using Theorem 2.2 and Theorem 3.7, we get the result. ■

REMARK 3. If $\Sigma = \{\emptyset, \Omega\}$, then every composition operator on X' is invertible, for some Banach function space X such that $C_T\colon X' \to X'$.

THEOREM 3.13. *Let X be a Banach function space with the associate space X' and T be a non-singular measurable transformation on Ω such that C_T is continuous on X' and $T(\Omega) = \Omega$. Then C_T is an injection on X' if and only if $f_T \neq 0$ a.e. and $T^{-1}(\Sigma) = \Sigma$.*

Proof. Using Theorem 2.2 and Theorem 3.8, we get the result. ■

COROLLARY 3.14. *Let $X = X(\mathbf{N}, \mu)$ be a Banach function space and T is a non-singular measurable transformation on $\mathbf{N}$ such that C_T is continuous on X'. Then C_T is an injection on X' with dense range if and only if T is invertible.*

COROLLARY 3.15. *Let $X = X(\mathbf{N}, \mu)$ be a Banach function space and T is a non-singular measurable transformation on $\mathbf{N}$ such that C_T is continuous on X'. If $a > 0$ and $b < \infty$, then C_T is an invertible composition operator on X' if and only if T is so.*

Remark 4. From the above results, we conclude that there are invertible composition operators induced by non-invertible transformations T on Ω.

The next theorem generalizes the result for Orlicz spaces given in [18, Proposition 3.3] on a σ-finite Borel measure space.

Definition 3. A measurable transformation $T\colon \Omega \to \Omega$ is said to be essentially surjective if $\mu(\Omega \setminus T(\Omega)) = 0$.

Theorem 3.16. *Let X be a Banach function space. Let T be a non-singular measurable transformation on Ω such C_T is continuous on X. Then C_T is one-to-one if and only if T is essentially surjective.*

Proof. Suppose that C_T is injective, that is, $\ker(C_T) = \{0\}$. Let

$$\Omega_\circ = \{x \in \Omega : f_T(x) = 0\} \quad \text{and} \quad X(\Omega_\circ) = \{f \in X : f(x) = 0 \text{ a.e. } x \in \Omega \setminus \Omega_\circ\}$$

and $\operatorname{support}(f) = \{f \in \Omega : f(x) \neq 0\}$, for each $f \in X$. Then, we have

$$X(\Omega_\circ) = \{f \in X : \operatorname{support}(f) \subseteq \Omega_\circ \text{ a.e.}\} = \{f \in X : f_T \mid_{\operatorname{support}(f)} = 0\}.$$

We claim that $X(\Omega_\circ) = \{0\}$. Let $f \in X(\Omega_\circ) \subseteq X(\Omega) \Rightarrow \|f\|_X < \infty$. Then, using the axiom (iii) in the definition of Banach function spaces, we see that

$$\begin{aligned}
\int_\Omega |(C_T f)(x)|\, d\mu(x) &= \int_{T(\Omega)} |f(y)| f_T(y)\, d\mu(y) \\
&\leq \int_\Omega |f(y)| f_T(y)\, d\mu(y) \\
&= \int_{\Omega\setminus\Omega_\circ} |f(y)| f_T(y)\, d\mu(y) + \int_{\Omega_\circ} |f(y)| f_T(y)\, d\mu(y) \\
&= \int_{\Omega\setminus\Omega_\circ} |f(y)| f_T(y)\, d\mu(y) = 0,
\end{aligned}$$

which implies that

$$C_T f(x) = 0 \text{ a.e. } x \in \Omega \Rightarrow f \in \ker(C_T).$$

Therefore $X(\Omega_\circ) \subseteq \ker(C_T) = \{0\}$. This shows that $X(\Omega_\circ) = \{0\}$. Thus $\mu(\Omega_\circ) = 0$. To complete the proof, it suffices to show that $\Omega \setminus \Omega_\circ = T(\Omega)$. Now $\Omega \setminus \Omega_\circ = \operatorname{support}(f_T) \subseteq T(\Omega)$, since $E \subseteq \Omega \setminus T(\Omega)$. Therefore, we see that

$$0 = \mu(T^{-1}(E)) = \int_E f_T(x)\, d\mu(x)$$

so that $f_T \mid_E = 0 \Rightarrow E \subseteq \Omega_\circ$. Thus $\Omega \setminus T(\Omega) \subseteq \Omega_\circ$. Therefore, we have

$$\mu(\Omega_\circ) = 0 \Rightarrow \mu(\Omega \setminus T(\Omega)) = 0.$$

This proves that T is essentially surjective.

Conversely, assume that T is essentially surjective so that $\Omega = T(\Omega) \cup B$ whenever $\mu(B) = 0$. Then, clearly, we have

$$\ker(C_T) = \{f \in X : C_T f = 0\} = \{f \in X : f \mid_{T(\Omega)} = 0\} = \{0\}. \quad \blacksquare$$

Corollary 3.17. *If (Ω, Σ, μ) is a non-atomic measure space, then the nullity of C_T on X is either 0 or ∞.*

Proof. Follows from $X(\Omega_\circ) \subseteq \ker(C_T)$ and $\Omega_\circ = \{x \in \Omega : f_T(x) = 0\}$. ■

4. Invertible composition operators induced by invertible transformations

The underlying σ-algebra of measurable sets plays an important role in the invertibility of C_T.

Definition 4. A topological space (Y, τ) is said to be an absolute Borel space if it is homeomorphic to a Borel subset of a Hilbert cube (see [22, p. 52] for the definition of a Hilbert cube). If the measure μ is σ-finite on the Borel subsets of an absolute Borel space (Y, τ), then (Y, τ, μ) is called an absolute measure space.

Definition 5. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces. Then a transformation $h\colon \Omega_1 \to \Omega_2$ is said to be a homomorphism if it satisfies the following properties:

$$h(A \cup B) = h(A) \cup h(B),\ h(A \cap B) = h(A) \cap h(B),\ \text{and } h(\Omega_1 \setminus A) = \Omega_2 \setminus h(A).$$

Also, if $h(\bigcup_{i\geq 1} A_i) = \bigcup_{i\geq 1} h(A_i)$, we say that h is a σ-homomorphism. If h is a one-to-one homomorphism, then h is called an automorphism.

Now we need a lemma to prove the main result of this section.

Lemma 4.1. *If T is a non-singular measurable transformation on Ω such that C_T is an invertible composition operator on a Banach function space $X = X_b$, then C_T^{-1} takes characteristic functions into characteristic functions.*

Proof. Let $E \in \Sigma$ be such that $\chi_E \in X$. Then, since C_T is onto, there exists a function $g \in X$ such that $C_T g = \chi_E$. Since C_T is one-to-one, by Corollary 2.5, we have $g = \chi_F$ for some $F \in \Sigma$. Then, we have

$$C_T^{-1}\chi_E = C_T^{-1}C_T\chi_F = \chi_F.$$

This proves our assertion. ∎

Theorem 4.2. *Let Ω be an absolute measure space and C_T is bounded on a Banach function space $X = X_b$. Then C_T is an invertible composition operator on X if and only if T is so with non-singular inverse and T^{-1} induces a composition operator.*

Proof. Proof is on similar lines as for L^2-spaces, see [25, p. 31]. ∎

Corollary 4.3. *If C_T is an invertible composition operator on a Banach function space X, then $C_T^{-1} = C_{T^{-1}}$.*

Remark 5. The above theorem is valid for each Banach function space X where every automorphism, or σ-homomorphism of σ-algebra of underlying measure space (Ω, Σ, μ) is induced by a unique point mapping. For examples of such measure space, we refer to [4, p. 123].

5. Fredholm composition operators

The study of Fredholm composition operators has been initiated by Cima, Thomson and Wogen on $H^2(D)$-spaces in [5], they have proved that a composition operator on $H^2(D)$ is Fredholm if and only if T is a conformal automorphism of the unit disc $D = \{z \in \mathbf{C} : |z| = 1\}$ in the complex plane $\mathbf{C}$. This study has been initiated on $L^2(\mu)$-spaces in [14] and [26]. In this section, we generalize this result to the general Banach function spaces.

Definition 6. An operator $A \in B(Y, Z)$, the space of all bounded linear operators from Y into Z, where Y and Z are Banach spaces, is said to be a Fredholm operator if range(A) is closed in Z, $\dim(\ker(A)) < \infty$ and $\operatorname{codim}(A) < \infty$.

Definition 7. A standard Borel space Ω is a Borel subset of a complete metric space (S, d), where d is metric on a set S. The class Σ will consist of all sets of the form $\Omega \cap E$, where E is a Borel subset of S.

We recall the next result from [28, p. 6] here.

Proposition 5.1. *Let Ω_1 and Ω_2 be two standard Borel spaces and T a Borel map of Ω_1 into Ω_2. Let λ be a finite measure on Ω_1 and q be the measure $E \mapsto \lambda(T^{-1}(E))$ on Ω_2. Then the range $T(\Omega_1)$ of T is q-measurable and its complement has q-measure 0. Moreover, there exist Borel sets A and Z such that*

(i) $A \subseteq \Omega_1,\ Z \subseteq T(\Omega_1)$,

(ii) $q(\Omega_2 \setminus Z) = 0$,

(iii) *A is a section for T over Z, that is, T is one-to-one on A and maps A onto Z.*

Theorem 5.2. *Let $X = X(\Omega, \Sigma, \mu)$ be a Banach function space with the associate space X'. Let $T\colon \Omega \to \Omega$ be a non-singular measurable transformation such that C_T is continuous on X. If C_T is invertible, then C_T is a Fredholm composition operator.*

Conversely, if (Ω, Σ, μ) is a σ-finite non-atomic standard Borel measure space, then C_T is a Fredholm composition operator on X'.

Proof. If C_T is invertible, then clearly C_T is a Fredholm operator on X.

Conversely, using Corollary 3.9 and Proposition 5.1, the rest of the proof is on similar lines as in [14, Theorem 1]. ■

Theorem 5.3. *Let T be a non-singular measurable transformation such that C_T is continuous on a Banach function space $X = X(\mathbf{N}, \mu)$, then C_T is a Fredholm operator on X if and only if range(C_T) contains all but finitely many elements of $\mathbf{N}$ and restriction of T to the complement of some finite set is one-to-one.*

Proof. Using Theorem 2.2, proof is on similar lines as in [14, Theorem 2]. ■

Remark 3. The adjoint of the unilateral shift on $l^2(\mathbf{N})$ is an example of a Fredholm composition operator.

The next theorem shows when does the adjoint of a composition operator become a composition operator.

THEOREM 5.4. *Let* (Ω, Σ, μ) *be an absolute measure space and* $T\colon \Omega \to \Omega$ *be a non-singular measurable transformation satisfying the hypothesis of Theorem 2.2 such that* C_T *is a composition operator on* X *and* X'. *Then the following statements are equivalent.*

(i) $C_T C_T^* = I$ *on* X' *and* C_T *is invertible on* X'.

(ii) $f_T(x) = 1$ μ*-a.e. and* T *is one-to-one.*

(iii) $f_T(x) = 1$ μ*-a.e. and* C_T *is invertible on* X'.

(iv) C_T^* *is a composition operator on* X'.

Proof. (i) $\Rightarrow$ (ii). By (i), we have

$$C_T C_T^* = M_{f_T} = I \text{ on } X'.$$

Since C_T is invertible, we see that

$$C_T^* = C_T^{-1} f_T = C_T^{-1} \text{ on } X'. \tag{5.1}$$

This implies that $f_T \circ T^{-1}(x) = T^{-1}(x)$ μ-a.e. $x \in \Omega$ and so $f_T(x) = 1$ μ-a.e. $x \in \Omega$, using Theorem 4.2.

(i) $\Rightarrow$ (iv). Using (5.1), we have $C_T^* = C_T^{-1}$ on X'. Also by Corollary 4.3, we have $C_T^* = C_{T^{-1}}$ on X'. This proves that C_T^* is a composition operator on X'.

(iv) $\Rightarrow$ (i). Suppose that C_T^* is a composition operator on X'. Then there exists a non-singular measurable transformation S such that $C_T^* = C_S$. Since $C_T^* C_T = M_{f_T}$, we get

$$C_S C_T = C_{T\circ S} = M_{f_T}.$$

Let $\Omega = \bigcup_{n=1}^{\infty} E_n$, where $\mu(E_n) < \infty$ for every n and $E_n \subseteq E_m$ if $m > n$. Let $f_n = \chi_{E_n}$. Then $C_{T\circ S} f_n = f_T f_n$, for all n. Equivalently $\chi_{(T\circ S)^{-1}(E_n)} = f_T \chi_{E_n}$ for all n. Since $\Omega = \bigcup_{n=1}^{\infty} (T\circ S)^{-1}(E_n)$, we conclude that $f_T = 1$ a.e. In view of the Theorem 3.12, it is enough to show that $T^{-1}(\Sigma) = \Sigma$ to complete the proof. Obviously $T^{-1}(\Sigma) \subseteq \Sigma$. For reverse inclusion, let $E \in \Sigma$ such that $\mu(E) < \infty$. Then, if χ_E is in the range of C_T, we have $\chi_E = C_T h$ for some $h \in X$. Since C_T is one-to-one, by Corollary 3.4, we have $h = \chi_F$, for some $F \in \Sigma$. Hence $\chi_E = C_T \chi_F = \chi_{T^{-1}(F)}$, which gives $E = T^{-1}(F)$. This shows that $E \in T^{-1}(\Sigma)$. In case χ_E does not belong to the range of C_T, we can write

$$\chi_E = f + g, \text{ for some } f \in X - Ran(C_T) \text{ and } g \in Ran(C_T).$$

Let $g = C_T g_1$ for some $g_1 \in X$. Then, since C_T^* is a composition operator, we have

$$C_T^* \chi_E = C_T^* f + C_T^* g = C_T^* g = C_T^* C_T g_1 = M_{f_T} g1 = g_1$$

is a characteristic function. Thus g is a characteristic function. Let $g = \chi_G$. Then $f = \chi_E - \chi_G = \chi_{E\setminus G} - \chi_{G\setminus E}$. Since

$$\langle f, g\rangle = \langle \chi_{E\setminus G} - \chi_{G\setminus E}, \chi_G\rangle = -\langle \chi_{G\setminus E}, \chi_G\rangle = 0,$$

so it follows that $G \subseteq E$. Let $F_1 = T^{-1}(F_2)$ for some $F_2 \in \Sigma$, that is, $\chi_{F_1} \in Ran(C_T)$, such that $E \setminus G \subseteq F_1$. Thus

$$\mu((E \setminus G) \cap F_1) = \langle f, \chi_{F_1} \rangle = 0.$$

This implies that $E \subseteq G$. Thus we have $\chi_E = \chi_G = g$ so that $E \in T^{-1}(\Sigma)$. Now, if $E \in \Sigma$ is of infinite measure, then we can write $E = \bigcup_{i=1}^{\infty} E_i$ where $\{E_i\}$ is a disjoint sequence of measurable sets of finite measure. For each E_i, there exists a set F_i such that $E_i = T^{-1}(F_i)$. Therefore

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} T^{-1}(F_i) = T^{-1}(\bigcup_{i=1}^{\infty} F_i) = T^{-1}(F),$$

where $F = \bigcup_{i=1}^{\infty} F_i$. Thus $T^{-1}(\Sigma) = \Sigma$. This completes the proof.

(ii) $\Rightarrow$ (i). Clearly, by Theorem 2.2 and by (ii), we have $C_T C_T^* = I$ on X'. By the equivalence of (i) and (iv), we see that C_T^* is a composition operator. Therefore, there is a measurable transformation S on Ω such that $C_T^* = C_S$ is a composition operator on X'. Thus, we see that $C_T C_T^* = I$ on X' which implies that $T \circ S(x) = x$ μ-a.e. $x \in \Omega$.

Also, since T is one-to-one, we get a measurable transformation U on Ω such that

$$(U \circ T)(x) = x \ \mu\text{-a.e. } x \in \Omega.$$

Thus, we conclude that T is an invertible transformation on Ω. By Theorem 4.2, we see that C_T is invertible on X'.

The equivalence of (i) and (iii) is obvious. ■

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Rajeev Kumar, 73, Vidhata Nagar, Near Bathindi Morh, Jammu–180 006, INDIA.

E-mail: `raj1k2@yahoo.co.in`