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ON A TYPE OF COMPACTNESS VIA GRILLS

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Abstract. In this paper, we introduce and study the idea of a new type of compactness, defined in terms of a grill $\mathcal{G}$ in a topological space X. Calling it $\mathcal{G}$-compactness, we investigate its relation with compactness, among other things. Analogues of Alexander's subbase theorem and Tychonoff product theorem are also obtained for $\mathcal{G}$-compactness. Finally, we exhibit a new method, in terms of the deliberations here, for construction of the well known one-point compactification of a T_2, locally compact and noncompact topological space.

1. Introduction

Choquet [1] in 1947 initiated the brilliant notion of a grill which subsequently turned out to be a very convenient tool for various topological investigations. It is also seen from the literature that in many situations, grills are more effective than certain similar concepts like nets and filters. According to Choquet [1], a grill $\mathcal{G}$ on a topological space X is a non-null collection of nonempty subsets of X satisfying two conditions: (i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$, and (ii) $(A, B \subseteq X)$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

In [2] we introduced a new topology on a topological space X, constructed by use of a grill on X, and is described as follows.

Let $\mathcal{G}$ be a grill on a topological space (X, τ). Consider the operator $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ (here and henceforth also, $\mathcal{P}(X)$ stands for the power set of X), given by $\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}$ for all open neighbourhoods U of $x\}$. Then the map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\Psi(A) = A \cup \Phi(A)$ for $A \in \mathcal{P}(X)$, is a Kuratowski closure operator and hence induces a topology $\tau_{\mathcal{G}}$ on X, strictly finer than τ, in general. An open base $\mathcal{B}$ for the topology $\tau_{\mathcal{G}}$ on X is given by $\mathcal{B} = \{U \setminus A : U \in \tau$ and $A \notin \mathcal{G}\}$.

In our earlier paper [2] we studied at length the above topology $\tau_{\mathcal{G}}$ for its different aspects and also its behaviours vis-a-vis the original topology τ on the underlying space. In the present article, we wish to introduce a kind of compactness property, termed as $\mathcal{G}$-compactness, on a topological space (X, τ), defined in terms

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of a grill $\mathcal{G}$ on X. In [6] Rancin defined I-compactness in a topological space X corresponding to an ideal I on the underlying set X. The concept of $\mathcal{G}$-compactness, as considered here, is a version of I-compactness by use of the notion of grills.

In Section 2, we shall endeavour to establish the relationships of $\mathcal{G}$-compactness of X first with the compactness of (X,τ) and of $(X,\tau_{\mathcal{G}})$, and then with an important weaker form of compactness, viz. (quasi) H-closedness. It will be shown that a parallel version of Alexender's subbase theorem holds for $\mathcal{G}$-compactness, and a sort of Tychonoff product theorem concerning $\mathcal{G}$-compactness can also be obtained. The proofs of these two theorems are patterned after the corresponding results in [6] concerning I-compactness.

In Section 3, we shall exhibit an interesting application of our study of $\tau_{\mathcal{G}}$-compactness by achieving the well known result on the one-point compactification of a locally compact, non-compact T_2 space, by way of a new method of construction.

Throughout the paper, by a space X we shall mean a topological space (X,τ), and τ-int$A$ and τ-cl$A$ (or simply int$A$ and cl$A$) will stand for the interior and closure respectively of a subset A of a space (X,τ). The system of all open neghbourhoods of a point x of a space (X,τ) will be denoted by $\tau(x)$.

2. $\mathcal{G}$-compactness

Definition 2.1. Let $\mathcal{G}$ be a grill on a topological space (X,τ). A cover $\{U_\alpha : \alpha\in\Lambda\}$ of X is said to be a $\mathcal{G}$-cover if there exists a finite subset Λ_0 of Λ such that $X\setminus\bigcup_{\alpha\in\Lambda_0}U_\alpha\notin\mathcal{G}$. A cover which is not a $\mathcal{G}$-cover of X will be called $\bar{\mathcal{G}}$-cover of X.

Definition 2.2. Let $\mathcal{G}$ be a grill on a topological space (X,τ). Then (X,τ) is said to be compact with respect to the grill $\mathcal{G}$ or simply $\mathcal{G}$-compact if every open cover of X is a $\mathcal{G}$-cover.

Remark 2.3. (a) Every compact space (X,τ) is clearly $\mathcal{G}$-compact for any grill $\mathcal{G}$ on X.

(b) If we take $\mathcal{G}=\mathcal{P}(X)\setminus\{\varnothing\}$, then $\mathcal{G}$-compactness of a space (X,τ) reduces to the compactness of (X,τ).

(c) If for any grill $\mathcal{G}$ on a space (X,τ), the space $(X,\tau_{\mathcal{G}})$ is $\mathcal{G}$-compact, then (X,τ) is compact (as $\tau\subseteq\tau_{\mathcal{G}}$) and hence is $\mathcal{G}$-compact (by (a) above).

Example 2.4. Let X be an uncountable set with co-countable topology τ defined on X. Then (X,τ) is not compact. Let $p\in X$. Consider $\mathcal{G}_p=\{A\subseteq X : p\in A\}$. Then $\mathcal{G}_p$ is a grill on X. It is easy to check that (X,τ) is $\mathcal{G}_p$-compact.

Example 2.5. Let τ denote the cofinite topology on an uncountable set X and $\mathcal{G}$ be the grill of all uncountable subsets of X. We claim that $\tau_{\mathcal{G}}=$ the co-countable topology σ on X. In fact, $V\in\sigma\Rightarrow X\setminus V=A\notin\mathcal{G}\Rightarrow V=X\setminus A$, where $X\in\tau$ and $A\notin\mathcal{G}\Rightarrow V\in\tau_{\mathcal{G}}$. Again $V\in\tau_{\mathcal{G}}$ with $V=U\setminus A$, where $U\in\tau$ and $A\notin\mathcal{G}\Rightarrow X\setminus U$ is finite and A is countable. Thus $X\setminus V=X\setminus(U\setminus A)=X\setminus(U\cap(X\setminus A))=(X\setminus U)\cup A$ which is countable. Hence $V\in\sigma$. We note here that (X,τ) is compact but $(X,\tau_{\mathcal{G}})$ is not compact. Also, it is easy to see that $(X,\tau_{\mathcal{G}})$ is $\mathcal{G}$-compact.

Theorem 2.6. *Let $\mathcal{G}$ be a grill on a topological space (X,τ). Then (X,τ) is $\mathcal{G}$-compact iff $(X,\tau_{\mathcal{G}})$ is $\mathcal{G}$-compact.*

Proof. As $\tau \subseteq \tau_{\mathcal{G}}$, it follows that (X,τ) is $\mathcal{G}$-compact if $(X,\tau_{\mathcal{G}})$ is $\mathcal{G}$-compact.

Conversely, let (X,τ) be $\mathcal{G}$-compact and $\{U_\alpha : \alpha \in \Lambda\}$ be a basic $\tau_{\mathcal{G}}$-open cover of X. Then for each $\alpha \in \Lambda$, $U_\alpha = V_\alpha \setminus A_\alpha$ where $V_\alpha \in \tau$ and $A_\alpha \notin \mathcal{G}$. Then $\{V_\alpha : \alpha \in \Lambda\}$ is a τ-open cover of X. Hence by $\mathcal{G}$-compactness of (X,τ), there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. Now, $X \setminus \bigcup_{\alpha\in\Lambda_0} U_\alpha = X \setminus \bigcup_{\alpha\in\Lambda_0}(V_\alpha \setminus A_\alpha) \subseteq (X \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha) \cup (\bigcup_{\alpha\in\Lambda_0} A_\alpha) \notin \mathcal{G}$ (as $A_\alpha \notin \mathcal{G}$, for all $\alpha \in \Lambda_0$). Thus $(X,\tau_{\mathcal{G}})$ is $\mathcal{G}$-compact. ■

Remark 2.7. In view of Remark 2.3, Examples 2.4 and 2.5, and Theorem 2.6 we have:

For a topological space (X,τ) and a grill $\mathcal{G}$ on X, the following implication-diagram holds, where no other implication than those displayed, is true in general.

$$\begin{array}{ccc} (X,\tau) \text{ is compact} & \Leftarrow & (X,\tau_{\mathcal{G}}) \text{ is compact} \\ \Downarrow & & \Downarrow \\ (X,\tau) \text{ is } \mathcal{G}\text{-compact} & \Leftrightarrow & (X,\tau_{\mathcal{G}}) \text{ is } \mathcal{G}\text{-compact} \end{array}$$

Having obtained some correlations among the concepts of compactness and $\mathcal{G}$-compactness of the spaces (X,τ) and $(X,\tau_{\mathcal{G}})$, we now recall the following well known weaker form of compactness.

Definition 2.8. [4] A topological space (X,τ) is said to be quasi H-closed (QHC, in short) if for every open cover $\mathcal{U}$ of X, there is a finite sub-collection $\mathcal{U}_0$ of $\mathcal{U}$ such that $X = \bigcup\{\mathrm{cl}U : U \in \mathcal{U}_0\}$. A Hausdorff quasi H-closed space is called an H-closed space.

In the next two Theorems we try to associate the notion of quasi H-closedness with that of $\mathcal{G}$-compactness.

Theorem 2.9. *Let $\mathcal{G}$ be a grill on a topological space (X,τ) such that $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$. If (X,τ) is $\mathcal{G}$-compact then (X,τ) is QHC.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of (X,τ). Then by $\mathcal{G}$-compactness of X, there exists a finite subset Λ_0 of Λ such that $(X \setminus \bigcup_{\alpha\in\Lambda_0} U_\alpha) \notin \mathcal{G}$. Then $\mathrm{int}(X \setminus \bigcup_{\alpha\in\Lambda_0} U_\alpha) = \varnothing$. For otherwise, $\mathrm{int}(X \setminus \bigcup_{\alpha\in\Lambda_0} U_\alpha) \in \tau \setminus \{\varnothing\}$ and hence $(X \setminus \bigcup_{\alpha\in\Lambda_0} U_\alpha) \in \mathcal{G}$, a contradiction. Hence $X = \bigcup_{\alpha\in\Lambda_0} \mathrm{cl}U_\alpha$ and X is QHC. ■

Theorem 2.10. *Let (X,τ) be a QHC space. Then (X,τ) is $\mathcal{G}_\delta$-compact, where $\mathcal{G}_\delta = \{A \subseteq X : \mathit{intcl}\ A \neq \varnothing\}$ (that $\mathcal{G}_\delta$ is a grill on X is clear).*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then by quasi H-closedness of (X,τ), there is a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha\in\Lambda_0} \mathrm{cl}U_\alpha = \varnothing$. We claim that $(X \setminus \bigcup_{\alpha\in\Lambda_0} U_\alpha) \notin \mathcal{G}_\delta$. In fact, $(X \setminus \bigcup_{\alpha\in\Lambda_0} U_\alpha) \in \mathcal{G}_\delta \Rightarrow \mathrm{intcl}(X \setminus \bigcup_{\alpha\in\Lambda_0} U_\alpha) \neq \varnothing$ $\Rightarrow X \setminus \bigcup_{\alpha\in\Lambda_0} \mathrm{cl}U_\alpha \neq \varnothing$, a contradiction. Hence (X,τ) is $\mathcal{G}_\delta$-compact. ■

It is known [5] that a Hausdorff topological space is compact iff it is H-closed and regular. Our aim now is to obtain an analogue of this result with compactness replaced by $\mathcal{G}$-compactness. This requires a suitable definition of a sort of regularity condition in terms of grills. We see that the idea of $\mathcal{G}$-regularity introduced in [3] serves our purpose. We recall the definition of $\mathcal{G}$-regularity below.

Definition 2.11. Let (X,τ) be a topological space and $\mathcal{G}$ be a grill on X. Then the space X is said to be $\mathcal{G}$-regular if for any closed set F in X with $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \setminus V \notin \mathcal{G}$.

Theorem 2.12. *Let $\mathcal{G}$ be a grill on a Hausdorff space (X,τ). If (X,τ) is $\mathcal{G}$-compact then it is $\mathcal{G}$-regular.*

Proof. Let F be a closed subset of X and $x \notin X$. By Hausdorffness of X, for each $y \in F$ there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Now, $\{V_y : y \in F\} \cup \{X \setminus F\}$ is an open cover of X. Thus by $\mathcal{G}$-compactness of X, there exist finitely many points $y_1, y_2, \ldots, y_n$ in F such that $X \setminus [(\bigcup_{1=1}^{n} V_{y_i}) \cup (X \setminus F)] \notin \mathcal{G}$. Let $G = X \setminus \bigcup_{i=1}^{n} \mathrm{cl} V_{y_i}$ and $H = \bigcup_{i=1}^{n} V_{y_i}$ Then G and H are two disjoint nonempty open sets in X such that $x \in G$, $F \setminus H = F \cap [X \setminus \bigcup_{i=1}^{n} V_{y_i}] = X \setminus [(\bigcup_{i=1}^{n} V_{y_i}) \cup (X \setminus F)] \notin \mathcal{G}$. Thus (X,τ) is $\mathcal{G}$-regular. ∎

It now follows from Theorems 2.9 and 2.12 that

Corollary 2.13. *Let $\mathcal{G}$ be a grill on a Hausdorff space (X,τ) such that $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$. If (X,τ) is $\mathcal{G}$-compact then it is H-closed and $\mathcal{G}$-regular.*

Theorem 2.14. *Let (X,τ) be an H-closed space and $\mathcal{G}$ be a grill on X. If (X,τ) is $\mathcal{G}$-regular then (X,τ) is $\mathcal{G}$-compact.*

Proof. Let $\mathcal{U}$ be an open cover of X. Then for each $x \in X$, there is some $U_x \in \mathcal{U}$ such that $x \in U_x$. Thus $x \notin (X \setminus U_x)$ where $(X \setminus U_x)$ is a closed set. By $\mathcal{G}$-regularity of X, there exist two disjoint open sets G_x and H_x such that $(X \setminus U_x) \setminus H_x \notin \mathcal{G}$ and $x \in G_x$. Let $A_x = (X \setminus U_x) \setminus H_x$. Now, $\mathrm{cl} G_x \cap H_x = \varnothing \Rightarrow \mathrm{cl} G_x \subseteq X \setminus H_x \subseteq (X \setminus H_x) \cup U_x = [X \setminus (H_x \cup U_x)] \cup U_x = A_x \cup U_x$.

Again, $\{G_x : x \in X\}$ being an open cover of the H-closed space X, there are finitely many points $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^{n} \mathrm{cl} G_{x_i}$. Then $X = \bigcup_{i=1}^{n} \mathrm{cl} G_{x_i} \subseteq \bigcup_{i=1}^{n} (A_{x_i} \cup U_{x_i}) \Rightarrow X \setminus \bigcup_{i=1}^{n} U_{x_i} \subseteq \bigcup_{i=1}^{n} A_{x_i} \notin \mathcal{G}$ (as $A_{x_i} \notin \mathcal{G}$ for $i = 1, 2, \ldots, n$). Thus (X,τ) is $\mathcal{G}$-compact. ∎

In view of Corollary 2.13 and the above theorem, we are led to our desired result:

Corollary 2.15. *Let $\mathcal{G}$ be a grill on a Hausdorff space (X,τ) such that $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$. Then (X,τ) is $\mathcal{G}$-compact iff (X,τ) is H-closed and $\mathcal{G}$-regular.*

Our aim now is to find an analogue of Alexander's subbase theorem for $\mathcal{G}$-compactness, which will help us to derive the result that the cartesian product of $\mathcal{G}$-compact spaces is $\mathcal{G}$-compact.

Theorem 2.16. *Let $\mathcal{U}$ be an open subbase for a topological space (X,τ). If X has an open $\bar{\mathcal{G}}$-cover then there is a $\bar{\mathcal{G}}$-cover of X, which consists of elements of $\mathcal{U}$.*

Proof. Let $\mathcal{C}$ be the collection of all open $\bar{\mathcal{G}}$-covers of X. Then by hypothesis, $\mathcal{C}$ is nonempty. Let $\{\mathcal{P}_\alpha\}$ be a linearly ordered subset of $\mathcal{C}$. Then $\bigcup_\alpha \mathcal{P}_\alpha$ is a covering of X. We claim that it is a $\bar{\mathcal{G}}$-covering. For, if not, then there exist $G_1, G_2, \ldots, G_n \in \bigcup \mathcal{P}_\alpha$ such that $X \setminus \bigcup_{i=1}^n G_i \notin \mathcal{G}$. Now, there exists a $\mathcal{P}_\beta \in \mathcal{C}$ such that $G_1, G_2, \ldots, G_n \in \mathcal{P}_\beta$. Thus $\mathcal{P}_\beta \notin \mathcal{C}$, a contradiction. Consequently by Zorn's Lemma, $\mathcal{C}$ contains a maximal element $\mathcal{P}$. Thus if H is open and $H \notin \mathcal{G}$, then there exist finitely many $G_1, G_2, \ldots, G_n \in \mathcal{P}$ such that $X \setminus (H \cup G_1 \cup G_2 \cup \cdots \cup G_n) \notin \mathcal{G}$.

We now show that the family of open sets which do not belong to $\mathcal{P}$ form a filter. For this, let $H_1, H_2 \in \tau$ and $H_1, H_2 \notin \mathcal{P}$. Then $X \setminus (H_1 \cup G_1 \cup G_2 \cup \cdots \cup G_n) = A_1 \notin \mathcal{G}$ and $X \setminus (H_2 \cup V_1 \cup V_2 \cup \cdots \cup V_m) = A_2 \notin \mathcal{G}$, for some finite subcollections $\{G_1, G_2, \ldots, G_n\}$ and $\{V_1, V_2, \ldots, V_m\}$ of $\mathcal{P}$. Consider $B = X \setminus [(H_1 \cap H_2) \cup (G_1 \cup G_2 \cup \cdots \cup G_n) \cup (V_1 \cup V_2 \cup \cdots \cup V_m)]$. Then $B \subseteq A_1 \cup A_2$. Since $A_1 \cup A_2 \notin \mathcal{G}$, we have $B \notin \mathcal{G}$. Thus $(H_1 \cap H_2) \in \tau \setminus \mathcal{P}$. Next, let $H \notin \mathcal{P}$ and $H \subseteq G$, where G and H are open sets. Then $X \setminus (H \cup G_1 \cup G_2 \cup \cdots \cup G_n) \notin \mathcal{G}$ for finitely many $G_1, G_2, \ldots, G_n \in \mathcal{P}$. Thus $X \setminus (G \cup G_1 \cup G_2 \cup \cdots \cup G_n) \notin \mathcal{G}$ and hence $G \in \tau \setminus \mathcal{P}$.

To complete the proof it is sufficient to show that $\mathcal{U} \cap \mathcal{P}$ is a $\bar{\mathcal{G}}$-cover of X. Let $x \in X$. Since $\mathcal{P}$ is an open cover of X, there exists a $G \in \mathcal{P}$ such that $x \in G$. Since $\mathcal{U}$ is a subbase for (X,τ), there exist $H_1, H_2, \ldots, H_n \in \mathcal{U}$ such that $x \in H_1 \cap H_2 \cap \cdots \cap H_n \subseteq G$. Then there exists an H_i (for some $i = 1, 2, \ldots, n$) such that $H_i \in \mathcal{P}$. For otherwise, if $H_i \notin \mathcal{P}$ for all $i = 1, 2, \ldots, n$, then $\bigcup_{i=1}^n H_i \notin \mathcal{P}$ (as the family of all open sets not in $\mathcal{P}$, is a filter). Thus $G \notin \mathcal{P}$, a contradiction. It then finally follows that $x \in H_i \in \mathcal{U} \cap \mathcal{P}$ and consequently, $\mathcal{U} \cap \mathcal{P}$ is a $\bar{\mathcal{G}}$-cover of X. This completes the proof. ∎

Corollary 2.17. *Let $\mathcal{G}$ be a grill on a topological space (X,τ). Then X is $\mathcal{G}$-compact iff there exists a subbase $\mathcal{S}$ of τ such that every cover of X by members of $\mathcal{S}$ is a $\mathcal{G}$-cover.*

Theorem 2.18. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces, and $\mathcal{G}_\alpha$ be a grill on X_α for each $\alpha \in \Lambda$. Let $\mathcal{G}$ be any grill on the cartesian product space $X = \prod_{\alpha\in\Lambda} X_\alpha$ such that $\pi_\alpha^{-1}(\mathcal{G}_\alpha) \supseteq \mathcal{G}$ for each $\alpha \in \Lambda$, where $\pi_\alpha : X \to X_\alpha$ is, as usual, the αth projection map. If X_α is $\mathcal{G}_\alpha$-compact for each $\alpha \in \Lambda$, then X is $\mathcal{G}$-compact.*

Proof. Let $\mathcal{U}$ be a subbasic open cover of X. In view of Theorem 2.16 it is sufficient to find a finite subset $\{U_1, U_2, \ldots, U_n\}$ of $\mathcal{U}$ such that $X \setminus \bigcup_{i=1}^n U_i \notin \mathcal{G}$. Let $\alpha_0 \in \Lambda$ and U_{α_0} denote the family of all those subsets V of X_{α_0} such that $\pi_{\alpha_0}^{-1}(V) \in \mathcal{U}$. We claim that for at least one $\alpha \in \Lambda$, $\mathcal{U}_\alpha$ is a covering of X_α. If not, then by choosing a point x_α from X_α (for each $\alpha \in \Lambda$), which is not covered by $\mathcal{U}_\alpha$, we would find a point in X not covered by $\mathcal{U}$, which contradicts the fact that $\mathcal{U}$ is a cover of X. Thus there exists a $\beta_0 \in \Lambda$ such that $\mathcal{U}_{\beta_0}$ is a cover of X_{β_0}. Then we can find finitely many $U_1^{\beta_0}, U_2^{\beta_0}, \ldots, U_n^{\beta_0} \in \mathcal{U}_{\beta_0}$ such that $(X_{\beta_0} \setminus \bigcup_{i=1}^n U_i^{\beta_0}) \notin \mathcal{G}_{\beta_0}$.

By considering $U_i = \pi_{\beta_0}^{-1}(U_i^{\beta_0}) \in \mathcal{U}$, we see that $(X \setminus \bigcup_{i=1}^n U_i) \notin \mathcal{G}$. Thus X is $\mathcal{G}$-compact.

3. One-point compactification via grills

As proposed in the introduction, the intent of this section is to exhibit a new method of construction of a one-point compactification of a locally compact Hausdorff space. To that end, we first give the general construction of a one-point extension of a topological space.

THEOREM 3.1. *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Let p be an object, not in X, and put $X^* = X \cup \{p\}$. Then the map $f : \mathcal{P}(X^*) \to \mathcal{P}(X^*)$, defined by*

$$f(A) = \begin{cases} clA, & \text{if } clA \notin \mathcal{G}, \text{ for } A \subseteq X, \\ clA \cup \{p\}, & \text{if } clA \in \mathcal{G}, \text{ for } A \subseteq X, \\ cl(A \setminus \{p\}) \cup \{p\}, & \text{if } p \in A \end{cases}$$

is a Kuratowski closure operator, inducing a topology τ^* on X^* such that

(a) every τ-open set in X is τ^*-open

(b) if $U (\subseteq X^*)$ is τ^*-open then $U \cap X$ is τ-open.

Proof. We first take up the verification, although straightforward to show that f indeed satisfies the Kuratowski closure axioms. Clearly $f(\varnothing) = \varnothing$ (as $\varnothing \notin \mathcal{G}$), and $A \subseteq f(A)$, for any $A \subseteq X^*$.

We now verify that for any $A, B \subseteq X^*, f(A \cup B) = f(A) \cup f(B)$.

Case-1. $A, B \subseteq X$.

If $\mathrm{cl}(A \cup B) \notin \mathcal{G}$, then $f(A \cup B) = \mathrm{cl}(A \cup B) = \mathrm{cl}A \cup \mathrm{cl}B = f(A) \cup f(B)$.

If $\mathrm{cl}(A \cup B) \in \mathcal{G}$, then $\mathrm{cl}A \in \mathcal{G}$ or $\mathrm{cl}B \in \mathcal{G}$, and hence $f(A \cup B) = \mathrm{cl}(A \cup B) \cup \{p\} = \mathrm{cl}A \cup \mathrm{cl}B \cup \{p\} = f(A) \cup f(B)$.

Case-2. $A \subseteq X$ and $p \in B$.

If $\mathrm{cl}A \notin \mathcal{G}$, then $f(A \cup B) = \mathrm{cl}((A \cup B) \setminus \{p\}) \cup \{p\} = \mathrm{cl}(A \cup (B \setminus \{p\})) \cup \{p\} = \mathrm{cl}A \cup \mathrm{cl}(B \setminus \{p\}) \cup \{p\} = f(A) \cup f(B)$.

If $\mathrm{cl}A \in \mathcal{G}$, then $f(A \cup B) = \mathrm{cl}((A \cup B) \setminus \{p\}) \cup \{p\} = \mathrm{cl}A \cup \{p\} \cup \mathrm{cl}(B \setminus \{p\}) \cup \{p\} = f(A) \cup f(B)$.

Case-3. $p \in A$ and $p \in B$.

Here $f(A \cup B) = \mathrm{cl}((A \cup B) \setminus \{p\}) \cup \{p\} = \mathrm{cl}(A \setminus \{p\}) \cup \mathrm{cl}(B \setminus \{p\}) \cup \{p\} = f(A) \cup f(B)$.

We next show that $f(f(A)) = f(A)$, for any $A \subseteq X^*$.

Case-(i): $A \subseteq X$.

In case $\mathrm{cl}A \notin \mathcal{G}$, we have $f(f(A)) = f(\mathrm{cl}A) = \mathrm{cl}A = f(A)$, while if $\mathrm{cl}A \in \mathcal{G}$, we get $f(f(A)) = f(\mathrm{cl}A \cup \{p\}) = f(\mathrm{cl}A) \cup f(\{p\}) = \mathrm{cl}A \cup \{p\} = f(A)$.

Case-(ii): $p \in A$.

If $\mathrm{cl}(A \setminus \{p\}) \notin \mathcal{G}$, then $f(f(A)) = f[\mathrm{cl}(A \setminus \{p\}) \cup \{p\}] = f[\mathrm{cl}(A \setminus \{p\})] \cup f(\{p\})$

$= \mathrm{cl}(A \setminus \{p\}) \cup \{p\} = f(A)$.

If $\mathrm{cl}(A \setminus \{p\}) \in \mathcal{G}$, then $f(f(A)) = f[\mathrm{cl}(A \setminus \{p\}) \cup \{p\}] = f[\mathrm{cl}(A \setminus \{p\})] \cup f(\{p\}) = \mathrm{cl}(A \setminus \{p\}) \cup \{p\} = f(A)$.

It follows that f is a Kuratowski closure operator on X^* which gives rise to a topology τ^* on X^* such that $f(A) = \tau^*\text{-cl}A$, for any $A \subseteq X^*$.

(a) Let $U(\subseteq X)$ be τ-open. Then $f(X^* \setminus U) = \mathrm{cl}[(X^* \setminus U) \cup \{p\}] \cup \{p\} = \mathrm{cl}(X \setminus U) \cup \{p\} = (X \setminus U) \cup \{p\} = X^* \setminus U$, so that U is τ^*-open.

(b) Since $U(\subseteq X^*)$ is τ^*-open, we have $f(X^* \setminus U) = X^* \setminus U \ \ldots$ (i). Now, $p \notin U \Rightarrow \mathrm{cl}[(X^* \setminus U) \cup \{p\}] \cup \{p\} = X^* \setminus U$ (by (i)) $\Rightarrow \mathrm{cl}(X \setminus U) \cup \{p\} = X^* \setminus U \Rightarrow \mathrm{cl}(X \setminus U) = (X \setminus U) \Rightarrow (X \setminus U)$ is τ-closed $\Rightarrow U(= U \cap X)$ is τ-open.

Again, $p \in U \Rightarrow \mathrm{cl}(X^* \setminus U) = X^* \setminus U$ (using (i) and since $p \notin (X^* \setminus U)) \Rightarrow \mathrm{cl}[(X \cup \{p\}) \cap (X^* \setminus U)] = (X \setminus \{p\}) \cap (X^* \setminus U) \Rightarrow \mathrm{cl}[X \cap (X \setminus U)] = X \cap (X \setminus U) \Rightarrow \mathrm{cl}(X \setminus (U \cap X)) = X \setminus U \cap X \Rightarrow U \cap X$ is τ-open. ∎

The next two theorems will carry us almost towards our goal for achieving the desired construction of the one-point compactification.

THEOREM 3.2. *Let $\mathcal{G}$ be a grill on a T_1 space (X, τ) such that for every $x \in X$, $\{x\} \notin \mathcal{G}$. Adjoin to X a new object $p \notin X$. Then there exists a topology on $X^* = X \cup \{p\}$ satisfying the the following properties:*

(a) X^* is T_1.

(b) X is dense in X^*.

Proof. Let us consider the space (X^*, τ^*) as constructed in Theorem 3.1. Now for any $x \in X$, $f(x) = x$ as $\mathrm{cl}(\{x\}) = \{x\} \notin \mathcal{G}$, and $f(p) = \mathrm{cl}(\{p\} \setminus \{p\}) \cup \{p\} = \{p\}$. This proves (a). Again, since $\mathrm{cl}X = X \in \mathcal{G}$, $f(X) = \mathrm{cl}X \cup \{p\} = X \cup \{p\} = X^*$, proving (b). ∎

THEOREM 3.3. *Let $\mathcal{G}$ be a grill on a Hausdorff space (X, τ) such that for every $x \in X$, $\{x\} \notin \mathcal{G}$. If for every point $x \in X$ there is an open neighbourhood U of x such that cl$U \notin \mathcal{G}$, then one can construct a one-point extension $X^* = X \cup \{p\}$ (where $p \notin X$) satisfying the following properties:*

(i) X^* is Hausdorff.

(ii) X is dense in X^*.

Proof. We consider again the one-point extension space (X^*, τ^*) of Theorem 3.1. We first note that as $\mathrm{cl}X \in \mathcal{G}$, $\tau^*\text{-cl}\ X = X^*$ (as in the proof of Theorem 3.2(b)) and (ii) follows.

To prove (i), let x, y be two distinct points of X. By Hausdorffness of (X, τ), x and y are strongly separated by sets U, V which are open in X and hence are open in X^* (by Theorem 3.1(a)). Now by hypothesis, for any $x \in X$ there is a τ-open neighbourhood U of x such that $\mathrm{cl}U \notin \mathcal{G}$. Let $N = X^* \setminus U$. Since $\mathrm{cl}U \notin \mathcal{G}$, we have $f(U) = U$. Thus N is an open neighbourhood of p in X^*. Consequently, U and N are the required disjoint τ^*-open neighbourhoods of x and p respectively in X^*. Hence X^* is Hausdorff, proving (i). ∎

As the final step, we now achieve the sought-after one-point compactification of a noncompact space.

THEOREM 3.4. *Let (X, τ) be a non-compact, locally compact, Hausdorff space. By adjoining a new point p $(\notin X)$ to X, one can construct an extension space $X^* = X \cup \{p\}$ having the following properties:*

(a) X^* is Hausdorff.

(b) X is dense in X^*.

(c) X^* is compact.

Proof. Let $\mathcal{G}$ be the family of subsets of X whose closures in X are not compact in X. It is easy to verify that $\mathcal{G}$ is a grill on X such that for each $x \in X$, $\{x\} \notin \mathcal{G}$. For each $x \in X$, by local compactness of X, there is an open neighbourhood U of x such that cl$U$ is compact, i.e., cl$U \notin \mathcal{G}$. Then $\mathcal{G}$ satisfies the conditions of Theorem 3.3. Hence (a) and (b) follow immediately from Theorem 3.3.

To prove (c) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be any cover of X^* by open sets of X^*. Then for some $\alpha_0 \in \Lambda$, $p \in U_{\alpha_0}$. Then $f(X^* \setminus U_{\alpha_0}) = (X^* \setminus U_{\alpha_0})$ and $p \notin X^* \setminus U_{\alpha_0} \Rightarrow \mathrm{cl}(X^* \setminus U_{\alpha_0}) \notin \mathcal{G}$ (see Theorem 3.1) $\Rightarrow X^* \setminus U_{\alpha_0} \notin \mathcal{G} \Rightarrow \mathrm{cl}(X^* \setminus U_{\alpha_0})$ is compact in X. Since $\{U_\alpha \cap X : \alpha \in \Lambda\}$ is an open cover of X (refer to Theorem 3.1(b)), $\mathrm{cl}(X^* \setminus U_{\alpha_0}) \subseteq \bigcup_{i=1}^{n} (U_{\alpha_i}) \cap X$, for finitely many sets $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ of $\mathcal{U}$. Then $X^* = (\bigcup_{i=1}^{n} U_{\alpha_i}) \cup U_{\alpha_0}$, and hence X^* is compact. ∎

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