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WEYL'S AND BROWDER'S THEOREM FOR AN ELEMENTARY OPERATOR

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Abstract. Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$ into itself. The generalized derivation $\delta_{A,B}$ is defined by $\delta_{A,B}(X) = AX - XB$. For pairs $C = (A_1, A_2)$ and $D = (B_1, B_2)$ of operators, we define the elementary operator $\Phi_{C,D}$ by $\Phi_{C,D}(X) = A_1XB_1 - A_2XB_2$. If $A_2 = B_2 = I$, we get the elementary operator $\Delta_{A_1,B_1}(X) = A_1XB_1 - X$. Let $d_{A,B} = \delta_{A,B}$ or $\Delta_{A,B}$. We prove that if A, B^* are log-hyponormal, then $f(d_{A,B})$ satisfies (generalized) Weyl's Theorem for each analytic function f on a neighborhood of $\sigma(d_{A,B})$, we also prove that $f(\Phi_{C,D})$ satisfies Browder's Theorem for each analytic function f on a neighborhood of $\sigma(\Phi_{C,D})$.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of bounded linear operators acting on infinite dimensional separable Hilbert space $\mathcal{H}$. If $A \in B(\mathcal{H})$ we shall write $\ker(A)$ and $\operatorname{ran}(A)$ for the null space and the range of A, respectively. By $\alpha(A)$ and $\beta(A)$ we denote the dimension of the kernel of A and the codimension of the range of A, respectively. Also write $\sigma(A)$, $\sigma_a(A)$, $\operatorname{iso}\sigma(A)$ for the spectrum, approximate point spectrum and the set of the isolated points of the spectrum of A, respectively. If $A \in B(\mathcal{H})$, we say that A has the single-valued extension property at λ_0, SVEP (for short), if for every open disk D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to \mathcal{H}$ which satisfies the equation $(\lambda I - A)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$.

Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$ and $0 < p \leq 1$. A is called a p-hyponormal operator if $(AA^*)^p \leq (A^*A)^p$. Especially, A is called a hyponormal operator if $p = 1$ and semi-hyponormal if $p = \frac{1}{2}$. A is called a log-hyponormal operator if A is invertible and $\log(AA^*) \leq \log(A^*A)$. Since $\log : (0,\infty) \to (-\infty,\infty)$ is operator monotone, every invertible p-hyponormal operator is log-hyponormal. But the converse is not true [7]. However it is interesting to regard log-hyponormal operators as 0-hyponormal operators [7, 21]. The idea

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of log-hyponormal operator is due to Ando [3] and the first paper in which log-hyponormality appeared is [13]. See [3, 7, 21] for properties of log-hyponormal operators. An operator $A \in B(\mathcal{H})$ has a unique polar decomposition $A = U|A|$, where $|A| = (A^*A)^{\frac{1}{2}}$ and U is a partial isometry. If $A = U|A|$, then the Aluthge transform of A is defined by $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$.

The class of operators A, B^* such that $\ker(d_{A,B}) \subseteq \ker(d_{A^*,B^*})$ is large and includes in particular the class of log-hyponormal operators [17]. It is well known that if A, B^* are log-hyponormal operators, then $\ker(d_{A,B}) \subseteq \ker(d_{A^*,B^*})$ and $\operatorname{asc}(d_{A,B}) \leq 1$; this implies that $d_{A,B}$ has the single valued extension property and hence satisfies Browder's Theorem [11]. Here $\operatorname{asc}(d_{A,B})$ denote the *ascent* of $d_{A,B}$.

The plan of this paper is as follows. In section 2 we prove that if $C = (A_1, A_2)$ and $D = (B_1, B_2)$ are pairs of operators and A_1, A_2, B_1^*, B_2^* are log-hyponormal such that A_1 doubly commutes with A_2 and B_1 doubly commutes with B_2, then $\operatorname{asc}(\Phi_{C,D}) \leq 1$. In section 3 we shall prove that if $A, B^* \in B(\mathcal{H})$ are log-hyponormal, then $d_{A,B}$ is isoloid and the range of $d_{A,B} - \lambda$ is closed for each isolated point λ in the spectrum of $d_{A,B}$. In section 4 we shall show that if $A, B^* \in B(\mathcal{H})$ are log-hyponormal, then the Weyl's Theorem holds for $f(d_{A,B})$ for every analytic function f defined on a neighborhood U of $\sigma(d_{A,B})$. Finally we shall prove the Browder's Theorem for the elementary operator $\Phi_{C,D}$.

2. The ascent of an elementary operator

Recall that the finite ascent property implies SVEP. In the following we prove that if $C = (A_1, A_2)$ and $D = (B_1, B_2)$ are pairs of operators and A_1, A_2, B_1^*, B_2^* are log-hyponormal such that A_1 doubly commutes with A_2 and B_1 doubly commutes with B_2, then $\operatorname{asc}(\Phi_{C,D}) \leq 1$.

LEMMA 2.1. *Let $A, B \in B(\mathcal{H})$ be* log-*hyponormal operators such that A doubly commutes with B. Then AB is* log-*hyponormal.*

Proof. If A and B are log-hyponormal, then A and B are invertible and

$$\log|A|^2 \geq \log|A^*|^2, \ \log|B|^2 \geq \log|B^*|^2.$$

Since $AB = BA$ and $AB^* = B^*A$ it follows that $|AB|^2 = |A|^2|B|^2 = |B|^2|A|^2$, and

$$|(AB)^*|^2 = |A^*|^2|B^*|^2 = |B^*|^2|A^*|^2.$$

Hence

$$\begin{aligned}
\log|AB|^2 &= \lim_{p\to 0^+} \frac{\big(|AB|^2\big)^p - 1}{p} = \lim_{p\to 0^+} \frac{\big(|A|^2\big)^p\big(|B|^2\big)^p - 1}{p} \\
&= \lim_{p\to 0^+} \frac{\big(\big(|A|^2\big)^p - 1\big)\big(\big(|B|^2\big)^p - 1\big) + \big(|A|^2\big)^p + \big(|B|^2\big)^p - 2}{p} \\
&= \log|A|^2 + \log|B|^2.
\end{aligned}$$

Similarly we have $\log|(AB)^*|^2 = \log|A^*|^2 + \log|B^*|^2$. Hence AB is invertible and

$$\log|AB|^2 - \log|(AB)^*|^2 = \log|A|^2 - \log|A^*|^2 + \log|B|^2 - \log|B^*|^2 \geq 0.$$

Thus AB is log-hyponormal. ∎

LEMMA 2.2. *If A and B^* are* log-*hyponormal, then* $\operatorname{asc}(\Delta_{A,B}) \leq 1$.

Proof. It is known that if A and B^* are log-hyponormal, then $\ker(d_{A,B}) \subseteq \ker(d_{A^*,B^*})$ [17] and this by [10] implies that $\ker(\Delta_{A,B}) \subseteq \ker(\Delta_{A^*,B^*})$ and the result follows by [10]. ∎

THEOREM 2.3. *Let $C = (A_1, A_2), D = (B_1, B_2)$ be pairs of operators in $B(\mathcal{H})$. If A_1, B_1^*, A_2, B_2^* are* log-*hyponormal operators such that A_1 doubly commutes with A_2 and B_1 doubly commutes with B_2, then* $\operatorname{asc}(\Phi_{C,D}) \leq 1$.

Proof. We have

$$\Phi_{C,D} = A_2\big(\Delta_{(A_2^{-1}A_1),(B_1B_2^{-1})}\big)B_2$$

Since A_2^{-1} and $(B_2^{-1})^* = (B_2^*)^{-1}$ are log-hyponormal [3, Lemma 1.1] and $A_2^{-1}A_1$ and $B_1B_2^{-1}$ are log-hyponormal by Lemma 2.1, then by applying Lemma 2.2 we obtain

$$\operatorname{asc}\left(\Delta_{(A_2^{-1}A_1),(B_1B_2^{-1})}\right) \leq 1.$$

Since

$$\begin{aligned}\ker\Phi_{C,D}^n &= A_2^n\big(\ker\Delta^n_{(A_2^{-1}A_1),(B_1B_2^{-1})}\big)B_2^n \\ &= A_2\big(\ker\Delta_{(A_2^{-1}A_1),(B_1B_2^{-1})}\big)B_2 = \ker\Phi_{C,D},\end{aligned}$$

it is $\operatorname{asc}(\Phi_{C,D}) \leq 1$. ∎

COROLLARY 2.4. *Let $C = (A_1, A_2), D = (B_1, B_2)$ be pairs of operators in $B(\mathcal{H})$. If A_1, B_1^*, A_2, B_2^* are* log-*hyponormal operators such that A_1 doubly commutes with A_2 and B_1 doubly commutes with B_2, then $\Phi_{C,D}$ has the single valued extension property.*

Proof. The proof follows from Theorem 2.3 and [18, Proposition 1.8]. ∎

COROLLARY 2.5. *Let $C = (A_1, A_2), D = (B_1, B_2)$ be pairs of operators in $B(\mathcal{H})$ and A_1, B_1^*, A_2, B_2^* be* log-*hyponormal operators such that A_1 doubly commutes with A_2 and B_1 doubly commutes with B_2. A necessary and sufficient condition for* $\operatorname{ran}(\Phi_{C,D})$ *to be closed is that* $\operatorname{ran}(\Phi_{C,D}) + \ker\Phi_{C,D}$ *is closed.*

Proof. Since $\operatorname{asc}(\Phi_{C,D}) \leq 1$, the proof follows from [18, Proposition 4.10.4]. ∎

3. The range of an elementary operator

Recall that $A \in B(\mathcal{H})$ is said to be isoloid if $\lambda \in \operatorname{iso}\sigma(A)$ implies $\lambda \in \sigma_p(A)$. In this section, we prove that if A and B^* are log-hyponormal, then $d_{A,B}$ is isoloid and $\operatorname{ran}(d_{A,B}-\lambda)$ is closed for each $\lambda \in \operatorname{iso}\sigma(d_{A,B})$. We denote by L_A the operator of left multiplication by A and by R_B the operator of right multiplication by B.

Lemma 3.1. *If $A=U|A|$ is the polar decomposition of A and $\widetilde{A}=|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ its Aluthge transform, then*

$$|A|^{\frac{1}{2}}A=\widetilde{A}|A|^{\frac{1}{2}} \text{ and } A\big(U|A|^{\frac{1}{2}}\big)=\big(U|A|^{\frac{1}{2}}\big)\widetilde{A}.$$

Lemma 3.2. *Let $A,B\in B(\mathcal{H})$. If A,B are invertible, then L_AR_B is invertible.*

Proof. Since $\sigma(L_AR_B)=\bigcup\{\sigma(zA): z\in\sigma(B)\}$ by [12, Theorem 3.2] we deduce that L_AR_B is invertible. ■

It is well known that $d_{A,B}$ is isoloid when A and B^* are hyponormal [11, Theorem 2.7]. The following theorem says that $d_{A,B}$ retains this property in the case in which A and B^* are log-hyponormal.

Theorem 3.3. *If A and B^* are* log*-hyponormal, then $d_{A,B}$ is an isoloid.*

Proof. The case $d_{A,B}=\Delta_{A,B}$.

Let $\lambda\in\operatorname{iso}\sigma(\Delta_{A,B})$ such that $\lambda\neq -1$, then $(\Delta_{A,B}-\lambda)(X)=AXB-(1+\lambda)X$ and it follows from [12, Theorem 3.2] that $\sigma(\Delta_{A,B}-\lambda)=\bigcup\{\sigma((-(1+\lambda)+zA): z\in\sigma(B)\}$. Since $\operatorname{iso}\sigma(A)=\operatorname{iso}\sigma(\widetilde{A})=\operatorname{iso}\sigma(\widetilde{\widetilde{A}})$ and $\operatorname{iso}\sigma(B)=\operatorname{iso}\sigma(\widetilde{B})=\operatorname{iso}\sigma(\widetilde{\widetilde{B}})$ by [2, Corollary 2.3], we deduce that $\operatorname{iso}\sigma(\Delta_{A,B})=\operatorname{iso}\sigma(\Delta_{\widetilde{A},\widetilde{B}})=\operatorname{iso}\sigma(\Delta_{\widetilde{\widetilde{A}},\widetilde{\widetilde{B}}})$. The operators A and B^* being log-hyponormal, it follows from [21] that $\widetilde{A}$ (resp. $\widetilde{B^*}$) is semi-hyponormal and from [2] that $\widetilde{\widetilde{A}}$ (resp. $\widetilde{\widetilde{B^*}}$) is invertible and hyponormal, and the result follows from [11, Theorem 2.7] since $\sigma_p(\Delta_{\widetilde{\widetilde{A}},\widetilde{\widetilde{B}}})=\sigma_p(\Delta_{A,B})$.

Let $\lambda=-1$,

$$\lambda\in\operatorname{iso}\sigma(\Delta_{A,B})\Rightarrow 0\in\operatorname{iso}\sigma(\Delta_{A,B}-\lambda)\Rightarrow 0\in\operatorname{iso}\sigma(L_AR_B).$$

This is a contradiction with Lemma 3.2.

The case $d_{A,B}=\delta_{A,B}$.

Let $\lambda\in\operatorname{iso}\sigma(\delta_{A,B})$. Then $0\in\operatorname{iso}\sigma(\delta_{A,B}-\lambda)$, where $\sigma(\delta_{A,B}-\lambda)=\sigma(A)-\sigma(B-\lambda)$ [12]. Hence $\operatorname{iso}\sigma(\delta_{A,B})=\operatorname{iso}\sigma(\delta_{\widetilde{A},\widetilde{B}})=\operatorname{iso}\sigma(\delta_{\widetilde{\widetilde{A}},\widetilde{\widetilde{B}}})$. The same arguments cited above guarantees that $\sigma_p(\Delta_{\widetilde{\widetilde{A}},\widetilde{\widetilde{B}}})=\sigma_p(\Delta_{A,B})$. ■

Remark 3.4. Note that in the above theorem we utilize the fact that $\widetilde{A^*}$ is p-hyponormal if and only if $\big(\widetilde{A}\big)^*$ p-hyponormal.

Theorem 3.5. *If A,B^* are log-hyponormal, then $d_{A,B}-\lambda$ has closed range for each $\lambda\in\operatorname{iso}\sigma\,(d_{A,B})$.*

Proof. Let $d_{A,B}=\Delta_{A,B}$ and $\lambda\in\operatorname{iso}\sigma(\Delta_{A,B})$ such that $\lambda\neq-1$. Let $Y\in\overline{\operatorname{ran}(\Delta_{\widetilde{A},\widetilde{B}}-\lambda)}$ for all $\lambda\in\operatorname{iso}\sigma(\Delta_{\widetilde{A},\widetilde{B}})$. Then there exists (X_n) in $B(\mathcal{H})$ such that

$$\widetilde{A}X_n\widetilde{B}-(1+\lambda)\,X_n\longrightarrow Y.$$

Let $\widetilde{B} = \widetilde{U}|\widetilde{B}|$ be the polar decomposition of $\widetilde{B}$. Then

$$|\widetilde{A}|^{\frac{1}{2}}\widetilde{A}X_n\widetilde{B}\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} - (1+\lambda)|\widetilde{A}|^{\frac{1}{2}}X_n\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} \longrightarrow |\widetilde{A}|^{\frac{1}{2}}Y\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}.$$

From Lemma 3.1 we have $|\widetilde{A}|^{\frac{1}{2}}\widetilde{A} = \widetilde{\widetilde{A}}|\widetilde{A}|^{\frac{1}{2}}$ and $\widetilde{B}(\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}) = (\widetilde{U}|\widetilde{B}|^{\frac{1}{2}})\widetilde{\widetilde{B}}$. Hence

$$\widetilde{\widetilde{A}}|\widetilde{A}|^{\frac{1}{2}}X_n\widetilde{B}\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} - (1+\lambda)|\widetilde{A}|^{\frac{1}{2}}X_n\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} \longrightarrow |\widetilde{A}|^{\frac{1}{2}}Y\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}$$
$$\widetilde{\widetilde{A}}|\widetilde{A}|^{\frac{1}{2}}X_n\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}\widetilde{\widetilde{B}} - (1+\lambda)|\widetilde{A}|^{\frac{1}{2}}X_n\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} \longrightarrow |\widetilde{A}|^{\frac{1}{2}}Y\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}.$$

Since the operators A and B^* are log-hyponormal, it follows from [21] that $\widetilde{A}$ (resp. $\widetilde{B^*}$) is semi-hyponormal and from [2] that $\widetilde{\widetilde{A}}$ (resp. $\widetilde{\widetilde{B^*}}$) is invertible and hyponormal, and so from [11, Theorem 2.7] $(\Delta_{\widetilde{\widetilde{A}},\widetilde{\widetilde{B}}} - \lambda)$ has closed range for each $\lambda \in \operatorname{iso}\sigma(\Delta_{\widetilde{\widetilde{A}},\widetilde{\widetilde{B}}}) = \operatorname{iso}\sigma(\Delta_{\widetilde{A},\widetilde{B}})$. Hence there exists $Z \in B(\mathcal{H})$ such that

$$\widetilde{\widetilde{A}}|\widetilde{A}|^{\frac{1}{2}}X_n\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}\widetilde{\widetilde{B}} - (1+\lambda)|\widetilde{A}|^{\frac{1}{2}}X_n\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} \longrightarrow \widetilde{\widetilde{A}}Z\widetilde{\widetilde{B}} - (1+\lambda)Z.$$

Since $|\widetilde{A}|^{\frac{1}{2}}$ and $\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}$ are invertible, from Lemma 3.2 $L_{|\widetilde{A}|^{\frac{1}{2}}}R_{\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}}$ is invertible. So there exists $X \in B(\mathcal{H})$ such that $Z = |\widetilde{A}|^{\frac{1}{2}}X\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}$. Hence

$$\widetilde{\widetilde{A}}Z\widetilde{\widetilde{B}} - (1+\lambda)Z = \widetilde{\widetilde{A}}|\widetilde{A}|^{\frac{1}{2}}X\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}\widetilde{\widetilde{B}} - (1+\lambda)|\widetilde{A}|^{\frac{1}{2}}X\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}.$$

From the uniqueness of the limit we obtain

$$\widetilde{\widetilde{A}}|\widetilde{A}|^{\frac{1}{2}}X\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}\widetilde{\widetilde{B}} - (1+\lambda)|\widetilde{A}|^{\frac{1}{2}}X\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} = |\widetilde{A}|^{\frac{1}{2}}Y\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}$$
$$|\widetilde{A}|^{\frac{1}{2}}\widetilde{A}X\widetilde{B}\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} - (1+\lambda)|\widetilde{A}|^{\frac{1}{2}}X\widetilde{U}|\widetilde{B}|^{\frac{1}{2}} = |\widetilde{A}|^{\frac{1}{2}}Y\widetilde{U}|\widetilde{B}|^{\frac{1}{2}}.$$

Then $\widetilde{A}X\widetilde{B} - (1+\lambda)X = Y$, and hence $Y \in \operatorname{ran}(\Delta_{\widetilde{A},\widetilde{B}} - \lambda)$, and thus $\Delta_{\widetilde{A},\widetilde{B}} - \lambda$ has closed range for $\lambda \in \operatorname{iso}\sigma(\Delta_{\widetilde{A},\widetilde{B}})$. The same argument implies that $(\Delta_{A,B} - \lambda)$has closed range for each $\lambda \in \operatorname{iso}\sigma(\Delta_{A,B})$ such that $\lambda \neq -1$.

The case $d_{A,B} = \delta_{A,B}$.

Let $\lambda \in \operatorname{iso}\sigma(\delta_{A,B})$. Then $0 \in \operatorname{iso}\sigma(\delta_{A,B} - \lambda)$. The same arguments implies that $(\delta_{A,B} - \lambda)$ has closed range for each $\lambda \in \operatorname{iso}\sigma(\delta_{A,B})$. ■

4. Weyl's and Browder's Theorem

An operator $A \in B(\mathcal{H})$ is called Fredholm if it has closed range, finite dimensional null space, and its range has finite co-dimension. The index of a Fredholm operator is given by

$$I(A) := \alpha(A) - \beta(A).$$

A bounded operator A is said to be a Weyl operator if it is Fredholm of index 0. Recall that the ascent of an operator A, denoted by $\operatorname{asc}(A)$, is the smallest nonnegative integer n such that $\ker(A^n) = \ker(A^{n+1})$. Analogously, the descent of an operator A, denoted by $\operatorname{des}(A)$, is the the smallest nonnegative integer n such that $\operatorname{ran}(A^n) = \operatorname{ran}(A^{n+1})$. It is well known that if $\operatorname{asc}(A)$ and $\operatorname{des}(A)$ are both finite then $\operatorname{asc}(A) = \operatorname{des}(A)$ [16]. $A \in B(\mathcal{H})$ is said to be a Browder operator if A is Fredholm with $\operatorname{asc}(A) = \operatorname{des}(A) < \infty$. Note that if A is Browder then A is Weyl, (see [15]). The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ are defined by (see [14])

$$\begin{aligned}
\sigma_e(A) &:= \{\, \lambda \in \mathbf{C} : A - \lambda \text{ is not Fredholm} \,\} \\
\sigma_w(A) &:= \{\, \lambda \in \mathbf{C} : A - \lambda \text{ is not Weyl} \,\} \\
\sigma_b(A) &:= \{\, \lambda \in \mathbf{C} : A - \lambda \text{ is not Browder} \,\}.
\end{aligned}$$

Let $\sigma_0(A)$ denote the set of Riesz points of A and $\sigma_{00}(A) = \{\, \lambda \in \operatorname{iso}\sigma(A) : 0 < \dim\ker A < \infty \,\}$. Then

$$\operatorname{iso}\sigma(A) \setminus \sigma_e(A) = \operatorname{iso}\sigma(A) \setminus \sigma_w(A) = \sigma_0(A) \subseteq \sigma_{00}(A).$$

Note that $A \in B(\mathcal{H})$ satisfies Weyl's Theorem (resp. Browder's Theorem) if $\sigma_w(A) = \sigma(A) \setminus \sigma_{00}(A)$ (resp. $\sigma_w(A) = \sigma(A) \setminus \sigma_0(A)$). A generalization of these notions are given in [4]; precisely, $A \in B(\mathcal{H})$ is said to be generalized Fredholm or B-Fredholm, if there exists a positive integer n for which the induced operator $A_n : \operatorname{ran}(A^n) \to \operatorname{ran}(A^n)$ is Fredholm in the usual sense, and generalized Weyl, if in addition A_n has index zero. The generalized Weyl's spectrum $\sigma_{Bw}(A)$ of A is defined to be the set

$$\{\, \lambda \in \mathbf{C} : (A - \lambda) \text{ is not generalized Weyl} \,\},$$

and we say that A satisfies generalized Weyl's Theorem if $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, where $E(A)$ is the set of all isolated eigenvalues of A. Note that if A satisfies generalized Weyl's Theorem then A satisfies generalized Browder's Theorem, see [4, Corollary 2.6]. Moreover, in [6] it is shown that if A satisfies generalized Weyl's Theorem, then A satisfies Weyl's Theorem, but the reverse implication in general fails [4, Example 4.1], and if A satisfies generalized Browder's Theorem, then A satisfies Browder's Theorem.

Let $A, B^* \in B(\mathcal{H})$ be log-hyponormal. Then the SVEP of $d_{A,B}$ implies that the Browder's Theorem holds for $d_{A,B}$. Recall [9, Theorem 2.5] that if an operator $A \in B(\mathcal{H})$ has SVEP, then A satisfies Weyl's Theorem if and only if $\operatorname{ran}(A - \lambda)$ is closed for every $\lambda \in \sigma_{00}(A)$. Hence in view of Theorem 3.2, $d_{A,B}$ satisfies Weyl's Theorem.

THEOREM 4.1. *Let $A, B^* \in B(\mathcal{H})$ be* log-*hyponormal. If f is analytic on a neighbourhood of $\sigma(d_{A,B})$, then $f(d_{A,B})$ satisfies Weyl's Theorem.*

Proof. SVEP being stable under the functional calculus [18], $d_{A,B}$ has SVEP $\Longrightarrow f(d_{A,B})$ has SVEP for each f analytic in a neighbourhood of $\sigma(d_{A,B})$. This implies that $\sigma_b(f(d_{A,B})) = \sigma_w(f(d_{A,B}))$ [14]. Since the spectral mapping theorem holds for σ_b, we have

$$\sigma_w(f(d_{A,B})) = \sigma_b(f(d_{A,B})) = f(\sigma_b(d_{A,B})) = f(\sigma_w(d_{A,B})).$$

To complete the proof we have to show that

$$f\left(\sigma_w\left(d_{A,B}\right)\right)=\sigma\left(f\left(d_{A,B}\right)\right)\diagdown\sigma_{00}\left(f\left(d_{A,B}\right)\right).$$

This follows from Theorem 3.1 and a limit argument applied to [20, Proposition 1]. ■

THEOREM 4.2. *Let* $C=(A_1,A_2)\,,D=(B_1,B_2)$ *be pairs of operators in* $B(\mathcal{H})$ *and* A_1,B_1^*,A_2,B_2^* *be* log-*hyponormal operators such that* A_1 *doubly commutes with* A_2 *and* B_1 *doubly commutes with* B_2*. If* f *is analytic on a neighborhood of* $\sigma\left(\Phi_{C,D}\right)$*, then* $f\left(\Phi_{C,D}\right)$ *satisfies Browder's Theorem.*

Proof. From Corollary 2.2 $\Phi_{C,D}$ has the single valued extension property. This implies that $\Phi_{C,D}$ satisfies Browder's Theorem [10]. SVEP being stable under the functional calculus [18], $\Phi_{C,D}$ has SVEP $\Longrightarrow f\left(\Phi_{C,D}\right)$ has SVEP for each f analytic in a neighborhood of $\sigma\left(\Phi_{C,D}\right)$. Hence $f\left(\Phi_{C,D}\right)$ satisfies Browder's Theorem. ■

The operator A is said to be *Drazin invertible* if there is an operator T and a nonnegative integer $n\in\mathbf{N}$ such that

$$A^nTA=A^n,TAT=T\ \text{ and } TA=AT.$$

It is known that A is Drazin invertible if and only if both asc(A) and des(A) are finite [19].

THEOREM 4.3. *Let* $A,B^*\in B(\mathcal{H})$ *be* log-*hyponormal. If* f *is analytic on a neighborhood of* $\sigma\left(d_{A,B}\right)$*, then* $f\left(d_{A,B}\right)$ *satisfies generalized Weyl's Theorem.*

Proof. Let $\lambda\in\sigma\left(d_{A,B}\right)\setminus\sigma_{Bw}\left(d_{A,B}\right)$. Since $d_{A,B}$ has SVEP, it follows upon arguing as in the proof of [4, Theorem 3.12] and an application of Theorem 3.1 that $\lambda\in\operatorname{iso}\sigma\left(d_{A,B}\right)=E\left(d_{A,B}\right)$. Conversely, if $\lambda\in E\left(d_{A,B}\right)$, then $d_{A,B}-\lambda$ is Fredholm of index 0 by Theorem 3.1 and Theorem 3.2. Hence $d_{A,B}$ satisfies generalized Weyl's Theorem. Let f be analytic on a neighborhood of $\sigma\left(d_{A,B}\right)$, and let $\sigma_D\left(d_{A,B}\right)=\left\{\lambda\in\mathbf{C}:\left(d_{A,B}-\lambda\right)\text{ is not Drazin invertible}\right\}$ denote the Drazin spectrum of $d_{A,B}$. Then $\sigma_D\left(f\left(d_{A,B}\right)\right)=f\left(\sigma_D\left(d_{A,B}\right)\right)$ [4, Corollary 2.4]. Since $d_{A,B}$ and $f\left(d_{A,B}\right)$ have SVEP, $\sigma_D\left(d_{A,B}\right)=\sigma_{Bw}\left(d_{A,B}\right)$ and $\sigma_D\left(f\left(d_{A,B}\right)\right)=\sigma_{Bw}\left(f\left(d_{A,B}\right)\right)$ [4, Theorem 3.12]. Hence

$$f\left(\sigma_{Bw}\left(d_{A,B}\right)\right)=f\left(\sigma\left(d_{A,B}\right)\diagdown E\left(d_{A,B}\right)\right)=\sigma_{Bw}\left(f\left(d_{A,B}\right)\right).$$

The isoloid property of $d_{A,B}$, Theorem 3.1 implies that

$$\sigma_{Bw}\left(f\left(d_{A,B}\right)\right)=\sigma\left(f\left(d_{A,B}\right)\right)\diagdown E\left(f\left(d_{A,B}\right)\right)$$

[5, Lemma 2.9], and the proof is complete. ■

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REFERENCES

[**1**] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$*, w-hyponormal operators*, Integral Equations Operator Theory, **13** (1990), 307–315.

[**2**] A. Aluthge and D. Wang, *w-hyponormal operators*, Integral Equations Operator Theory, **36** (2000), 1–10.

[**3**] T. Ando, *Operators with a norm condition*, Acta. Sci. Math. (Szeged), **33** (1972), 169–178.

[**4**] M. Berkani, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appli., **272** (2002), 596–603.

[**5**] M. Berkani and A. Arroud, *Generalized Weyl's theorem and hyponormal operators*, J. Aust. Math. Soc., **76** (2004), 291–302.

[**6**] M. Berkani and J. Koliha, *Weyl type theorems for bounded linear operators*, Acta. Sci. Math. (Szeged), **69** (2003), 359–376.

[**7**] M. Cho and K. Tanahashi, *Isolated point of spectrum of p-hyponormal operators,* log-*hyponormal opertors*, pre-print.

[**8**] R.E. Curto and Y.M. Han, *Weyl's theorem, a-Weyl's theorem and local spectral theory*, J. London Math. Soc. **67** (2003), 499–509.

[**9**] B.P. Duggal, *A remark on generalized Putnam-Fuglede theorem*, Proc. Amer. Math. Soc. **129** (2000), 83–87.

[**10**] B.P. Duggal and S.V. Djordjević, *Dunford's property (C) and Weyl's Theorem*, Integral Equations Operator Theory, **43** (2002), 290–297.

[**11**] B.P. Duggal, *Weyl's Theorem for generalized derivation and an elementary operator*, Math. Vesnik **54** (2002), 71–81.

[**12**] M.R. Embry and M. Rosemblum, *Spectra, tensor product and linear operator equation*, Pac. J. Math. **53** (1974) 95–107.

[**13**] M. Fujii, C. Hemeji and A. Matsumoto, *Theorems of Ando and Saito for p-hyponormal operators*, Math. Japonica **39** (1994), 595–598.

[**14**] R.E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Irish Acad. **A 85**(2) (1985), 151–176.

[**15**] R.E. Harte and W.Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), 2115–2124.

[**16**] H. Heuser, *Functional Analysis*, Marcel Dekker, New York, 1982.

[**17**] I.H. Jeon, K. Tanahashi and A. Uchiyama, *On quasisimilarity for* log-*hyponormal operators*, Glasgow Math. J., **46** (2004), 169–176.

[**18**] K.B. Laursen, *Operators with finite ascent*, Pacific J. Math. **152**, 2(1992), 323–336.

[**19**] D.C. Lay, *Spectral analysis using ascent, descent, nullity and defect*, Math. Ann. **184** (1970), 197–214.

[**20**] K.K. Oberai, *On the Weyl spectrum II*, Illinois J. Math. **21**, 84–90.

[**21**] K. Tanahashi, *On* log-*hyponormal operators*, Integral Equations Operator Theory, **34** (1999), 364–372.

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