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GENERALIZED (CO)HOMOLOGY AND MORSE COMPLEX

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Abstract. In this paper we analyse Morse complex in the framework of generalized (co)homology theory. This analysis was suggested in [7].

1. Introduction

It is well known that using a Morse function on a compact smooth manifold one can recover integral cohomology of this manifold. It is interesting to see what can be said about generalized homology. Of course, since a Morse function actually determines appropriate CW complex homotopy equivalent to the given manifold, one knows a priori that it should give information about generalized homology as well, but one wants to know how generalized (co)homology can be computed using a Morse function.

We give first some background on spectral sequences. Here we follow the treatment as in [4] which we find most illuminating. After that we remind the reader about the basic notions of generalized homology and give a spectral sequence which one uses to compute generalized homology or cohomology of a given filtered space—the Atiyah-Hirzebruch spectral sequence.

We apply these techniques in the case when we have a Morse function on a given compact manifold. We know from general considerations that this spectral sequence converges toward (generalized) (co)homology of our manifold, but our task is to show what is the second term of our spectral sequence and how it is related to the Morse function in question. We also give an example which shows how one computes using this sequence. For the reason of simplicity, when we deal with actual computations, we only use generalized homology with $\mathbf{Z}/2$ coefficients, although more general results remain valid.

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At the end, we ponder on higher differentials in this spectral sequence in order to better understand geometry of the situation which ordinary homology leaves behind.

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2. Spectral sequences and generalized (co)homology

This material is fairly standard. For the reader's convenience we supply some basic definitions and facts.

Look at the following diagram consisting of graded abelian groups and homomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & A_{p-2} \arrow[rr, "i"] & & A_{p-1} \arrow[rr, "i"] \arrow[dl, "j"] & & A_p \arrow[rr, "i"] \arrow[dl, "j"] & & A_{p+1} \arrow[r, "i"] \arrow[dl, "j"] & \cdots \\
\cdots & & E_{p-1} \arrow[ul, "k"] & & E_p \arrow[ul, "k"] & & E_{p+1} \arrow[ul, "k"] & & \cdots
\end{tikzcd}$$

We are using here homology type notation and we assume that each triangle

$$\cdots \to A_{p-1} \to A_p \to E_p \to A_{p-1} \to \cdots$$

from this diagram is a long exact sequence. This is what Boardman (see [4]) calls *unrolled exact couple*. Since we express a bias toward homology, i and j have degree 0 while k has degree -1. Of course, all constructions work as well for the case when k has degree $+1$.

If we denote by $i^{(r-1)}$ $(r-1)$-fold iterate of i, we get the following diagram.

$$\begin{tikzcd}
A_{p-r-1} \arrow[rr, "i"] & & A_{p-r} \arrow[r, "i^{(r-1)}"] \arrow[dl, "j"] & A_{p-1} \arrow[rr, "i"] & & A_p \arrow[dl, "j"] \\
& E_{p-r} \arrow[ul, "k"] & & & E_p \arrow[ul, "k"] &
\end{tikzcd}$$

We use this diagram to introduce the following notation

$$\begin{aligned}
Z_p^r &:= k^{-1}(\operatorname{Im}[i^{(r-1)}\colon A_{p-r} \to A_{p-1}]) \\
B_p^r &:= j \operatorname{Ker}[i^{(r-1)}\colon A_p \to A_{p+r-1}] \\
E_p^r &:= Z_p^r / B_p^r
\end{aligned}$$

The (sub)group Z_p^r is known as the group of r-cycles, and B_p^r as the group of r-boundaries of E_p. The group E_p^r is a component of the E_r-term of the spectral sequence which we construct. Of course, $E_1^r = E^r$. The following inclusions hold

$$0 = B_p^1 \subseteq B_p^2 \subseteq B_p^3 \subseteq \cdots \subseteq \operatorname{Im} j = \operatorname{Ker} k \subseteq \cdots \subseteq Z_p^3 \subseteq Z_p^2 \subseteq Z_p^1 = E_p.$$

Differential d^r in this spectral sequence is induced by the additive relation $j(i^{(r-1)})^{-1}k$ (for additive relations see [15]. Namely, we would like to take an element of E_p, apply k, then lift this result $(r-1)$-times (using i) and, finally, apply j to what we got (look at the previous diagram). We cannot do this for all elements and even if we could, the result would only be defined up to a certain subgroup—that is how one introduces the groups of r-cycles and r-boundaries. One can check that in this way d^r maps E^r_p into E^r_{p-r}. In conclusion, some diagram chase shows that $H(E^r, d^r) \cong E^{r+1}$ and we really have a spectral sequence.

We can also introduce the following groups

$$Z^\infty_p := \bigcup_r Z^r_p, \qquad B^\infty_p := \bigcap_r B^r_p, \qquad E^\infty_p := Z^\infty_p / B^\infty_p$$

If we are able to relate this E^∞_p to some filtered group G than we have solved *the convergence problem* for this spectral sequence. Namely, suppose that G is filtered by subgroups $F_p = F_pG$ in such a way that $E^\infty_p = F_p/F_{p-1}$ (so we are using a filtration in which $F_{p-1} \subseteq F_p$—increasing filtration). Then we say that our spectral sequence *converges* to G and we denote this by

$$E^{r_0} \Longrightarrow G,$$

where r_0 corresponds to the last term in our spectral sequence for which we are able to find a decent presentation (usually, it is the second term!).

Next we discuss generalized (co)homology. Basic reference for this material is [6] or [1]. As we know, a generalized (co)homology theory satisfies all Eilenberg-Steenrod axioms except for the dimension axiom, namely homology of a point need not be trivial in nonzero dimensions. This is rather important for what follows. We will be working with *multiplicative* homology theory, therefore homology of a space $h_*(X)$ is a *module* over the *ring* of coefficients $h_* = h_*(pt)$ (or $\widetilde{h}_*(S^0)$ in the case of reduced theory). Let us just mention a few examples of generalized (co)homology theories to illustrate their difference from ordinary homology.

Example 1. Complex K-theory. The ring of coefficients KU_* for the homology theory is the ring of finite Laurent series $\mathbf{Z}[t, t^{-1}]$ with generator t in degree 2. This generator corresponds to the isomorphism arising from Bott periodicity.

Example 2. Complex cobordism. Here the ring of coefficients for the homology theory is $MU_* = \mathbf{Z}[x_1, x_2, \dots]$ where the polynomial generators x_i are in degree $2i$.

Example 3. Morava K-theories. The ring of coefficients here is $K(n)^* = \mathbf{F}_p[v_n, v_n^{-1}]$, where the degree of the generator v_n is $-2(p^n - 1)$. We would like to point out that we have infinitely many Morava K-theories—for different n and p and they are all periodic theories.

When we are given a filtered space X and a (multiplicative) homology theory h_* (as usual, h_* denotes both the theory in question and its ring of coefficients) we can construct a spectral sequence as follows. Denote by X_p the p-th space in the filtration of X. Then, we have the following unrolled exact couple

$$\begin{tikzcd}
h_*(X_{p-2}) \arrow[rr, "i"] & & h_*(X_{p-1}) \arrow[rr, "i"] \arrow[dr, "j"] & & h_*(X_p) \arrow[rr, "i"] \arrow[dr, "j"] & & h_*(X_{p+1}) \arrow[dl, "j"] \\
& h_*(X_{p-1}, X_{p-2}) \arrow[ul, "k"] & & h_*(X_p, X_{p-1}) \arrow[ul, "k"] & & h_*(X_{p+1}, X_p) \arrow[ul, "k"] &
\end{tikzcd}$$

The resulting spectral sequence is *the Atiyah-Hirzebruch* spectral sequence. It converges to the $h_*(X)$ (filtration on this group is induced by the filtration on X). The first differential d_1 of this spectral sequence is given by the composition jk:

$$d_1 : h_*(X_p, X_{p-1}) \xrightarrow{k} h_{*-1}(X_{p-1}) \xrightarrow{j} h_{*-1}(X_{p-1}, X_{p-2}).$$

We examine a version of this in sections that follow.

3. Morse functions and the resulting spectral sequences

Let us first recall a few general facts about Morse functions. Let $f\colon M \to \mathbf{R}$ be a Morse function on a smooth closed manifold M. We denote by $Crit_p(f)$ the set of its critical points of index p. For two critical points x_-, x_+ of f, we denote by $\mathcal{M}(x_-, x_+)$ the set of all $\gamma\colon \mathbf{R} \to M$ satisfying

$$\frac{d\gamma}{dt} = -\nabla f(\gamma), \quad \gamma(t) \to x_\pm \text{ as } t \to \pm\infty.$$

This equation defines one-parameter flow ψ_t on M. Note that

$$\mathcal{M}(x_-, x_+) = W^u(x_-) \cap W^s(x_+),$$

where

$$W^s(x_+) = \{\, x \in M \mid \lim_{t\to+\infty} \psi_t(x) = x_+ \,\}$$
$$W^u(x_-) = \{\, x \in M \mid \lim_{t\to-\infty} \psi_t(x) = x_- \,\},$$

are the stable and unstable manifolds of the negative gradient flow. For a generic choice of Riemannian metric (used to define the gradient ∇f), or, equivalently, for a generic choice of Morse function f, the intersection above is transverse (Morse-Smale condition). Hence, the set $\mathcal{M}(x_-, x_+)$ is a smooth manifold of dimension $m(x_+) - m(x_-)$, where $m(x)$ denotes the Morse index of a critical point x.

The group $\mathbf{R}$ acts on $\mathcal{M}(x_-, x_+)$ via $\gamma \mapsto \gamma(\cdot + s)$. We denote the orbit space of this action by

$$\widehat{\mathcal{M}}(x_-, x_+) := \mathcal{M}(x_-, x_+)/\mathbf{R}.$$

The manifolds $\widehat{\mathcal{M}}(x_-, x_+)$ can be given a coherent orientation σ (see [11]) for more details). Obviously, $\dim \widehat{\mathcal{M}}(x_-, x_+) = \dim \mathcal{M}(x_-, x_+) - 1$. It is known that when $m(x_+) - m(x_-) = 1$ the zero dimensional manifold $\widehat{\mathcal{M}}(x_-, x_+)$ is compact, and thus a finite set.

Let $f\colon M \to \mathbf{R}$ be a Morse function on a smooth closed manifold M. We denote by $Crit_p(f)$ the set of its critical points of index p and define

$$C_p(f) := \text{free abelian group generated by } Crit_p(f).$$

The graded abelian groups $C_*(f)$ can be turned into the chain complex in the following way. For a generator $x \in Crit_p(f)$ we define

$$\partial_p x := \sum_{y \in Crit_{p-1}(f)} n(x, y) y,$$

and extend it to the map

$$\partial_p \colon C_p(f) \to C_{p-1}(f).$$

Here $n(x,y)$ is the signed (with respect to the orientation σ) number of points in (zero dimensional compact) manifold $\widehat{\mathcal{M}}(x,y)$. Of course, $H_p(f) := \operatorname{Ker}\partial_p / \operatorname{Im}\partial_p$. It turns out that $H_*(f) \cong H_*(M)$, where $H_*(M)$ denotes the singular homology of M. One also derives from this chain complex the well-known Morse inequalities. This is the classical point of view.

As we see, in the classical case, only the one-dimensional moduli spaces of trajectories are important. Thus, we can say that (ordinary) Morse homology algebraizes the one-dimensional moduli spaces. In the case of generalized homology, moduli spaces of any dimension play their role and the generalized homology is computed using a spectral sequence as is shown below. That construction therefore give an algebraization of the higher moduli spaces. Another way to algebraize them is given by Barraud and Cornea [3].

In order to get to our spectral sequence we follow the idea that goes back to Smale [12, 13, 14], Milnor [9], Witten [16] and Floer [7]. Let $f\colon M \to \mathbf{R}$ be a Morse function such that $f(x) = m(x)$ for every critical point x (see Chapter 4 in [9] for the proof of existence of such function). Let $M_p = f^{-1}((-\infty, p + \frac{1}{2}])$. We use this filtration for the spectral sequence we are going to construct. Using notation from the previous section we put

$$A_p := h_*(M_p);\ E_p := h_*(M_p, M_{p-1}).$$

Deforming along the gradient lines of f we get the homotopy equivalence $(M_p, M_{p-1}) \simeq (\bigcup_{x\in Crit_p(f)} W^u(x) \cup M_{p-1}, M_{p-1})$.

The first differential in our spectral sequence is

$$d_1 : h_*(M_p, M_{p-1}) \to h_{*-1}(M_{p-1}) \to h_{*-1}(M_{p-1}, M_{p-2}), \tag{1}$$

namely the connecting homomorphism for the triple (M_p, M_{p-1}, M_{p-2}). By excision, we have the following isomorphism

$$h_*(M_p, M_{p-1}) \cong \bigoplus_{x\in Crit_p(f)} h_*(W^u(x) \cup M_{p-1}, M_{p-1}),$$

so in order to determine d_1 we have to determine its xy-component, where x stands for a critical point of index p, while y stands for a critical point of index $p-1$. For that, let

$$S^u(x) := W^u(x) \cap f^{-1}(p - \frac{1}{2}), \quad S^s(x) := W^s(x) \cap f^{-1}(p - \frac{1}{2}).$$

The group $h_*(W^u(x) \cup M_{p-1}, M_{p-1})$ is generated by the class $[\Sigma S^u(x)]$ (suspension of the unstable sphere). Namely,

$$h_*(W^u(x) \cup M_{p-1}, M_{p-1}) \cong \widetilde{h}_*(W^u(x) \cup M_{p-1}/M_{p-1}) \cong \widetilde{h}_*(\Sigma S^u(x)).$$

Under the connecting homomorphism of the pair $(W^u(x) \cup M_p, M_{p-1})$, this class goes to $[S^u(x)]$. Let $L = f^{-1}(p - \frac{1}{2}) \subset M_{p-1}$ and consider the composition

$$\begin{aligned} h_{*-1}(S^u(x)) &\to h_{*-1}(L) \to h_{*-1}(M_{p-1}) \to h_{*-1}(M_{p-1}, M_{p-2}) \to \\ &\to h_{*-1}(\bigcup_{y\in Crit_{p-1}(f)} W^u(y) \cup M_{p-2}, M_{p-2}). \end{aligned}$$

If we proceed with the projection to the y-component and use the homotopy equivalence

$$(W^u(y) \cup M_{p-2}, M_{p-2}) \simeq (L, L \setminus S^s(y)),$$

provided by the gradient flow, we see that in order to determine our differential we have to find out where does the generator of $\widetilde{h}_{*-1}(S^u(x))$ go under the homomorphism $h_{*-1}(S^u(x)) \to h_{*-1}(L, L \setminus S^s(y))$ induced by the inclusion

$$\imath\colon S^u(x) \to L. \tag{2}$$

Using Poincaré duality, we take a look at the homomorphism $h^{n-*}(L, L \setminus S^s(y)) \to h^{n-*}(S^u(x))$. To get what we need we invoke the Thom isomorphism theorem. Namely, after restriction to some tubular neighborhood of $S^s(y)$ in L, we may assume that L is a vector bundle over $S^s(y)$ and that the generator of $h^{n-*}(L, L \setminus S^s(y))$ is the Thom class τ. Let $D[S^u(x)]$ be the generator of $h^{n-*}(S^u(x))$ and let $\imath^*\tau = \lambda D[S^u(x)]$, where $\lambda \in h_*$. Then

$$\langle \imath^*\tau, [S^u(x)]\rangle = \lambda\langle D[S^u(x)], [S^u(x)]\rangle = \lambda.$$

But τ is the Poincaré dual of $[S^s(y)]$, and hence

$$\langle \imath^*\tau, [S^u(x)]\rangle = [S^s(y)] \cdot [S^u(x)] = [S^s(y) \cap S^u(x)] \in h_*.$$

From (1) and (2) we conclude that the first differential of our spectral sequence is the differential of the complex

$$C_*(f) := \bigoplus_{p+q=*} h_q\langle Crit_p(f)\rangle, \tag{3}$$

where $h_q\langle Crit_p(f)\rangle$ is a free h_q-module over the set $Crit_p(f)$. Therefore, we have proved the following theorem.

THEOREM 1. *Let M be a closed manifold; let f be a self-indexing Morse function on M and h_* a multiplicative homology theory. There exists a spectral sequence converging to $h_*(M)$, with the second term given by*

$$E^2_{p,q} = H^f_p(M; h_q),$$

where by $H^f_p(M; h_q)$ we denote a homology calculated from the following Morse complex:

$$C_*(f) := \bigoplus_{p+q=*} h_q\langle Crit_p(f)\rangle,$$

where the differential is given on generators by

$$d(vx) = \sum_{y \in Crit_{p-1}(f)} v[S^s(y) \cap S^u(x)]y,$$

for $v \in h_*$ and $x \in Crit_p(f)$.

In case this spectral sequence collapses at its second term, we get that $h_n(M) \cong \bigoplus_{p+q=n} H^f_p(M; h_q)$. This happens in the case of ordinary homology.

The reason is simple—all higher differentials become zero since they either originate from the trivial group, or target the trivial group.

Note that (3) may be written, using the Conley index ([5]), as

$$C_*(f) = \bigoplus_{x \in Crit_*(f)} h_*(N_x, L_x),$$

where (N_x, L_x) is the Conley's Index pair of an isolated compact invariant set $\{x\}$ (see [5, 2, 7, 8, 10] for more details concerning the Conley index).

Critical points here are a special case of a more general notion of the Morse decomposition (see [2] for the definition) associated to a flow on a compact manifold. This suggests that the Theorem 1 might be generalized to the Conley index setting although it is not immediately clear whether the differentials will be so explicitly tractable in the most general case.

4. Examples and higher differentials

We illustrate our construction with the following simple example.

Let us define a generalized homology theory as follows:

$$h_*(X) := H_*(\mathcal{S}^1 \times X).$$

Let us consider the case $X = \mathcal{S}^1$. If we choose the height function on $\mathcal{S}^1$ as our Morse function we get only two singular points, one of index 1 and one of index 0. There are only two trajectories. Our spectral sequence in this case reduces to a complex in which all differentials are zero. Therefore, $h_*(\mathcal{S}^1) \cong H_*(\mathcal{S}^1) \otimes \Sigma H_*(\mathcal{S}^1)$. Here $\Sigma H_*(\mathcal{S}^1)$ denotes a copy of $H_*(\mathcal{S}^1)$ shifted by one dimension. We therefore recover ordinary homology of torus $\mathcal{S}^1 \times \mathcal{S}^1$, but by using a simpler complex than in the classical case.

Let us at the end say a few words about higher differentials in this spectral sequence. Naturally, it is very difficult in the general case to deal with them. But the case in which there are no critical points at a certain level, it is possible to say something general about them.

Suppose that we have a manifold and a Morse function on it such that there are no critical points of odd index.

Complex projective spaces $\mathbf{C}P^n$ provide examples of such manifolds. Namely, if we see $\mathbf{C}P^n$ as the quotient space $\mathcal{S}^{2n+1}/\mathcal{S}^1$ we may define the following function:

$$f[z_0, \ldots, z_n] := \sum_{k=1}^{n} k|z_k|^2.$$

It turns out that this is a Morse function which has $n+1$ critical points with indices $0, 2, \ldots, 2n$. We know that in the case of ordinary homology we get a trivial complex and ordinary homology is isomorphic to this complex. In the case of generalized homology, this does not happen. We cannot recover generalized homology immediately. We can only conclude that $d_2 = 0$ and move to d_3. What we get for d_3 in this case is the expression very much like the one we have had for d_2. The only, but very important, difference is that in this case $S^s(y) \cap S^u(x)$ is not a finite number of points but is homotopy equivalent to a compact one-dimensional manifold, namely to a finite number of copies of $\mathcal{S}^1$. Of course, it depends on the generalized homology one uses to proceed further with calculations.

In general, in case there are no critical points of index, say $p+1, \dots, q-1$, several differentials become trivial and one has to deal with the case in which $S^s(y) \cap S^u(x)$ is a compact manifold of dimension higher than two. Its fundamental class in the generalized theory then becomes part of an expression for higher differentials.

REFERENCES

[**1**] J. F. Adams, *Stable Homotopy and Generalised Homology*, Chicago Lectures in Math., 1974.

[**2**] A. Banyaga, D. Hurtubise, *Lectures on Morse Homology*, Kluwer Texts in the Mathematical Sciences, Vol. 29, Kluwer Academic Publishers, 2004.

[**3**] J.-F. Barraud, O. Cornea, *Homotopical Dynamics in Symplectic Topology*, in: *Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology*, P. Biran, O. Cornea and F. Lalonde (Eds.), NATO Science Series, Springer 2006, pp. 109–148.

[**4**] J. M. Boardman, *Conditionally convergent spectral sequences*, Contemp. Math. **239** (1999), 49–84.

[**5**] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference in Mathematics, Vol. 38 (1978), American Mathematical Society, Providence, RI.

[**6**] E. Dyer, *Cohomology Theories*, W. A. Benjamin, 1969.

[**7**] A. Floer, *Witten's complex and infinite dimensional Morse theory*, J. Diff. Geom., **30** (1989), 207–221.

[**8**] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York 1988.

[**9**] J. Milnor, *Lectures on the h-cobordism Theorem*, Princeton Univ. Press, 1965.

[**10**] D. Salamon, *Connected simple systems and the Conley Index of isolated invariant sets*, Trans. Amer. Math. Soc. **291** (1985), 1–41.

[**11**] M. Schwarz, *Morse Homology*, Birkhäuser, Basel, 1993.

[**12**] S. Smale, *The generalized Poincaré conjecture in dimensions greater than 4*, Ann. of Math., **64** (1956), 399–405.

[**13**] S. Smale, *On gradient dynamical systems*, Ann. of Math., **74** (1961), 199–206.

[**14**] S. Smale, *On the structure of manifolds*, Amer. J. of Math., **84** (1962), 387–399.

[**15**] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, 1978.

[**16**] E. Witten, *Supersymmetry and Morse theory*, J. Diff. Geom., **17** (1982), 661–692.

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