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DENSE SETS, NOWHERE DENSE SETS AND AN IDEAL IN GENERALIZED CLOSURE SPACES

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Abstract. In this paper, concepts of various forms of dense sets and nowhere dense sets in generalized closure spaces have been introduced. The interrelationship among the various notions have been studied in detail. Also, the existence of an ideal in generalized closure spaces has been settled.

1. Introduction

Structure of closure spaces is more general than that of topological spaces. Hammer studied closure spaces extensively in [8,9], and a recent study on these spaces can be found in Gnilka [5,6], Stadler [14,15], Harris [10], Habil and Elzenati [7]. Although the applications of general topology is not available so much in digital topology, image analysis and pattern recognition; the theory of generalized closure spaces has been found very important and useful in the study of image analysis [3,13]. In [14,15], Stadler studied separation axioms on generalized closure spaces. The following definition of a generalized closure space can be found in [7] and [15].

Let X be a set. $\wp(X)$ be its power set and $cl\colon \wp(X) \longrightarrow \wp(X)$ be any arbitrary set-valued set function, called a closure function. We call clA, $A \subset X$, the closure of A and we call the pair (X, cl) a generalized closure space.

The closure function in a generalized closure space (X, cl) is called:

(a) grounded if $cl(\emptyset) = \emptyset$,

(b) isotonic if $A \subset B \Rightarrow clA \subset clB$,

(c) expanding if $A \subset clA$ for all $A \subset X$,

(d) sub-additive if $cl(A \cup B) \subset clA \cup clB$,

(e) idempotent if $cl(clA) = clA$,

(f) additive if $\bigcup_{\lambda\in\Lambda} cl(A_\lambda) = cl(\bigcup_{\lambda\in\Lambda}(A_\lambda))$.

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A generalized closure space will be called:

(i) grounded, if its closure function is grounded,

(ii) isotonic, if its closure function is grounded and isotonic,

(iii) expanding, if its closure function is expanding,

(iv) idempotent, if its closure function is idempotent.

A isotonic expanding space (X, cl) is called a *neighbourhood space*. An idempotent neighbourhood space is called a *closure space*. A sub-additive closure space is a topological space.

The interior function $int : \wp(X) \longrightarrow \wp(X)$ is defined by $intA = X - cl(X - A)$. It follows that $clA = X - (int(X - A))$ for all $A \subset X$. A set $A \in \wp(X)$ is called closed in the generalized closure space (X, cl) if $clA = A$ holds. A is called open if $X - A$ is closed or if $A = intA$.

We note that in a topological space(X, τ), a dense subset D of X satisfies the following properties:

$(1D)$ for every nonempty open set V in τ, $V \cap D \neq \emptyset$;

$(2D)$ $clD = X$;

$(3D)$ for any superset B of D, B is dense in τ;

$(4D)$ $int(X - D) = \emptyset$.

Now the question naturally arises whether the concept of a dense set in a generalized closure space can be given which will, under certain restrictions on the generalized closure space, satisfy the above four parallel properties. Again we note that in a topological space, a nowhere dense set A satisfies the following properties:

$(1N)$ for every nonempty open set V, there exists a nonempty open set $W \subset V$ such that $W \cap A = \emptyset$,

$(2N)$ $int\ clA = \emptyset$,

$(3N)$ for every subset B of A, B is nowhere dense,

$(4N)$ $cl\ int(X - A) = X$.

Now the problem is to define nowhere dense sets in a generalized closure space which, under certain restrictions on the generalized closure space, will satisfy the above four parallel properties.

Ideals play an important role in topological spaces. Ideals are used as an indispensable tool in constructing new topologies from old [11], in the study of I-resolvability [2], I-compactification and local I-compactness [12], in the study of I-continuity [1], in the study of Baire spaces and Volterra spaces [4] etc. Various collections of ideals have been investigated and used in the study of the topics just mentioned.

One of the most important and useful example of an ideal in a topological space (X, τ) is the collection of nowhere dense sets in (X, τ). Now the following question arises:

Whether there exists an ideal in non-topological generalized closure spaces?

A partial answer to this question has been given in this paper. Henceforth, a generalized closure space will be written as a gc-space.

2. Dense sets in gc-spaces

It is natural to define a dense set in a gc-space as follows:

Definition 2.1. A nonempty subset D of X is called *gc-dense* in a gc-space (X, cl) if $V \cap D \neq \emptyset$ for every nonempty open set V in (X, cl).

Then it follows that if D is gc-dense then for any superset B of D, B is gc-dense in (X, cl). But property $(2D)$ may not hold.

Example 2.1. Let $X = \{a, b, c\}$. Define $cl\colon \wp(X) \longrightarrow \wp(X)$ by $cl\emptyset = \emptyset$, $clX = X$, $cl\{a\} = X$, $cl\{b\} = \{b\}$, $cl\{c\} = \{c\}$, $cl\{a, b\} = \{a, b\}$, $cl\{a, c\} = \{a, b\}$, $cl\{b, c\} = \{b, c\}$. Note that nonempty open sets are X, $\{a, c\}$, $\{a, b\}$, $\{c\}$, $\{a\}$. Now if $A = \{a, c\}$ then $V \cap A \neq \emptyset$ for every nonempty open set V in (X, cl), but $clA = \{a, b\} \neq X$.

The above example allows us to define a new concept of dense set in a gc-space.

Definition 2.2. A nonempty subset D of X will be called *sgc-dense* in a gc-space (X, cl) if $clD = X$.

Example 2.1 shows that in a gc-space (X, cl), an sgc-dense set may not be a gc-dense set. Note that if $A = \{a\}$ then $clA = X$ but $V \cap A = \emptyset$ for the nonempty open set $V = \{c\}$.

In an isotonic space (X, cl), if A is an sgc-dense set then for any superset B of A, B is also an sgc-dense set. But if we consider example 2.1 then (X, cl) is not isotonic and if $A = \{a\}$, $B = \{a, b\}$, we see $clA = X$, but $clB \neq X$.

Theorem 2.1. *If (X, cl) is isotonic then an sgc-dense set is gc-dense.*

Proof is easy.

The converse of the above theorem may not stand.

Example 2.2. Let $X = \{a, b, c\}$, $cl\emptyset = \emptyset$, $cl\{a\} = \{a, b\}$, $cl\{b\} = \{b, c\}$, $cl\{c\} = \{b, c\}$, $cl\{a, b\} = X$, $cl\{a, c\} = X$, $cl\{b, c\} = \{b, c\}$, $clX = X$. Then (X, cl) is isotonic and $\{a\}$ is gc-dense but not sgc-dense.

Theorem 2.2. *Let (X, cl) be a closure space. Then the following statements are equivalent for any subset A of X.*

(i) A is gc-dense

(ii) A is sgc-dense.

Proof. Clearly $(ii) \Rightarrow (i)$. For $(i) \Rightarrow (ii)$, let A be gc-dense but $clA \neq X$. Then $X - clA \neq \emptyset \Rightarrow int(X - A) \neq \emptyset$. Let $V = int(X - A) = X - clA$. Then V is open and nonempty, but $V \cap clA = \emptyset$. Since $A \subset clA$, then $V \cap A = \emptyset$, a contradiction to the fact that A is gc-dense. Thus $(i) \Rightarrow (ii)$. ∎

Idempotent property and expanding property are not redundant in the above theorem. Consider Example 2.2. Let $A = \{a\}$. Then $cl(clA) = cl\{a, b\} = X \neq clA$. So (X, cl) is not idempotent but it is isotonic. We have already noted that A is gc-dense but not sgc-dense.

Example 2.3. Let $X = \{a, b, c\}$. $cl\emptyset = \emptyset$, $cl\{a\} = \{a, b\}$, $cl\{b\} = \{c\}$, $cl\{c\} = \{b, c\}$, $cl\{a, b\} = X$, $cl\{a, c\} = X$, $cl\{b, c\} = \{b, c\}$, $clX = X$. Then (X, cl) is isotonic but not expanding since $\{b\}$ is not a subset of $cl\{b\}$. Note that $\{a\}$ is gc-dense which is not sgc-dense.

In case of a topological space (X, τ) it is true that for any nonempty open set V and for any nonempty subset A of X, $V \cap clA \neq \emptyset \Leftrightarrow V \cap A \neq \emptyset$. This is not true in case of gc-spaces. See example 2.1. Here $\{c\}$ is open $\{c\} \cap cl\{a\} = \{c\}$ but $\{c\} \cap \{a\} = \emptyset$.

So it is quite natural to consider another type of dense set in a gc-space.

Definition 2.3. In a gc-space (X, cl), a subset A of X is said to be a *wgc-dense set* if for every nonempty open set V in (X, cl), $V \cap clA \neq \emptyset$.

In Example 2.1, since $cl\{a\} = X$, for every non-empty open set V, $V \cap cl\{a\} \neq \emptyset$, but for the open set $\{c\}$, $\{c\} \cap \{a\} = \emptyset$. Thus $\{a\}$ is a wgc-dense set but not gc-dense. In the same example, note that $A = \{a, c\}$ is gc-dense but $clA = \{a, b\}$. Hence A is not wgc-dense because $\{c\}$ is open and $\{c\} \cap clA = \emptyset$.

In a gc-space, an sgc-dense set is necessarily wgc-dense. The converse may not be true. Consider Example 2.2. Here the non-empty open sets are $\{a\}$ and X. Consider $A = \{a\}$. Then $V \cap clA \neq \emptyset$ for every non-empty open set V, because $clA = \{a, b\}$. But $clA \neq X$.

Theorem 2.3. *Let (X, cl) be an idempotent gc-space. The following statements are equivalent for any non-empty subset A of X.*

(*i*) *A is wgc-dense,*

(*ii*) *A is sgc-dense.*

Proof is easy.

We note that if a gc-space is expanding then the class of all gc-dense sets is contained in the class of all wgc-dense sets. These classes may not be equal in an expanding gc-space. Consider the following example.

Example 2.4. Let $X = \{a, b, c\}$, $cl\emptyset = \emptyset$, $cl\{a\} = \{a, b\}$, $cl\{b\} = \{b, c\}$, $cl\{c\} = \{b, c\}$, $cl\{a, b\} = X$, $cl\{b, c\} = X$, $cl\{a, c\} = \{a, c\}$, $clX = X$. Then (X, cl) is expanding. The non-empty open sets are $\{b\}$ and X. $\{b\} \cap cl\{a\} \neq \emptyset$ but $\{b\} \cap \{a\} = \emptyset$. Hence $\{a\}$ is wgc-dense but not gc-dense.

The following conclusion can be drawn:

Theorem 2.4. *In a closure space (X, cl) the following statements are equivalent for a subset A of X.*

(*i*) *A is gc-dense,*

(ii) *A is sgc-dense,*

(iii) *A is wgc-dense.*

Proof follows from Theorems 2.2 and 2.3.

3. Nowhere dense sets in gc-spaces

It is natural to define a nowhere dense set in a gc-space by the following

DEFINITION 3.1. A subset B of X in a gc-space (X, cl) is called *gc-nowhere dense* (in short, *gc-nwdense*) if for every non-empty open set V in (X, cl) there exists a non-empty open set W with $W \subset V$ such that $W \cap B = \emptyset$.

It follows that every subset B of a gc-nwdense set A in a gc-space (X, cl) is gc-nwdense. But property $(2N)$ is not, in general, satisfied. Consider Example 2.1. Note that $\{b\}$ is gc-nwdense in (X, cl) but $int\ cl\{b\} = \{c\} \neq \emptyset$.

The above example suggests the following

DEFINITION 3.2. A subset B of X in a gc-space (X, cl) is said to be *sgc-nwdense* if $int\ clB = \emptyset$.

Example 2.1 shows that a gc-nwdense set in a gc-space may not be sgc-nwdense. The same example shows that an sgc-nwdense set in a gc-space may not be gc-nwdense. Note that $int\ cl\{b, c\} = int\{b, c\} = X - cl(X - \{b, c\}) = X - cl\{a\} = X - X = \emptyset$. So $\{b, c\}$ is sgc-nwdense. But $\{b, c\}$ is not gc-nwdense, for $\{c\}$ is a non-empty open set which fails to satisfy the condition for $\{b, c\}$ to be a gc-nwdense set in (X, cl).

THEOREM 3.1. *If (X, cl) is an isotonic gc-space then any subset of an sgc-nwdense set is sgc-nwdense.*

Proof. Let A be sgc-nwdense in (X, cl) and $B \subset A$. Then $int\ clA = \emptyset$. Since (X, cl) is isotonic, $B \subset A \Rightarrow clB \subset clA \Rightarrow int\ clB \subset int\ clA = \emptyset \Rightarrow int\ clB = \emptyset$. ∎

In an isotonic space there may exist an sgc-nwdense set which is not gc-nwdense. Consider the following example.

EXAMPLE 3.1. Let $X = \{a, b, c\}$, $cl\emptyset = \emptyset$, $cl\{a\} = \{a\}$, $cl\{b\} = \{a, b\}$, $cl\{c\} = X$, $cl\{a, b\} = X$, $cl\{a, c\} = X$, $cl\{b, c\} = X$, $clX = X$. Then (X, cl) is isotonic and non-empty open sets are $\{b, c\}$ and X. Note that $int\ cl\{b\} = \emptyset$. So $\{b\}$ is sgc-nwdense in (X, cl). But $\{b\}$ is not gc-nwdense in (X, cl).

THEOREM 3.2. *In an isotonic space (X, cl), if A is an sgc-nwdense set then for every nonempty open set V, V is not a subset of clA.*

Proof. Let A be sgc-nwdense in the isotonic space (X, cl). Then $int\ cl\ A = \emptyset$. So $X - int\ clA = X$ or $cl(X - clA) = X$ or $cl\ int(X - A) = X \Rightarrow int(X - A)$ is sgc-dense in (X, cl). Since (X, cl) is isotonic, $int(X - A)$ is gc-dense by Theorem 2.1.

Thus $X - clA$ is gc-dense. Hence for every nonempty open set V, $V \cap (X - clA) \neq \emptyset$ implies that V is not a subset of clA. ■

We now arrive at another concept of a nowhere dense set in a gc-space.

Definition 3.3. A subset B of X in a gc-space (X, cl) is said to be *wgc-nwdense* in (X, cl) if for every nonempty open set V in (X, cl), V is not a subset of clB.

It follows that an sgc-nwdense set in an isotonic space is a wgc-nwdense set. The converse may not hold. Consider example 2.2. Here non-empty open sets are $\{a\}$ and X. Hence $\{c\}$ is wgc-nwdense. But $int\ cl\{c\} \neq \emptyset$. So $\{c\}$ is not sgc-nwdense.

Theorem 3.3. *In an idempotent isotonic gc-space* (X, cl) *the following statements are equivalent for a subset A of X :*

(i) *A is sgc-nwdense,*

(ii) *A is wgc-nwdense.*

Proof is easy.

In an idempotent isotonic space (X, cl) an sgc-nwdense set may not be gc-nwdense. Consider the following example.

Example 3.2. Let $X = \{a, b, c\}$, $cl\emptyset = \emptyset$, $cl\{a\} = \{a\}$, $cl\{b\} = \{b\}$, $cl\{c\} = \{a, c\}$, $cl\{a, b\} = X$, $cl\{a, c\} = \{a, c\}$, $cl\{b, c\} = X$, $clX = X$. Here (X, cl) is a closure space. Nonempty open sets are $\{b, c\}$, $\{a, c\}$, $\{b\}$, X. Note that $int\ cl\{a\} = \emptyset$. So $\{a\}$ is sgc-nwdense. But the open set $\{a, c\}$ has no nonempty open subset V for which $V \cap \{a\} = \emptyset$. So $\{a\}$ is not gc-nwdense in (X, cl).

If the condition of sub-additivity is added to a closure space, i.e., if we consider a topological space then the class of sgc-nwdense sets and the class of gc-nwdense sets coincide.

We conclude this section with the following theorem.

Theorem 3.4. *In a topological space* (X, τ) *the following statements are equivalent for a subset A of X:*

(i) *A is gc-nwdense in* (X, τ)*,*

(ii) *A is sgc-nwdense in* (X, τ)*,*

(iii) *A is wgc-nwdense in* (X, τ)*.*

4. Existence of an ideal in gc-spaces

The following question arises: Which of the above three classes of nowhere dense sets do form an ideal in a gc-space? A partial answer to this question will be given in this section.

First we have the following observation. There exists a closure space (X, cl) in which there exists an open set V and a subset A of X such that V is not a subset of clA but $V - clA$ is not a nonempty open set.

Example 4.1. Let X be the space from Example 3.2. Now let $A = \{b\}, V = \{b, c\}$. Then V is not a subset of clA. Also $V - clA = \{c\}$ which is not open.

Definition 4.1. A gc-space (X, cl) is said to be an *ogc-space* if the following condition holds in (X, cl):

For any nonempty open set V and any subset A of X, if V is not a subset of clA then $V - clA$ is a nonempty open set.

We shall now cite an example of a sub-additive ogc-space which is not a topological space.

Example 4.2. Let $X = \{a, b, c\}$, $cl\emptyset = \emptyset$, $cl\{a\} = X$, $cl\{b\} = \{b\}$, $cl\{c\} = \{c\}$, $cl\{a, b\} = \{a, b\}$, $cl\{a, c\} = \{a, c\}$, $cl\{b, c\} = \{b, c\}$, $clX = X$. (X, cl) is sub-additive and ogc but not a topological space.

Theorem 4.1. *In a sub-additive, isotonic ogc-space (X, cl) the collection $W(N)$ of all wgc-nodense sets form an ideal in (X, cl).*

Proof. From Definition 3.3 it clearly follows that $A \in W(N)$ and $B \subset A$ imply $B \in W(N)$. Again, if A, $B \in W(N)$ then we claim that $A \cup B \in W(N)$. For, if there exists a nonempty open set $V \subset cl(A \cup B)$ then $V \subset clA \cup clB$ (since (X, cl) is sub-additive) and since $A \in W(N)$, therefore, V is not contained in ClA. Then $V - clA$ is a nonempty open set in (X, cl) since (X, cl) is ogc. So $V - clA \subset clB$, a contradiction to the fact that $B \in W(N)$. Thus for every nonempty open set V in (X, cl), V is not a subset of $cl(A \cup B)$. Hence $A \cup B \in W(N)$. Thus $W(N)$ forms an ideal in (X, cl). ■

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