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NOTES ON DOUBLY WARPED AND DOUBLY TWISTED PRODUCT CR-SUBMANIFOLDS OF KAEHLER MANIFOLDS

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Abstract. Recently, B. Y. Chen studied warped product CR-submanifolds [7] and twisted product CR-submanifolds [6] in Kaehler manifolds. In this paper, we have checked the existence of other product CR-submanifolds such as doubly warped and doubly twisted products in Kaehler manifolds.

1. Introduction

Warped product manifolds were introduced by R. L. Bishop and B. O'Neill in [2] to construct new examples of negatively curved manifolds. Later, this notion has been generalized in several ways. Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds of dimensions m and n, respectively and let $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ be the canonical projections. Also let $b : B \times F \to (0, \infty)$, $f : B \times F \to (0, \infty)$ be smooth functions. Then the doubly twisted product ([10], [15]) of (B, g_B) and (F, g_F) with twisting functions b and f is defined to be the product manifold $M = B \times F$ with metric tensor $g = f^2\, g_B \oplus b^2\, g_F$. We denote this kind manifolds by ${}_fB \times_b F$. Denote by $F(B)$ the algebra of smooth functions on B and by $\Gamma(E)$ the $F(B)$ module of smooth sections of a vector bundle E (same notation for any other bundle) over B. If $X \in \Gamma(TB)$ and $V \in \Gamma(TF)$, then from Proposition 1 of [10], we have

$$\nabla_X V = V(lnf)X + X(lnb)V, \tag{1.1}$$

where ∇ denotes the Levi-Civita connection of the doubly twisted product ${}_fB \times_b F$ of (B, g_B) and (F, g_F). In particular, if $f = 1$, then $B \times_b F$ is called the twisted product of (B, g_B) and (F, g_F) with twisting function b. We note that the notion of twisted products was introduced in [5]. If $M = B \times_b F$ is a twisted product manifold, then (1.1) becomes

$$\nabla_X V = X(lnb)V. \tag{1.2}$$

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Moreover, if b only depends on the points of B, then $B\times_b F$ is called warped product of (B,g_B) and (F,g_F) with warping function b. In this case, for $X\in\Gamma(TB)$ and $V\in\Gamma(TF)$, from Lemma 7.3 of [2], we have

$$\nabla_X V = X(lnb)V, \tag{1.3}$$

where b depends on the points of B and $X\in\Gamma(TB)$, $V\in\Gamma(TF)$.

As a generalization of the warped product of two semi-Riemannian manifolds, ${}_fB\times_b F$ is called the doubly warped product of semi-Riemannian manifolds (B,g_B) and (F,g_F) with warping functions b and f if only depend on the points of B and F, respectively.

In [7], B. Y. Chen considered warped product CR-submanifolds of Kaehler manifolds and showed that there exist no warped product CR-submanifolds in the form $M_\perp\times_f M_T$, where $M_\perp$ is a totally real submanifold and M_T is a holomorphic submanifold of a Kaehler manifold $\bar{M}$. Then he introduced CR-warped products which are warped product CR-submanifolds in the form $M_T\times_f M_\perp$ such that M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $\bar{M}$. CR-warped products (or CR-products) have been also studied in [3], [4], [8], [9], [11], [12], [13], [14].

B. Y. Chen [6] also introduced twisted product CR-submanifolds in Kaehler manifolds and showed that a twisted product CR-submanifold in the form $M_\perp\times_\lambda M_T$ is a CR-product. Then he considered twisted product CR-submanifolds in the form $M_T\times_\lambda M_\perp$ and established a general sharp inequality for twisted product CR-submanifolds in Kaehler manifolds.

In this paper, we investigate the existence of other product (doubly warped and doubly twisted product) CR-submanifolds in Kaehler manifolds. In fact, we show that there do not exist doubly warped and doubly twisted product CR-submanifolds in Kaehler manifolds.

2. Preliminaries

Let $(\bar{M},g)$ be a Kaehler manifold. This means that $\bar{M}$ admits a tensor field J of type (1,1) on $\bar{M}$ such that, $\forall X,Y\in\Gamma(T\bar{M})$, we have

$$J^2=-I,\quad g(X,Y)=g(JX,JY),\quad (\bar{\nabla}_X J)Y=0 \tag{2.1}$$

where g is Riemannian metric and $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$.

Let $\bar{M}$ be a Kaehler manifold with complex structure J and M is a Riemannian manifold isometrically immersed in $\bar{M}$. Then M is called holomorphic (complex) if $J(T_pM)\subset T_pM$, for every $p\in M$, where T_pM denotes the tangent space to M at the point p. M is called totally real if $J(T_pM)\subset T_pM^\perp$ for every $p\in M$, where $T_pM^\perp$ denotes the normal space to M at the point p. In 1978, A. Bejancu introduced [1] a new class of submanifolds of Kaehler manifolds as follows. A submanifold M is called a CR-submanifold if there exists on M a differentiable distribution $D:p\rightarrow D_p\subset T_pM$ such that D is invariant with respect to J and its orthogonal complement $D^\perp$ is totally real distribution, i.e, $J(D_p^\perp) \subseteq T_pM^\perp$. Obviously, holomorphic and totally real submanifolds are CR-submanifolds having $D_p = T_pM$ and $D_p = 0$, respectively. A CR-submanifold is called proper if it is neither holomorphic nor totally real.

Let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let TM be the Lie algebra of vector fields in M and $TM^\perp$ the set of all vector fields normal to M. Denote by ∇ the Levi-Civita connection of M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.2}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{2.3}$$

for any $X, Y \in TM$ and any $N \in TM^\perp$, where $\nabla^\perp$ is the connection in the normal bundle $TM^\perp$, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form and the shape operator A are related by

$$g(A_N X, Y) = g(h(X, Y), N). \tag{2.4}$$

For Kaehler manifolds and their submanifolds, see [17].

3. Doubly warped and doubly twisted product CR-submanifolds

In this section, we consider CR-submanifolds which are doubly warped or doubly twisted products in the form ${}_fM_T \times_b M_\perp$, where M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $\bar{M}$.

Theorem 3.1. *Let $\bar{M}$ be a Kaehler manifold. Then there do not exist doubly warped product CR-submanifolds which are not (singly) warped product CR-submanifolds in the form ${}_fM_T \times_b M_\perp$ such that M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $\bar{M}$.*

Proof. Let us suppose that M be a doubly warped product CR-submanifold of a Kaehler manifold $\bar{M}$. Then from (1.1) we have

$$g(\nabla_{JX} V, X) = V(lnf) g(JX, X) = 0$$

for $X \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$. Using (2.2) we get $g(\bar{\nabla}_{JX} V, X) = 0$. Since D and $D^\perp$ are orthogonal, we obtain $g(\bar{\nabla}_{JX} X, V) = 0$. Then from (2.1) we have $g(\bar{\nabla}_{JX} JX, JV) = 0$. Thus from (2.2), we have

$$g(h(JX, JX), JV) = 0 \tag{3.1}$$

for $X \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$. Now, substituting X by $Y + Z$, $Z, Y \in \Gamma(D)$, in (3.1) and taking into account that h is symmetric, we get

$$g(h(JY, JZ), JV) = 0 \tag{3.2}$$

for $Y, Z \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. On the other hand, from (2.2) we have $g(h(JY, JZ), JV) = g(\bar{\nabla}_{JY} JZ, JV)$. Using (2.1) we obtain

$$g(h(JY, JZ), JV) = -g(Z, \nabla_{JY} V)$$

for $Y, Z \in \Gamma(D)$ and $V \in \Gamma(D^{\perp}$. Then from (1.1) we get

$$g(h(JY, JZ), JV) = -V(lnf)g(JY, Z) \tag{3.3}$$

for $Y, Z \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. Thus using (3.2) in (3.3), we obtain

$$V(lnf)g(JY, Z) = 0.$$

Hence, for $Z = JY$, we get

$$V(lnf)g(Y, Y) = 0.$$

Then, we derive $V(lnf) = 0$ due to $g(Y, Y) \neq 0$. $V(lnf) = 0$ implies that f is constant. Thus M is a warped product CR-submanifold in the form $M_T \times_b M_{\perp}$ which is called CR-warped product (see [7]). Thus, proof is complete. ∎

Theorem 3.1 tells us that there exist no doubly warped product CR-submanifolds in Kaehler manifolds other than warped product CR-submanifolds. In the rest of this section, we investigate the existence of doubly twisted product CR-submanifolds in Kaehler manifolds.

THEOREM 3.2. *Let $\bar{M}$ be a Kaehler manifold. Then there do not exist doubly twisted product CR-submanifolds of $\bar{M}$ which are not (singly) twisted product CR-submanifolds in the form ${}_f M_T \times_b M_{\perp}$ such that M_T is a holomorphic submanifold and $M_{\perp}$ is a totally real submanifold of $\bar{M}$.*

Proof. From (1.1), we have $g(\nabla_X V, Y) = V(lnf)g(X, Y)$ for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. Since D and $D^{\perp}$ are orthogonal, we get $-g(\nabla_X Y, V) = V(lnf)g(X, Y)$. Using (2.2) and (2.1), we obtain

$$-g(\bar{\nabla}_X JY, JV) = V(lnf)g(X, Y).$$

Thus, from (2.2), we derive

$$-g(h(X, JY), JV) = V(lnf)g(X, Y) \tag{3.4}$$

for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. On the other hand, from (2.2), we have

$$g(h(X, JY), JV) = g(\bar{\nabla}_{JY} X, JV)$$

for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. Using(2.1), we get

$$g(h(X, JY), JV) = -g(\bar{\nabla}_{JY} JX, V).$$

Hence, (2.2) implies that

$$g(h(X, JY), JV) = g(JX, \nabla_{JY} V).$$

Thus from (1.1), we obtain

$$g(h(X, JY), JV) = V(lnf)g(JX, JY).$$

Using (2.1) we arrive at

$$g(h(X, JY), JV) = V(lnf)g(X, Y) \tag{3.5}$$

for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. Then from (3.4) and (3.5) we obtain

$$V(lnf)g(X, Y) = 0.$$

Since D is Riemannian, we get $V(lnf) = 0$. This implies that f only depends the points of M_T. Thus we can write

$$g = \tilde{g}_{M_T} \oplus b^2 g_{M_\perp}, \quad \text{where} \quad \tilde{g}_{M_T} = f^2\, g_{M_T}.$$

Thus it follows that M is a twisted product CR-submanifold in the form $M = M_T \times_b M_\perp$, (see [6] for twisted product CR-submanifolds). Hence, we conclude that there are no doubly twisted product CR-submanifolds in Kaehler manifolds, other than twisted product CR-submanifolds. ∎

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