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GLUING AND PIUNIKHIN-SALAMON-SCHWARZ ISOMORPHISM FOR LAGRANGIAN FLOER HOMOLOGY

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Abstract. We prove Floer gluing theorem in the case of objects of mixed type, that incorporate both Morse gradient trajectories and holomorphic discs with Lagrangian boundary conditions.

1. Introduction

Let M be a smooth compact manifold of dimension n and $f\colon M \to \mathbf{R}$ a Morse function. Let $P = T^*M$ be a cotangent bundle over M, $L_0 = O_M$ a zero section, $H\colon T^*M \to \mathbf{R}$ compactly supported Hamiltonian and $L_1 = \phi_1^H(L_0)$ a Hamiltonian deformation of O_M. Denote by $CM_*(f)$ Morse chain groups generated by the critical points of f and by $CF_*(H)$ Floer chain groups generated by the set $L_0 \cap L_1$ (both with $\mathbf{Z}_2$-coefficients). Let $HM_*(f)$ and $HF_*(H)$ be the corresponding Morse and Floer homology groups (the latter are well defined in this situation). The isomorphism between $HM_*(f)$ and $HF_*(H)$ was constructed in [9], following Piunikhin, Salamon and Schwarz's construction for the case of periodical orbits [16]. The purpose of [9] was to prove that isomorphisms in Floer homology for Lagrangian intersections naturally intertwine with analogous isomorphisms in Morse homology. The isomorphism constructed there was based on counting the objects of mixed type. More precisely, we are interested in the following three moduli spaces. For p and q two critical points of Morse function f, denote by $\mathcal{M}(p,q,f)$ the set of negative gradient flows of f, i.e. solutions of equation:

$$\begin{cases} \frac{d\gamma}{ds} + \nabla f(\gamma) = 0 \\ \gamma(-\infty) = p,\ \gamma(+\infty) = q. \end{cases} \tag{1}$$

For two Hamiltonian paths x and y that begin and end at O_M, let $\mathcal{M}(x,y,H)$ be

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the set of solutions of:

$$\begin{cases} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0 \\ u(s,i) \in L_0,\ i \in \{0,1\} \\ u(-\infty,t) = x(t), \\ u(+\infty,t) = y(t). \end{cases} \tag{2}$$

The equation (2) is the negative gradient flow of the action functional

$$\mathcal{A}_H\colon \Omega_0 \to \mathbf{R}, \quad \mathcal{A}_H(\alpha) := \int_0^1 \alpha^*\theta - H_t(\alpha(t))\,dt$$

(where θ is the Liouville $1-$ form on T^*M) defined on

$$\Omega_0 \colon = \{\alpha\colon [0,1] \to T^*M \mid \alpha(0), \alpha(1) \in O_M\}.$$

Denote by $\widehat{\mathcal{M}}(p,q,f)$ and $\widehat{\mathcal{M}}(x,y,H)$ these sets modulo $\mathbf{R}-$ actions $\gamma(\cdot) \mapsto \gamma(\cdot+\tau)$, and $u(\cdot,\cdot) \mapsto u(\cdot+\tau,\cdot)$.

Now consider the space of pairs (γ, u)

$$\gamma\colon (-\infty,0] \to M, \quad u\colon [0,+\infty) \times [0,1] \to T^*M$$

that satisfy

$$\begin{cases} \frac{d\gamma}{ds} = -\nabla f(\gamma(s)), \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)) = 0 \\ u(0,t), u(s,0), u(s,1) \in O_M, \\ \gamma(-\infty) = p, u(+\infty,t) = x(t), \\ \gamma(0) = u(0,\frac{1}{2}). \end{cases} \tag{3}$$

Here $\rho_R\colon [0,+\infty) \to \mathbf{R}$ is a smooth function such that, for a fixed R,

$$\rho_R(s) = \begin{cases} 1, & s \ge R+1 \\ 0, & s \le R. \end{cases}$$

Denote by $\mathcal{M}(p,f;x,H)$ the set of solutions of (3) (see Fig. 1).

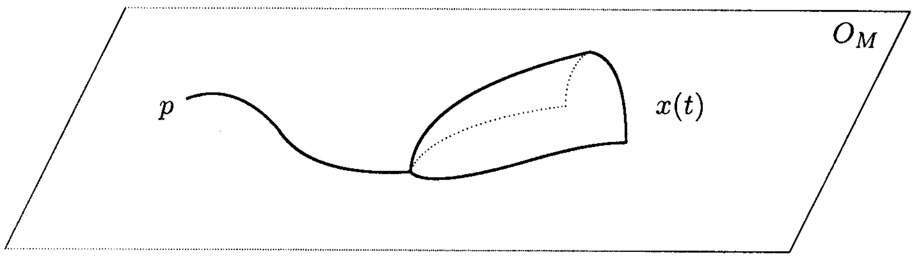

Fig. 1. Mixed object from $\mathcal{M}(p,f;x,H)$

If $m_f(p)$ is the Morse index of critical point p and $\mu_H(x)$ the Maslov index of Hamiltonian path x (see[2, 17, 18]) for definition of Maslov index, [14] for its application to grading of Floer homology groups and [12] for generalizations), then

$\dim \mathcal{M}(p,q,f) = m_f(p) - m_f(q)$ and $\dim \mathcal{M}(x,y,H) = \mu_H(x) - \mu_H(y)$. In [9] we derived the following Proposition by relying upon the convergence results given in [8] and gluing results that are the subject of this paper.

PROPOSITION 1. [9] *For generic f and H, $\mathcal{M}(p,f;x,H)$ is a smooth manifold of dimension $m_f(p) - (\mu_H(x) + \frac{n}{2})$. When $m_f(p) = (\mu_H(x) + \frac{n}{2})$, it is compact, hence a finite set. When $m_f(p) = (\mu_H(x) + \frac{n}{2}) + 1$ then we have the following identification for the boundary of one-dimensional manifold $\mathcal{M}(p,f;x,H)$:*

$$\begin{aligned}\partial\mathcal{M}(p,f;x,H) = &\bigcup_{m_f(q)=m_f(p)-1} \widehat{\mathcal{M}}(p,q,f) \times \mathcal{M}(q,f;x,H) \cup \\ &\cup \bigcup_{\mu_H(y)=\mu_H(x)+1} \mathcal{M}(p,f;y,H) \times \widehat{\mathcal{M}}(y,x,H).\end{aligned} \tag{4}$$

A sketch of the proof of Proposition 1 is the following: let $W^u(p,f)$ be the unstable manifold of the critical point p of a Morse function f and let $W^s(x,H)$ be the set of solutions of

$$\begin{cases} u\colon [0,+\infty) \times [0,1] \to T^*M \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)) = 0 \\ u(0,t), u(s,0), u(s,1) \in O_M, \\ u(+\infty,t) = x(t), \end{cases} \tag{5}$$

Then $\dim W^u(p,f) = m_f(p)$ (see [13]) and $\dim W^s(x,H) = -\mu_H(x) + \frac{n}{2}$ (see Appendix in [15]). For generic f, H the evaluation map

$$\mathrm{ev}\colon W^u(p,f) \times W^s(x,H) \to M \times M, \quad (\gamma, u) \mapsto \Big(\gamma(0), u(0, \frac{1}{2})\Big)$$

is transversal to the diagonal Δ and therefore $\mathcal{M}(p,f;x,H) = \mathrm{ev}^{-1}(\Delta)$ is a smooth manifold of codimension n in $W^u(p,f) \times W^s(x,H)$, so we obtain the dimension formula.

Let $CM_k(f)$ be the set of all critical points of Morse index k and $CF_k(H)$ the set of all Hamiltonian paths with ends in zero section of Maslov index $\mu_H(x) + \frac{n}{2}$. For $m_f(p) = \mu_H(x) + \frac{n}{2}$ we denote the cardinal number $\pmod 2$ of $\mathcal{M}(p,f;x,H)$ by $n(p,f;x,H)$ and define homomorphism

$$\Phi\colon CM_k(f) \to CF_k(H) \quad \text{by} \quad p \mapsto \sum_{\mu_H(x)=m_f(p)-\frac{n}{2}} n(p,f;x,H)x \tag{6}$$

on the generators. It follows from Proposition 1 that the homomorphism Φ is well defined and that it is well defined also on $HM_*(f)$ (see [16, 9]). By using the analysis of the boundary of some auxiliary one-dimensional mixed moduli spaces (see [9]), one can prove that Φ is an isomorphism and that the diagram

$$\begin{array}{ccc} HF_*(H^\alpha) & \xrightarrow{S^{\alpha\beta}} & HF_*(H^\beta) \\ {\scriptstyle\Phi^\alpha}\uparrow & & \uparrow{\scriptstyle\Phi^\beta} \\ HM_*(f^\alpha) & \xrightarrow[T^{\alpha\beta}]{} & HM_*(f^\beta) \end{array}$$

commutes (see [9]). Here $S^{\alpha\beta}$ and $T^{\alpha\beta}$ are the natural isomorphisms in, respectively, Floer and Morse theory.

Piunikhin-Salamon-Schwarz morphisms in more general cases were investigated by Albers in [1] where it was shown that it does not have to be an isomorphism in general, Barraud and Cornea in [3], Leclercq in [11], Lalonde in [10] and others.

The crucial analytical tools used in the proof of Proposition 1 (and the similar characterizations of the boundary of moduli spaces) are Gromov compactness and gluing techniques. From Gromov compactness it follows that the sequence of mixed objects from $\mathcal{M}(p,f;x,H)$ (if it does not converge to the object from $\mathcal{M}(p,f;x,H)$) must converge to a broken object. It proves one inclusion in (4). Gluing gives the opposite: it assigns to a broken object the sequence of non-broken mixed objects that converges (in the Gromov weak sense) toward it.

In this paper we carry out the proof of gluing theorem for the case of the mixed type objects. The natural idea that occurs is to reduce the gluing construction to the Morse (or Floer) case, such that the holomorphic strip (or gradient trajectory) part in mixed object stays fixed. Indeed, this is the idea that we use in the construction of the pre-glued objects. But then we confront the problem of the attaching point: the exact solutions of combined (ordinary and partial differential) equation that converge to the previously given, broken one, do not have the same fixed attaching point, but only the ends (Morse critical point at one end and Hamiltonian path, at the other). Thus we choose the point of view that encircles Morse and Floer trajectories, so first establish the analytical setting and construct Banach manifolds of mixed-type mappings. The tangent spaces to these manifolds will be Sobolev $W^{1,r}$ and L^r spaces, for $r>2$. Since we use some Hilbert space techniques, it would be more convenient to take $W^{1,2}$ and L^2, as it can be done in Morse case (see [19]). But we deal with two-dimensional domains (such as for u in (3), so these maps do not have to be continuous, which is the case for $r>2$, due to the Sobolev Embedding Theorem.

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2. Notation and results

We state the main result in the following

THEOREM 2. a) *Let K be a compact subset of $\mathcal{M}(p,f;x,H)\times\widehat{\mathcal{M}}(x,y,H)$. Then there exists a lower parameter bound $\rho_0=\rho_0(K)$ and a smooth embedding*

$$\begin{aligned}\sharp\colon K\times[\rho_0,+\infty)&\hookrightarrow\mathcal{M}(p,f;y,H)\\ ((\gamma,u),v,\rho)&\mapsto(\gamma,u)\sharp_\rho v.\end{aligned}$$

We call the map $\sharp$ gluing *of a mixed-type object (γ,u) and holomorphic strip v. For an arbitrary sequence of gluing parameters $\rho_n\to\infty$, we have the weak convergence*

$$(\gamma,u)\sharp_{\rho_n}v\xrightarrow{C^\infty_{loc}}((\gamma,u),v).$$

b) *Let K be a compact subset of $\widehat{\mathcal{M}}(p,q,f)\times\mathcal{M}(q,f;x,H)$. Then there exists a lower parameter bound $\rho_0=\rho_0(K)$ and a smooth embedding*

$$\begin{aligned}\sharp\colon K\times[\rho_0,+\infty)&\hookrightarrow\mathcal{M}(p,f;x,H)\\ (\alpha,(\gamma,u),\rho)&\mapsto\alpha\sharp_\rho(\gamma,u).\end{aligned}$$

The map $\sharp$ is a gluing *of a gradient trajectory α and mixed-type object (γ,u). For an arbitrary sequence of gluing parameters $\rho_n\to\infty$, we have the weak convergence*

$$\alpha\sharp_{\rho_n}(\gamma,u)\xrightarrow{C^\infty_{loc}}(\alpha,(\gamma,u)).$$

Here we say that the sequence $(\gamma_n,u_n)\in\mathcal{M}(p,f;y,H)$ weakly converges to the broken trajectory $((\gamma,u),v)\in\mathcal{M}(p,f;x,H)\times\widehat{\mathcal{M}}(x,y;H)$ if $(\gamma_n,u_n)\xrightarrow{C^\infty_{loc}}(\gamma,u)$ and there is a sequence $\tau_n\in\mathbf{R}$, such that

$$u_n(\cdot+\tau_n,\cdot)\xrightarrow{C^\infty_{loc}}v(\cdot,\cdot).$$

Similarly, we say that the sequence $(\gamma_n,u_n)\in\mathcal{M}(p,f;y,H)$ weakly converges to the broken trajectory $(\alpha,(\gamma,u))\in\widehat{\mathcal{M}}(p,q,f)\times\mathcal{M}(q,f;x,H)$ if there is a sequence $\tau_n\in\mathbf{R}$, such that

$$\gamma_n(\cdot+\tau_n)\xrightarrow{C^\infty_{loc}}\alpha(\cdot)$$

and $(\gamma_n,u_n)\xrightarrow{C^\infty_{loc}}(\gamma,u)$.

In order to abbreviate notations, we will assume $w=(\gamma,u)$ through the rest of the paper.

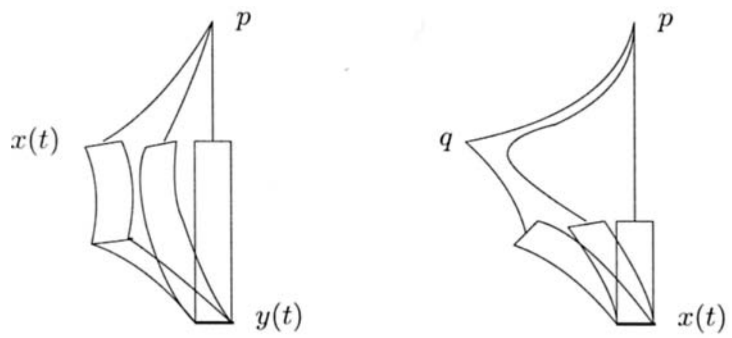

Fig. 2. Gluing of broken trajectories

We will prove Theorem 2 in the next section. Before doing that, we introduce concepts and notation that we will use.

In order to analyze gluing of mixed objects, we need to equip the moduli spaces $\mathcal{M}(x,y,H)$ and $\mathcal{M}(p,f;x,H)$ with the smooth Banach manifold structure.

The topology and Banach manifold structure of $\mathcal{M}(x, y, H)$ is induced by the Sobolev norm in the following way. Let

$$D := \mathbf{R} \times [0,1] \subset \mathbf{R}^2. \tag{7}$$

For given Hamiltonian paths $x(t)$ and $y(t)$ with ends in O_M, denote by $C^\infty(x, y)$ the set of all smooth maps u that satisfy

$$\begin{aligned} &u\colon D \to T^*M \\ &u(s,0), u(s,1) \in O_M \\ &u(-\infty, t) = x(t),\ u(+\infty, t) = y(t). \end{aligned} \tag{8}$$

The tangent space space $T_u C^\infty(x, y)$ to $C^\infty(x, y)$ at point u consists of all vector fields ξ such that

$$\begin{aligned} &\xi\colon D \to u^*(TT^*M), \\ &\xi(s,t) \in T_{u(s,t)}T^*M,\ \xi(s,0), \xi(s,1) \in TM, \\ &\xi(-\infty, t) = \xi(+\infty, t) = 0. \end{aligned}$$

For $r > 2$, let $\|\xi\|_{L^r}$ and $\|\xi\|_{W^{1,r}}$ stand for standard Sobolev norms

$$\|\xi\|_{L^r} = \left(\iint_D |\xi|^r \, ds\, dt\right)^{\frac{1}{r}}, \quad \|\xi\|_{W^{1,r}} = \left(\iint_D (|\xi|^r + |\nabla_s \xi|^r + |\nabla_t \xi|^r) \, ds\, dt\right)^{\frac{1}{r}}. \tag{9}$$

Denote by $W^{1,r}_u(x, y)$ and $L^r_u(x, y)$ the completions of this tangent space in $W^{1,r}$- and L^r-Sobolev norms. Finally, let $\mathcal{P}^{1,r}(x, y)$ be the space of all u that satisfy (8) such that the tangent space at u is given by

$$T_u \mathcal{P}^{1,r}(x, y) = W^{1,r}_u(x, y).$$

The topology and Banach manifold structure of $\mathcal{P}^{1,r}(x, y)$ (hence the topology of $\mathcal{M}(x, y, H) \subset \mathcal{P}^{1,r}(x, y)$) is induced by the topology of $W^{1,r}_u(x, y)$ by means of the exponential map

$$(\exp_u \xi)(s,t) = \exp_{u(s,t)} \xi(s,t)$$

(see [5] for details).

The set $\mathcal{M}(x, y, H)$ is a zero set of a smooth section:

$$K(u) = \overline{\partial}_{H,J}(u) = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - J(u)X_H(u) \tag{10}$$

of a Banach bundle $\mathcal{E}(x, y) \to \mathcal{P}^{1,r}(x, y)$ with a fibre $L^r_u(x, y)$ over $u \in \mathcal{P}^{1,r}(x, y)$ (see [4, 5]).

Its linearization at u is a Fredholm operator defined on $W^{1,r}_u(x, y)$ with values in $L^r_u(x, y)$. It has the form

$$DK_u(\xi) = \nabla_s \xi + J(u)\nabla_t \xi + \nabla_\xi J(u)\partial_t u - \nabla_\xi \nabla H(u) \tag{11}$$

and its index is equal to the difference of Maslov indices at the ends $\mu_H(x) - \mu_H(y)$. In the local coordinates (10) is of the form

$$\zeta(s,t) \mapsto \frac{\partial \zeta}{\partial s}(s,t) + J(s,t)\frac{\partial \zeta}{\partial t}(s,t) + A(s,t)\zeta(s,t).$$

The situation with $\mathcal{M}(p, f; x, H)$ is somewhat different. Although we initially defined it as a transversal intersection set of two manifolds, we need to consider it as the zero set of some Fredholm operator, in order to get one consistent picture including all three moduli spaces (gradient trajectories, holomorphic discs and mixed objects). Let p and $x(t)$ be as before. For $r > 2$, denote by $\mathcal{P}^{1,r}(p)$ the Sobolev $W^{1,r}$-completion of the space of all smooth trajectories γ that satisfy:

$$\begin{cases} \gamma\colon (-\infty, 0] \to O_M \\ \gamma(-\infty) = p, \end{cases}$$

and by $\mathcal{P}^{1,r}(x)$ the Sobolev $W^{1,r}$-completion of the space of all smooth maps u that satisfy:

$$\begin{cases} u\colon [0, +\infty) \times [0,1] \to T^*M \\ u(0,t), u(s,0), u(s,1) \in O_M, \\ u(+\infty, t) = x(t). \end{cases}$$

By Sobolev $W^{1,r}$-completion we assume the previous discussion: the completion of the tangent space and the topology induced by exponential map (for more details about this construction is Morse theory see [19]. We denote tangent spaces by

$$W^{1,r}_\gamma(p) = T_\gamma \mathcal{P}^{1,r}(p), \qquad W^{1,r}_u(x) = T_u \mathcal{P}^{1,r}(x).$$

Let $\widetilde{\mathrm{ev}}$ be the following evaluation map:

$$\widetilde{\mathrm{ev}}\colon \mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x) \to M \times M, \quad \widetilde{\mathrm{ev}}(\gamma, u) = \left(\gamma(0), u\left(0, \frac{1}{2}\right)\right).$$

For generic choices of a Morse function (or a Riemannian metric) $\widetilde{\mathrm{ev}}$ is transversal to the diagonal $\Delta \subset M \times M$ and

$$\mathcal{P}^{1,r}(p, x) := \widetilde{\mathrm{ev}}^{-1}(\Delta)$$

is infinite-dimensional smooth Banach submanifold of $\mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$. As before, denote by $W^{1,r}_{(\gamma,u)}(p, x) = T_{(\gamma,u)} \mathcal{P}^{1,r}(p, x)$ the corresponding tangent space. The space $\mathcal{M}(p, f; x, H)$ is the zero set of a restriction of a smooth section

$$\begin{gathered} F = \widetilde{F}|_{\mathcal{P}^{1,r}(p,x)}, \quad \widetilde{F} = (F_1, F_2), \\ F_1(\gamma) = \frac{d\gamma}{ds} + \nabla f(\gamma), \quad F_2(u) = \overline{\partial}_{\rho_R H, J}\, u \end{gathered} \tag{12}$$

of a Banach bundle

$$\mathcal{E}^{0,r}(p, x) \to \mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x) \tag{13}$$

with a fibre $L^r_\gamma(p) \times L^r_u(x)$ over a point $(\gamma, u) \in \mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$. Here as before, by $L^r_\gamma(p)$ we denote the space of all ξ that satisfy

$$\begin{cases} \xi\colon (-\infty, 0] \to \gamma^*(TM), \\ \xi(s) \in T_{\gamma(s)}M, \ \xi(-\infty) = 0, \\ \|\xi\|_{L^r} < \infty \end{cases}$$

and by $L^r_u(x)$ the space of all ξ such that

$$\begin{cases} \xi\colon [0, +\infty) \times [0,1] \to u^*(TT^*M), \quad \xi(s,t) \in T_{u(s,t)}T^*M, \\ \xi(0,t), \xi(s,0), \xi(s,1) \in TM, \quad \xi(+\infty, t) = 0, \\ \|\xi\|_{L^r} < \infty. \end{cases}$$

The linearization of (12) at the point (γ, u) is

$$\begin{aligned}
&(DF)_{(\gamma,u)}\colon W^{1,r}_{(\gamma,u)}(p,x) \to L^r_\gamma(p) \times L^r_u(x), \quad (DF)_{(\gamma,u)} = ((DF_1)_\gamma, (DF_2)_u),\\
&(DF_1)_\gamma\colon W^{1,r}(p) \to L^r(p), \quad (DF_1)_\gamma \eta = \nabla_{\frac{d\gamma}{ds}}\eta + \nabla_\eta \nabla f(\gamma),\\
&(DF_2)_u\colon W^{1,r}(x) \to L^r(x),\\
&(DF_2)_u\xi = \nabla_s \xi + J(u)\nabla_t \xi + \nabla_\xi J(u)\partial_t u - \nabla_\xi \nabla(\rho_R H)(u).
\end{aligned} \tag{14}$$

Proposition 3. *The operator* (14) *is Fredholm hence the map* (12) *is a Fredholm map.*

Proof. We will omit the subscripts u, γ, (γ, u) in order to abbreviate notations. First we observe that the manifold $\mathcal{P}^{1,r}(p,x)$ is of finite codimension in $\mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$. Indeed, the tangent space $W^{1,r}(p,x)$ of $\mathcal{P}^{1,r}(p,x)$ is the kernel of the differential $D\widetilde{\mathrm{ev}}$ of the evaluation map. It holds:

$$W^{1,r}(p) \times W^{1,r}(x) / \operatorname{Ker}(D\widetilde{\mathrm{ev}}) \cong \operatorname{Im}(D\widetilde{\mathrm{ev}}).$$

The image $\operatorname{Im}(D\widetilde{\mathrm{ev}})$ is of finite dimension since the target space of $D\widetilde{\mathrm{ev}}$ is so. It follows that $W^{1,r}(p,x)$ is of finite codimension and from Hahn-Banach theorem that it can be complemented in $W^{1,r}(p) \times W^{1,r}(x)$ by some finite-dimensional space, denote it by X. Since $\operatorname{codim}_{M\times M}(\Delta) = n$ it holds $\dim X = n$.

We consider the auxiliary operator

$$\begin{aligned}
&D\widetilde{F}\colon W^{1,r}(p) \times W^{1,r}(x) \to L^r(p) \times L^r(x)\\
&D\widetilde{F} := (DF_1, DF_2)
\end{aligned}$$

defined on the product $W^{1,r}(p) \times W^{1,r}(x)$ in order to compute the index of the operator DF in the terms of the indices of DF_1 and DF_2. Since the operators DF_1 and DF_2 are Fredholm (see [19, 15]) and it holds

$$\operatorname{Ker} D\widetilde{F} = \operatorname{Ker} DF_1 \times \operatorname{Ker} DF_2, \quad \operatorname{Coker} D\widetilde{F} = \operatorname{Coker} DF_1 \times \operatorname{Coker} DF_2$$

we conclude that $D\widetilde{F}$ is also Fredholm with the Fredholm index $\operatorname{Ind}(D\widetilde{F}) = \operatorname{Ind}(DF_1) + \operatorname{Ind}(DF_2)$. The operator DF is a restriction of $D\widetilde{F}$ to the space $W^{1,r}(p,x)$. Consider the following (disjoint) decompositions of the spaces $W^{1,r}(p) \times W^{1,r}(x)$ and $L^r(p) \times L^r(x)$:

$$W^{1,r}(p) \times W^{1,r}(x) = X_1 \oplus X_2 \oplus X_3 \oplus X_4, \quad L^r(p) \times L^r(x) = Y_1 \oplus Y_2 \oplus Y$$

where the subspaces X_i, Y_i and Y are defined in the following way:

$$\begin{aligned}
&X_3 := W^{1,r}(p,x) \cap \operatorname{Ker}(D\widetilde{F}), \quad X_1 := W^{1,r}(p,x) \ominus X_3\\
&X_4 := X \cap \operatorname{Ker}(D\widetilde{F}), \quad X_2 := X \ominus X_4\\
&Y_i := D\widetilde{F}(X_i), \text{ for } i = 1,2, \quad Y := L^r(p) \times L^r(x) \ominus (Y_1 \oplus Y_2)\,.
\end{aligned} \tag{15}$$

The sign $A \ominus B$ stands for the complement of B in A and all the spaces in (15) are well defined due to the Hahn-Banach theorem. All the spaces except X_1 and

Y_1 are of finite dimension: X_3 and X_4 are subspaces of $\operatorname{Ker} D\widetilde{F}$—which is of finite dimension; X_2 is the subspace of X—which is of finite dimension; Y_2 is the isomorphic image of X_2; finally Y is of finite dimension since it is a co-kernel of Fredholm operator $D\widetilde{F}$. Set

$$m_2 := \dim(X_2) = \dim(Y_2), \quad m_3 := \dim(X_3), \quad m_4 := \dim(X_4), \quad m := \dim(Y).$$

We see that

$$\operatorname{Ker}(D\widetilde{F}) = X_3 \oplus X_4, \quad \operatorname{Coker}(D\widetilde{F}) = Y$$

and, since $DF = D\widetilde{F}|_{X_1 \oplus X_3} \colon X_1 \oplus X_3 \to Y_1 \oplus Y_2 \oplus Y$,

$$\operatorname{Ker}(DF) = X_3, \quad \operatorname{Coker}(DF) = Y_2 \oplus Y.$$

We conclude that DF is also Fredholm, and moreover, we compute its index:

$$\begin{aligned}
\operatorname{Ind}(DF) &= \dim(\operatorname{Ker}(DF)) - \dim(\operatorname{Coker}(DF)) = \dim(X_3) - \dim(Y_2 \oplus Y) \\
&= m_3 - (m_2 + m) = (m_3 + m_4) - m - (m_2 + m_4) \\
&= \dim(\operatorname{Ker}(D\widetilde{F})) - \dim(\operatorname{Coker}(D\widetilde{F})) - \dim(X_2 \oplus X_4) \\
&= \operatorname{Ind}(D\widetilde{F}) - \dim(X) = \operatorname{Ind}(DF_1) + \operatorname{Ind}(DF_2) - n \\
&= m_f(p) - \mu_H(x) - \frac{n}{2}. \quad \blacksquare
\end{aligned}$$

3. Proof of Theorem 2

3.1. Pre-gluing. We first define a *pre-gluing* map, i.e. an approximate solution. Let $w = (\gamma, u) \in \mathcal{M}(p, f; x, H)$, i.e. $\gamma \in W^u(p, f)$, $u \in W^s(x, H)$ and $v \in \mathcal{M}(x, y, H)$. Denote by

$$\beta^+ \colon \mathbf{R} \to [0, 1] \tag{16}$$

a smooth non-decreasing cut-off function, equal to 0 for $s \leq 0$ and to 1 for $s \geq 1$. Let $u(s,t) = \exp_{x(t)}(\xi(s,t))$ for all t and $s \geq s_0$, and $v(s,t) = \exp_{x(t)}(\zeta(s,t))$ for all t and $s \leq -s_0$. For $\rho \geq \max\{2s_0, s_0 + 1\}$, define

$$u \sharp_\rho^0 v(s,t) := \begin{cases} u(s,t), & 0 \leq s \leq \frac{\rho}{2}, \\ \exp_{x(t)}(\beta^+(-s + \frac{\rho}{2} + 1)\xi(s,t)), & \frac{\rho}{2} \leq s \leq \frac{\rho}{2} + 1, \\ x(t), & \frac{\rho}{2} + 1 \leq s \leq \rho \\ \exp_{x(t)}(\beta^+(s - \rho)\zeta(s - 2\rho, t)), & \rho \leq s \leq \rho + 1, \\ v(s - 2\rho, t), & s \geq \rho + 1. \end{cases} \tag{17}$$

Now set

$$\varpi := w \sharp_\rho^0 v := (\gamma, u \sharp_\rho^0 v). \tag{18}$$

The linearization of the pre-gluing map (17), for $\xi \in \operatorname{Ker}(DF_2)_u = T_u W^s(x, H)$,

$\zeta \in \mathrm{Ker}(DK)_v = T_v\mathcal{M}(x,y,H)$, is given by

$$\xi\hat{\sharp}\zeta := D\sharp^0_\rho(u,v)(\xi,\zeta) = \begin{cases} \xi(s,t), & 0 \le s \le \frac{\rho}{2}, \\ \nabla_2 \exp(\beta^+(-s+\frac{\rho}{2}+1)\nabla_2 \exp^{-1}\xi(s,t)), & \frac{\rho}{2} \le s \le \frac{\rho}{2}+1, \\ 0, & \frac{\rho}{2}+1 \le s \le \rho \\ \nabla_2 \exp(\beta^+(s-\rho)\nabla_2 \exp^{-1}\zeta(s-2\rho,t)), & \rho \le s \le \rho+1, \\ \zeta(s-2\rho,t), & s \ge \rho+1. \end{cases} \tag{19}$$

Here $\nabla_2 \exp$ is a fibre linearization of exponential map. More precisely, for $P = T^*M$, let

$$K\colon T(TP) \to TP, \quad \pi\colon TP \to P$$

denote the unique Levi-Civita connection with respect to the given Riemannian metric g and the canonical projection. For $\xi \in TP$ in the injectivity neighborhood associated to exp, denote by:

$$\nabla_1 \exp(\xi) : = D\exp(\xi) \circ \left(D\pi|_{\mathrm{Ker}(K(\xi))}\right)^{-1} : T_{\pi(\xi)}P \xrightarrow{\cong} T_{\exp(\xi)}P$$
$$\nabla_2 \exp(\xi) : = D\exp(\xi) \circ \left(K|_{\mathrm{Ker}(D\pi(\xi))}\right)^{-1} : T_{\pi(\xi)}P \xrightarrow{\cong} T_{\exp(\xi)}P$$

(see Appendix A.2 in [19] for details).

The linearization of (18) is, when $\varsigma = (\eta,\xi) \in T_w\mathcal{M}(p,f;x,H)$, i.e. $\eta \in \mathrm{Ker}\,D_\gamma = T_\gamma W^u(p,f)$,

$$\varsigma\hat{\sharp}\zeta = (\eta,\xi)\hat{\sharp}\zeta = (\eta,\xi\hat{\sharp}\zeta).$$

We will use the abbreviation χ for the triple (w,u,ρ) and (in the case when we want to emphasize the relation between ϖ and χ) the notation ϖ_χ for ϖ as in (18).

The local representation of the operator F from (12) with respect to ϖ gives rise to the bundle mapping

$$F_\varpi\colon \nabla_2 \exp_\varpi^{-1} \circ F \circ \exp_\varpi\colon \mathcal{E}^{1,r} \supset \mathcal{O}_\varpi \to \mathcal{E}^{0,r} \tag{20}$$

where $\mathcal{E}^{1,r}$ and $\mathcal{E}^{0,r}$ are vector bundles over $K \times [\rho_0,\infty)$ with a fibres $W^{1,r}_\varpi(p,x)$ and $L^r_\gamma(p) \times L^r_{u\,\sharp^0_\rho\,v}(x)$ respectively. The fibre derivative

$$DF_{\varpi_\chi}(0)\colon W^{1,r}_\varpi(p,x) \to L^r_\gamma(p) \times L^r_{u\,\sharp^0_\rho\,v}(x)$$

is exactly the linearization $(DF)_\varpi$ at the point ϖ of a map F from (12).

We will use the abbreviation D_χ for $DF_{\varpi_\chi}(0)$. Similarly, we use the notation K_u for the local representation of the operator K from (10) and D_u for $DK_u(0) = (DK)_u$.

The exact solution of combined equation of type (3) will exist due to the Banach contraction principle (more precisely, its existence is the content of the abstract Lemma 5. The proof of the Theorem 2 is long and we will divide it in several steps. The first step is to prove that the linearized operator D_χ is onto for ρ large enough. This will imply the existence of the right inverse needed for the application of the abstract Lemma 5. It is done in the auxiliary Proposition 4. The second step is to prove that the mentioned right inverse is bounded and to check the other conditions required in Lemma 5—this is done in the Lemma 6 and in the rest of the Subsection 3.2. The last step is the proof of the embedding property which is given in the Subsection 3.3.

We will use the following notations. If $\langle\cdot,\cdot\rangle_{L^2}$ stands for standard scalar product:

$$\langle\xi,\zeta\rangle_{L^2} := \int \langle\xi(\tau),\zeta(\tau)\rangle_g \, d\tau$$

where g is Riemannian metric appearing in (9), then the following sets are well defined:

$$\widetilde{L}_w^{\perp} := \{\eta \in W_w^{1,r}(p,x) \mid \langle\eta,\xi\rangle_{L^2} = 0 \text{ for all } \xi \in \operatorname{Ker} D_w\}$$
$$\widetilde{L}_v^{\perp} := \{\eta \in W_v^{1,r}(x,y) \mid \langle\eta,\zeta\rangle_{L^2} = 0 \text{ for all } \zeta \in \operatorname{Ker} D_v\}$$
$$L_\chi^{\perp} := \{\eta \in W_\varpi^{1,r}(p,y) \mid \langle\eta,\xi \,\hat{\sharp}\, \zeta\rangle_{L^2} = 0 \text{ for all } (\xi,\zeta) \in \operatorname{Ker} D_w \times \operatorname{Ker} D_v\}.$$

Indeed, from the standard fact that the solutions of Morse gradient and Floer perturbed Cauchy-Riemann equations of types (1) and (2) have the exponential decay it follows that $\xi \in L^q(p,x)$, $\zeta \in L^q(x,y)$ and $\xi \,\sharp\, \zeta \in L^q(p,y)$, where $\frac{1}{r}+\frac{1}{q}=1$.

First we prove the following auxiliary Proposition.

Proposition 4. *There is a lower parameter bound ρ_1 such that for all gluing parameters $\rho \geq \rho_1$ and $(w,v) \in K$ the Fredholm operator $D_\chi \colon W_\varpi^{1,r}(p,y) \to L_\gamma^r(p) \times L^r_{u\,\sharp_\rho^0\, v}(y)$ of the type (14) is onto. Since $\operatorname{Ker} D_\chi$ is of finite dimension, there is a complement Z of $\operatorname{Ker} D_\chi$ in $W_\varpi^{1,r}(p,y)$ and the projection onto $\operatorname{Ker} D_\chi$, denote it by $\operatorname{Proj}_{\operatorname{Ker} D_\chi}$. Then the following composition induces an isomorphism:*

$$\varphi_\chi := \operatorname{Proj}_{\operatorname{Ker} D_\chi} \circ \hat{\sharp} \colon \operatorname{Ker} D_w \times \operatorname{Ker} D_v \xrightarrow{\cong} \operatorname{Ker} D_\chi.$$

Proof. From indices formulae

$$\operatorname{Ind}(D_w) = m_f(p) - \Big(\mu_H(x) + \frac{n}{2}\Big)$$
$$\operatorname{Ind}(D_v) = \mu_H(x) - \mu_H(y)$$
$$\operatorname{Ind}(D_\chi) = m_f(p) - \Big(\mu_H(x) + \frac{n}{2}\Big)$$

and the fact that D_w and D_v are onto, so

$$\operatorname{Ind} D_w = \dim \operatorname{Ker} D_w, \quad \operatorname{Ind} D_v = \dim \operatorname{Ker} D_v$$

it follows

$$\dim \operatorname{Ker} D_\chi \geq \operatorname{Ind}(D_\chi) = \operatorname{Ind} D_w + \operatorname{Ind} D_v = \dim \operatorname{Ker} D_w + \dim \operatorname{Ker} D_v. \tag{21}$$

It is sufficient to prove that, for fixed $w,v \in K$ there exists a lower parameter bound $\rho(w,v)$, such that, for $\rho \geq \rho(w,v)$ the map φ_χ is onto (since K is compact, this would imply the existence of the uniform lower bound ρ_K such that φ_χ is onto for any $(w,v) \in K$, $\rho \geq \rho_K$). Indeed, if we suppose φ_χ is onto, then

$$\dim \operatorname{Ker} D_w + \dim \operatorname{Ker} D_v \geq \dim \operatorname{Ker} D_\chi. \tag{22}$$

From (21) and (22) we have

$$\operatorname{Ind} D_\chi = \operatorname{Ind} D_w + \operatorname{Ind} D_v = \dim \operatorname{Ker} D_w + \dim \operatorname{Ker} D_v = \dim \operatorname{Ker} D_\chi$$

so the proof of the Proposition follows.

So we need to prove that φ_χ is onto. It is enough to show that for some $C > 0$ and ρ large enough it holds

$$\|D_\chi \varsigma\|_{L^r} \geq C \|\varsigma\|_{W^{1,r}} \tag{23}$$

for all $\varsigma \in L_\chi^\perp$. Indeed, if φ_χ is not surjective, it follows from the decomposition $W^{1,r}_\varpi(p,y) = L_\chi^\perp \oplus \operatorname{Ker} D_w \hat{\sharp} \operatorname{Ker} D_u$ that there exists $\varsigma \in \operatorname{Ker} D_\chi$ such that $\varsigma \in L_\chi^\perp$. But from (23) it follows that ς must be the zero vector. Suppose (23) is not true. Then there exist sequences $\rho_n \to \infty$, $\varsigma_n \in L_{\chi_n}^\perp$ such that

$$\|\varsigma_n\|_{W^{1,r}} = 1 \text{ and } \|D_{\chi_n} \varsigma_n\|_{L^r} \to 0. \tag{24}$$

Here $\varsigma_n = (w, v, \rho_n)$ $\chi_n = (w, v, \rho_n)$ and $\varpi_n = \varpi_{\chi_n}$, i.e w and v are fixed, and $\rho_n \to \infty$.

We can assume we are working with trivial case $\mathbf{R}^{2n}$ instead of T^*M. Indeed, let Θ stands for the domain of w, i.e.

$$\Theta := (-\infty, 0] \cup ([0, +\infty) \times [0, 1]) . \tag{25}$$

We start from a trivialization $\phi_x \colon TT^*M|_{N(x)} \to N(x) \times \mathbf{R}^{2n}$ where $N(x)$ is the normal neighborhood of the path $x(t)$ and trivializations ϕ_w of w^*TT^*M , ϕ_v of w^*TT^*M such that $\phi_x(s,t)$, $\phi_w(s,t)$, $\phi_v(s,t)$ are bounded uniformly in (s,t) in the operator norm. For $\rho \geq \rho_0$ we define a trivialization ϕ_ρ of $(w\sharp^0_\rho v)^*TT^*M$ such that

$$\begin{aligned} \phi_\rho|_{[\frac{\rho}{2}, \rho+1]} &\equiv \phi_x|_{[\frac{\rho}{2}, \rho+1]}, \\ \phi_\rho|_{(-\infty, \frac{\rho}{2}]} &\equiv \phi_1 \circ \phi_w|_{(-\infty, \frac{\rho}{2}]}, \\ \phi_\rho|_{[\rho+1, +\infty)} &\equiv \phi_2 \circ \phi_v|_{[\rho+1, +\infty)} \end{aligned}$$

where

$$\phi_1(s,t) := \phi_x\left(\frac{\rho}{2}, t\right) \circ \phi_w^{-1}\left(\frac{\rho}{2}, t\right), \quad \phi_2(s,t) := \phi_x(\rho+1, t) \circ \phi_v^{-1}(\rho+1, t)$$

are uniformly bounded in (s,t). Thus we obtain the trivialization

$$\phi \colon \bigcup_{\rho \geq \rho_0} (w\sharp^0_\rho v)^* TT^*M \to [\rho_0, +\infty) \times \Theta \times \mathbf{R}^{2n}$$

which is uniformly bounded in the operator norm. So the estimates we want to prove stay unchanged when we transfer the settings into trivial framework.

The strategy is the following: near the breaking Hamiltonian $x(t)$ we use the fact that the asymptotic linearized operators are isomorphisms and away from $x(t)$ we use that the pre-glued objects are equal (up to the shifting) to the fixed ones w and u. More precisely, let $\beta \colon \to [0,1]$ be a smooth cut-off function equal to 0 for $s \in (-\infty, \frac{1}{2} - \varepsilon] \cup [1+\varepsilon, \infty)$ and equal to 1 for $s \in [\frac{1}{2} - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}]$ for fixed, suitably chosen $\varepsilon > 0$. Let $\beta_n(s) := \beta(\frac{s}{\rho_n})$. From the construction of the pre-gluing map one can obtain

$$\|D_{\chi_n} \varsigma_n\|_{L^r} \to 0 \quad \Rightarrow \quad \|\frac{\partial}{\partial s}(\beta_n \varsigma_n) + J(\infty, \cdot) \frac{\partial \beta_n \varsigma_n}{\partial t} + A(\infty, \cdot) \beta_n \varsigma_n \|_{L^r} \to 0$$

(see [4, 6] or [19] for the details). Since $\frac{\partial}{\partial s}+J(\infty)\frac{\partial}{\partial t}+A(\infty)$ is an isomorphism we see that $\beta_n\varsigma_n \xrightarrow{W^{1,r}} 0$, so

$$\|\varsigma_n|_{[\frac{\rho_n}{2},\rho_n+1]}\|_{W^{1,r}}\to 0. \tag{26}$$

Denote by $\beta^-(s):=\beta^+(-s)$, where β^+ is given in (16) and by $\beta^\pm_\tau(s):=\beta^\pm(s+\tau)$, $\varsigma_\tau(s,t):=\varsigma(\tau+s,t)$. Now it follows from (24) and (26) that

$$\|D_w(\beta^-_{-\frac{\rho_n}{2}-1}\varsigma_n)\|_{L^r}\to 0 \text{ and } \|D_v(\beta^+_{\rho_n}(\varsigma_n)_{2\rho_n})\|_{L^r}\to 0. \tag{27}$$

But $\beta^-_{-\frac{\rho_n}{2}-1}\varsigma_n\in\widetilde{L}_w^\perp$ and $\beta^+_{\rho_n}(\varsigma_n)_{2\rho_n}\in\widetilde{L}_u^\perp$ and for such vectors it holds:

$$\begin{aligned}\|D_w(\beta^-_{-\frac{\rho_n}{2}-1}\varsigma_n)\|_{L^r}&\geq c_1\|\beta^-_{-\frac{\rho_n}{2}-1}\varsigma_n\|_{W^{1,r}}\\ \|D_v(\beta^+_{\rho_n}(\varsigma_n)_{2\rho_n})\|_{L^r}&\geq c_2\|\beta^+_{\rho_n}(\varsigma_n)_{2\rho_n}\|_{W^{1,r}}\end{aligned} \tag{28}$$

since $D_w|_{\widetilde{L}_w^\perp}$ and $D_v|_{\widetilde{L}_u^\perp}$ are isomorphisms.

We now conclude:

$$\begin{aligned}1&=\lim_{n\to\infty}\|\varsigma_n\|_{W^{1,r}_{\varpi_n}}\\ &=\lim_{n\to\infty}\|\beta^+_{-\rho_n}\varsigma_n+\beta^-_{-\frac{\rho_n}{2}-1}\varsigma_n+(1-\beta^+_{-\rho_n}-\beta^-_{-\frac{\rho_n}{2}-1})\varsigma_n\|_{W^{1,r}}\\ &\leq\lim_{n\to\infty}\left(\|\beta^+_{-\rho_n}\varsigma_n\|_{W^{1,r}}+\|\beta^-_{-\frac{\rho_n}{2}-1}\varsigma_n\|_{W^{1,r}}+\|(1-\beta^+_{\rho_n}-\beta^-_{-\frac{\rho_n}{2}-1})\varsigma_n\|_{W^{1,r}}\right)\\ &\overset{(i)}{=}\lim_{n\to\infty}\left(\|\beta^+_{\rho_n}(\varsigma_n)_{2\rho_n}\|_{W^{1,r}}+\|\beta^-_{-\frac{\rho_n}{2}-1}\varsigma_n\|_{W^{1,r}}\right)\\ &\overset{(ii)}{\leq}\lim_{n\to\infty}\left(\frac{1}{c_1}\|D_w(\beta^-_{-\frac{\rho_n}{2}-1}\varsigma_n)\|_{L^r}+\frac{1}{c_2}\|D_v(\beta^+_{\rho_n}(\varsigma_n)_{2\rho_n})\|_{L^r}\right)\\ &\overset{(iii)}{=}0.\end{aligned}$$

The equality (i) follows from (26), the fact that

$$\operatorname{supp}(1-\beta^+_{-\rho_n}-\beta^-_{-\frac{\rho_n}{2}-1})\subset[\frac{\rho_n}{2},\rho_n+1]$$

and the equality $\|\beta^+_{-\rho_n}\varsigma_n\|_{W^{1,r}}=\|\beta^+_{\rho_n}(\varsigma_n)_{2\rho_n}\|_{W^{1,r}}$; (ii) follows from (28) and (iii) follows from (27). We obtain the contradiction so the proof follows. ∎

3.2. The existence of the exact solution. Reducing the problem of finding an exact solution to the following abstract Lemma is standard ingredient in all gluing problems.

Lemma 5. [7, 19] *Assume that a smooth map $f\colon E\to F$ between Banach spaces E and G has an expansion*

$$f(\varsigma)=f(0)+Df(0)\varsigma+N(\varsigma)$$

so that $Df(0)$ has a finite dimensional kernel and a right inverse G and so that for $\varsigma,\ \zeta\in E$ it holds:

$$\|GN(\varsigma)-GN(\zeta)\|_E\leq C\left(\|\varsigma\|_E+\|\zeta\|_E\right)\|\varsigma-\zeta\|_E$$

for some constant C. Set $\varepsilon = \frac{1}{5C}$. If $\|Gf(0)\|_E \le \frac{\varepsilon}{2}$, then there exist the unique zero point of the map f

$$x_0 \in B_\varepsilon(0) \cap G(F)$$

such that $\|x_0\|_E \le 2\|Gf(0)\|_E$.

The proof of Lemma 5 is based on Banach contraction mapping principle and can be found in [19].

From Proposition 4 it follows that there exists a vector bundle isomorphism

$$D|_{L^\perp} : L^\perp \xrightarrow{\cong} \mathcal{E}^{0,r}$$

where

$$L^\perp := \bigcup_{\chi \in K \times [\rho_0, +\infty)} L^\perp_\chi$$

and $\mathcal{E}^{0,r}$ is defined below the formula (20). Its fibre-wise restriction is

$$D_\chi|_{L^\perp_\chi} : L^\perp_\chi \xrightarrow{\cong} L^r_\gamma(p) \times L^r_{u\,\sharp^0_\rho\,v}(y).$$

Let G_χ be the inverse map $G_\chi := (D_\chi|_{L^\perp_\chi})^{-1}$. In order to apply Lemma 5 to our case, we need the estimate of the right inverse. We prove the following

LEMMA 6. *There exist constant C and a lower parameter bound $\rho_2 \ge \rho_0$ such that G satisfies the estimate*

$$\|G_\chi \eta\|_{W^{1,r}} \le C\|\eta\|_{L^r} \tag{29}$$

for all $\chi \in K \times [\rho_2, +\infty)$, $\eta \in L^r_\gamma(p) \times L^r_{u\,\sharp^0_\rho\,v}(y)$.

Proof. The estimate (29) is equivalent to

$$\|\varsigma\|_{W^{1,r}} \le C\|D_\chi \varsigma\|_{L^r}. \tag{30}$$

Suppose (30) does not hold, then there exist sequences $\rho_n \to \infty$, $(u_n, v_n) \in K$ and $\varsigma_n \in L^\perp_{\chi_n}$ (where $\chi_n = (w_n, v_n, \rho_n)$) such that

$$\|\varsigma_n\|_{W^{1,r}} = 1, \quad \|D_{\chi_n}\varsigma_n\|_{L^r} \to 0. \tag{31}$$

The rest of the proof can be reduced to the arguments similar to the ones from the proof of Proposition 4. The main difference is the fact that here we have the sequence (w_n, v_n) of the broken trajectories and the sequence $\rho_n \to \infty$, instead of the fixed one (w, v) and the sequence $\rho_n \to \infty$ as it was there. This difficulty can be solved by assuming, due to the compactness of K that $(w_n, v_n) \xrightarrow{W^{1,r}} (w, v) \in K$. Denote by

$$\chi_{m,n} := (w_n, v_n, \rho_m), \quad \chi_n := (w, v, \rho_n)$$
$$\varpi_{m,n} := w_n\,\sharp^0_{\rho_m}\,v_n, \quad \varpi_n := w\,\sharp^0_{\rho_n}\,v.$$

Let $U \subset T^*M$ be the open neighborhood of $w(\Theta) \cup u(D)$ (see (25) and (7) and

$$\Phi \colon TT^*M|_U \xrightarrow{\approx} U \times \mathbf{R}^{2n} \tag{32}$$

such that $\varpi_{\chi_{m,n}}(\Theta) \subset U$ for $m, n \geq n_0$. Define

$$\Phi_{m,n} \colon \xi \mapsto \Phi^{-1}(w \,\sharp^0_{\rho_m} v, \mathrm{Proj}_2 \circ \Phi(\xi))$$

where Proj_2 denotes the projection to the second component in the right side of (32). We see that $\Phi_{n,n}$ establishes the isomorphism between $W^{1,r}_{\varpi_{n,n}}$ and $W^{1,r}_{\varpi_n}$ as well as between $L^r_{\gamma_n} \times L^r_{u_n \sharp^0_{\rho_n} v_n}$ and $L^r_\gamma \times L^r_{u \sharp^0_{\rho_n} v}$ such that (due to the condition $(w_n, v_n) \xrightarrow{W^{1,r}} (w, v)$) $\phi_{n,n} \to \mathrm{Id}_{W^{1,r}_{\varpi_n}}$ when $n \to \infty$. So, for given $\varepsilon > 0$ we transform the condition (31) to

$$1 - \varepsilon \leq \|(\mathrm{Proj}_{L^\perp_{\chi_n}} \circ \Phi_{n,n})\varsigma_n\|_{W^{1,r}} \leq 1 + \varepsilon, \quad \|D_{\chi_n}\varsigma_n\|_{L^r} \to 0$$

for $n \geq n_0$. The map $\mathrm{Proj}_{L^\perp_{\chi_n}}$ is well defined due to the fact that $L^\perp_{\chi_n}$ complemented in $W^{1,r}_{\varpi_n}(p, y)$ (since its complement is $\mathrm{Ker}\, D_\chi$ hence finite-dimensional). But $(\mathrm{Proj}_{L^\perp_{\chi_n}} \circ \Phi_{n,n})\varsigma_n$ is the sequence of vectors in $L^\perp_{\chi_n}$ so we are in the situation from Proposition 4. The rest of the proof is as there (see [19] for more details). ∎

Now we apply Lemma 5 to our situation. The map f from Lemma 5 is, in our case, the fibre-wise restriction

$$F_\chi \colon W^{1,r}_{\varpi_\chi}(p, y) \to L^r_\gamma(p) \times L^r_{u \sharp^0_\rho v}(y)$$

of a bundle map (20). The maps Df and N from Lemma 5 are the corresponding derivation terms in the expansion of F_χ. So Df is D_χ. The map G from Lemma 5 is the map G_χ from Lemma 6. The only assumption from Lemma 5 left to check is the assumption about the non-linear term N. From the following three expansions:

$$\begin{aligned} F_\chi(\xi) &= F_\chi(0) + D_\chi(0)\xi + N_\chi(0, \xi) \\ F_\chi(\eta) &= F_\chi(0) + D_\chi(0)\eta + N_\chi(0, \eta) \\ F_\chi(\xi) &= F_\chi(\eta) + D_\chi(\eta)(\xi - \eta) + N_\chi(\eta, \xi - \eta) \end{aligned}$$

we compute

$$\begin{aligned} N_\chi(\xi) - N_\chi(\eta) &= F_\chi(\xi) - F_\chi(\eta) - D_\chi(0)(\xi - \eta) \\ &= (D_\chi(\eta) - D_\chi(0))(\xi - \eta) + N_\chi(\eta, \xi - \eta). \end{aligned}$$

Since K is compact, the set

$$\bigcup_{\chi \in K \times [\rho_0, +\infty)} \varpi_\chi(\Theta) \tag{33}$$

is relatively compact in T^*M so we have an estimate:

$$\|N_\chi(\xi) - N_\chi(\eta)\| \leq C_1 \left(\|\xi\|_{W^{1,r}_\chi} + \|\eta\|_{W^{1,r}_\chi}\right) \|\xi - \eta\|_{W^{1,r}_\varpi} \tag{34}$$

where the constant C_1 depends on the C^2- norm of the $\nabla_2 \exp$ on a relatively compact set (33). Now the required estimate for GN in Lemma 5 follows from (34) and the fact that G_χ from Lemma 6 is bounded.

It follows from Lemma 5 that there exists the unique solution $\Gamma(\chi) \in B_\varepsilon(0) \cap L^\perp_\chi$ satisfying $F_\chi(\Gamma(\chi)) = 0$. If we set $w \,\sharp_\rho\, v := \exp_{w \sharp^0_\rho v} \Gamma(\chi)$ we obtain the exact solution satisfying $F(w \,\sharp_\rho\, v) = 0$. From the estimates

$$\|F(\varpi_\chi)\|_{L^r} \leq \alpha e^{-m\rho} \tag{35}$$

for some $\alpha > 0$, $m > 0$ and

$$\|\Gamma(\chi)\|_{W^{1,r}} \leq 2C\|F(\varpi_\chi)\|_{L^r} \tag{36}$$

it follows that

$$\|\Gamma(\chi)\|_{W^{1,r}} \leq C(K)e^{-m\rho} \tag{37}$$

for some $C(K) > 0$ depending on K. The estimate (36) follows from Lemma 5 with C depending on right inverse G. The estimate (35) can be easily obtained from the construction of the pre-glued trajectory ϖ and the exponential convergence of w, v toward $x(t)$ when $s \to \pm\infty$ (see [5] for details).

We deduce from (37) that the difference of the approximately glued trajectories $w \sharp_\rho^0 v$ in (18) and the exact solution $w \sharp_\rho v$ tends to zero fast enough when $\rho \to \infty$. Obviously the pre-glued trajectories $w \sharp_\rho^0 v$ already converge to the given broken one in the weak sense.

The exact solution of (3) is smooth, this follows from elliptic regularity theory. The only part in Theorem 2 left to prove is the embedding property.

3.3. Embedding property. To show that the gluing map $\sharp$ is an embedding, for $\rho \geq \rho(K)$ we need to prove that its linearization $D\sharp$ is an isomorphism at every point $(w, \hat{v}, \rho)$ and that $\sharp$ is injective.

In section 3.1 we defined pre-gluing for elements from $\mathcal{M}(p, f; x, H) \times \mathcal{M}(x, y, H)$. Let a be fixed regular value of $\mathcal{A}_H$. There is an identification

$$\mathcal{M}^a(x, y; H) \cong \widehat{\mathcal{M}}(x, y, H) \tag{38}$$

where

$$\mathcal{M}^a(x, y; H) = \{v \in \mathcal{M}(x, y; H) \mid \mathcal{A}_H(v(0, \cdot)) = a\}.$$

Using the inclusion

$$\iota\colon \mathcal{M}^a(x, y, H) \hookrightarrow \mathcal{M}(x, y, H)$$

we can define gluing for $(w, \hat{v}) \in \mathcal{M}(p, f; x, H) \times \widehat{\mathcal{M}}(x, y, H)$ as $\sharp \circ (\mathrm{Id}, \iota)$ (assuming the identification (38)).

Since K is compact, and regularity is open condition, it is enough to find $\rho(w, \hat{v})$ for a fixed pair $(w, \hat{v})$ such that, for $\rho \geq \rho(w, \hat{v})$, $D\sharp(w, \hat{v}, \rho)$ is an isomorphism. Then we take $\rho(K) := \max_{(w,\hat{v}) \in K} \rho(w, \hat{v})$.

To prove that $D\sharp(w, \hat{v}, \rho)$ is an isomorphism, it is enough to prove that it is injective, since, from indices formulae, we have:

$$\begin{aligned}\dim\left(K \times [\rho(K), \infty)\right) &= m_f(p) - \left(\mu_H(x) + \frac{n}{2}\right) + (\mu_H(x) - \mu_H(y) - 1) + 1\\ &= m_f(P) - \left(\mu_H(y) + \frac{n}{2}\right) = \dim \mathcal{M}(p, f; y, H).\end{aligned}$$

Assume it is not true, i.e. that there exists no $\rho(w, \hat{v})$ such that $D\sharp(w, \hat{v}, \rho)$ is injective for $\rho \geq \rho(w, \hat{v})$. Then, there exist sequences $\xi_n \in T_w\mathcal{M}(p, f; x, H)$, $\zeta_n \in T_{\hat{v}}\widehat{\mathcal{M}}(x, y, H)$, $\rho_n \to \infty$ and $t_n \in T_{\rho_n}[\rho_0, \infty)$ such that

$$(\xi_n, \zeta_n, t_n) \neq (0, 0, 0), \quad (\xi_n, \zeta_n, t_n) \in \operatorname{Ker} D\sharp(u, v, \rho_n). \tag{39}$$

Assume, since kernel is a vector subspace, that

$$\|\xi_n\| + \|\zeta_n\| + |t_n| = 1. \tag{40}$$

We have

$$D\sharp(w,\hat{v},\rho)(\xi,\zeta,t) = D\sharp_\rho(w,v)(\xi,\zeta) + D\sharp(w,v)\cdot(\dot{w},-\dot{v})\cdot t,$$

where first D denotes derivative in (w,v) and the second D derivative in (w,v,ρ). Further:

$$D\sharp_\rho(w,v)(\xi,\zeta) = \nabla_1 \exp(\Gamma(\chi))(\xi\sharp_\rho\zeta) + \nabla_2\exp(\Gamma(\chi))\,(D\Gamma(\chi)(\xi,\zeta))$$

and

$$D\sharp(w,v)(\dot{w},-\dot{v})t = \nabla_1\exp(\Gamma(\chi))(\dot{w}\sharp_\rho(-\dot{v})t)$$

so from (39) we conclude

$$\begin{aligned} 0 &= D\sharp(\xi_n,\zeta_n,t_n) \\ &= \nabla_1\exp(\Gamma(\chi_n))[(\xi_n\sharp_{\rho_n}\zeta_n) + \dot{w}\sharp_{\rho_n}(-\dot{v})t_n] + \nabla_2\exp(\Gamma(\chi_n))\,(D\Gamma(\chi_n)(\xi_n,\zeta_n)) \end{aligned}$$

i.e.

$$\nabla_1\exp(\Gamma(\chi_n))[(\xi_n\sharp_{\rho_n}\zeta_n) + \dot{w}\sharp_{\rho_n}(-\dot{v})t_n] = -\nabla_2\exp(\Gamma(\chi_n))\,(D\Gamma(\chi_n)(\xi_n,\zeta_n))\,.$$

But from the identity $(\nabla_2\exp^{-1}\circ\nabla_1\exp)(p,0) = \mathrm{Id}_{T_pP}$ and the exponential decrease $\|\Gamma(\chi_n)\|$, $\|D\Gamma(\chi_n)\| \stackrel{n\to\infty}{\longrightarrow} 0$, we conclude

$$\|(\xi_n\sharp_{\rho_n}\zeta_n) + \dot{w}\sharp_{\rho_n}(-\dot{v})t_n\|_{W^{1,r}} \to 0.$$

So it follows

$$\|\xi_n + \dot{w}t_n\|_{W^{1,r}(-\infty,\frac{\rho_n}{2}]} \to 0, \quad \|\zeta_n - \dot{v}t_n\|_{W^{1,r}[-2\rho_n,+\infty)} \to 0.$$

Since $\zeta_n \in T_{\hat{v}}\widehat{\mathcal{M}}(x,y,H)$ and $T_{\hat{v}}\widehat{\mathcal{M}}(x,y,H)\cap\mathbf{R}\dot{v} = \{0\}$ we conclude $t_n\to 0$, so

$$\|\xi_n\|, \|\zeta_n\| \to 0$$

which is in contradiction with (40). Hence we proved the regularity of $D\sharp$.

The injectivity of $\sharp$ (for ρ large enough) follows from already proved regularity and the compactness of K. We argue by contradiction, so assume that there exist sequences $\rho_n\to\infty$,

$$(w_n,\hat{v}_n) \neq (\delta_n,\hat{\sigma}_n) \tag{41}$$

such that $w_n\sharp_{\rho_n}\hat{v}_n = \delta_n\sharp_{\rho_n}\hat{\sigma}_n$. Since K is compact we can assume that

$$w_n\to w, \quad \hat{v}_n\to\hat{v}, \quad \delta_n\to\delta, \quad \hat{\sigma}_n\to\hat{\sigma}.$$

It follows from exponential decrease of $\Gamma(\chi_n) = \Gamma(w_n,\hat{v}_n,\rho_n)$ and $\Gamma(\lambda_n) = \Gamma(\delta_n,\hat{\sigma}_n,\rho_n)$ that in local coordinates:

$$\|w\sharp^0_{\rho_n}v - \delta\sharp^0_{\rho_n}\sigma\|_{W^{1,r}} \stackrel{n\to\infty}{\longrightarrow} 0$$

so $w=\delta$, $\hat{v}=\hat{\sigma}$. Since we already proved that $\sharp$ is local diffeomorphism, we conclude that there exists $n_0\in\mathbf{N}$ such that for all $n\geq n_0$ it holds

$$(w_n,\hat{v}_n) = (\delta_n,\hat{\sigma}_n)$$

which contradicts (41).

We have finished the proof of the part a) of Theorem 2.

The proof of the part b) goes analogously. ∎

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