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AN ITERATIVE METHOD FOR VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we introduce a new iterative scheme to investigate the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a relaxed (γ, r)-cocoercive, Lipschitz continuous mapping. Our results improve and extend the corresponding results of many others.

1. Introduction and preliminaries

Variational inequalities introduced by Stampacchia [7] in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences. It is well known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation has been used to suggest and analyze in variational inequalities. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. It is well known that the convergence of a projection method requires the operator to be strongly monotone and Lipschitz continuous. Gabay [1] has shown that the convergence of a projection method can be proved for cocoercive operators. Note that cocoercivity is a weaker condition than strong monotonicity.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and $A\colon C \to H$ a nonlinear map. Let P_C be the projection of H onto the convex subset C. The classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \tag{1.1}$$

for all $v \in C$. Next, we denote the solution of the variational inequality (1.1) by $VI(C, A)$. We now recall some well-known concepts and results.

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Lemma 1.1. *For a given $z \in H$, $u \in C$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{1.2}$$

for $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y \rangle \geq 0$ for all $y \in C$.

Using Lemma 1.1, one can show that the variational inequality (1.1) is equivalent to a fixed point problem.

Lemma 1.2. *The element $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda A u)$, where $\lambda > 0$ is a constant.*

It is clear from Lemma 1.2 that variational inequalities and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Recall that the following definitions:

(1) A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0,$$

for all $u, v \in C$.

(2) A is called μ-strongly monotone, if each $x, y \in C$, we have

$$\langle Ax - Ay, x - y \rangle \geq \mu \|x - y\|^2,$$

for a constant $v > 0$. This implies that

$$\|Ax - Ay\| \geq \mu \|x - y\|,$$

that is, A is μ-expansive and when $\mu = 1$, it is expansive.

(3) A is said to be γ-cocoercive [10,11], if for each $x, y \in C$, we have

$$\langle Ax - Ay, x - y \rangle \geq \gamma \|Ax - Ay\|^2, \quad \text{for a constant } \gamma > 0.$$

Clearly, every γ-cocoercive map A is $1/\gamma$-Lipschitz continuous.

(4) A is called relaxed γ-cocoercive, if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\gamma) \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

(5) A is said to be relaxed (γ, r)-cocoercive, if there exist two constants $\gamma, r > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\gamma) \|Ax - Ay\|^2 + r \|x - y\|^2, \quad \forall x, y \in C.$$

(6) A mapping $S : C \to C$ is called nonexpansive if $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$. Next, we denote by $F(S)$ the set of fixed points of S.

(7) A set-valued mapping $T\colon H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \ge 0$. A monotone mapping $T\colon H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \ge 0,\ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [6].

For finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of variational inequalities for α-cocoerceive map, Takahashi and Toyoda [9] introduced the following iterative process:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n) \tag{1.3}$$

for every $n = 0, 1, 2, \dots$, where A is α-cocoerceive, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.3) converges weakly to some $z \in F(S) \cap VI(C, A)$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbf{R}^n$ under the assumption that a set $C \subset \mathbf{R}^n$ is closed and convex, a mapping A of C into $\mathbf{R}^n$ is monotone and k-Lipschitz-continuous and $VI(C, A)$ is nonempty, Korpelevich [3] introduced the following so-called extra-gradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{1.4}$$

for every $n = 0, 1, 2, \dots$, where $\lambda \in (0, 1/k)$. He proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative process converge to the same point $z \in VI(C, A)$. Nadezhkina and Takahashi [4], Zeng and Yao [15] introduced some new iterative schemes for finding elements in $F(S) \cap VI(C, A)$ by combining (1.3) and (1.4).

Recently, Iiduka and Takahashi [2] proposed another iterative scheme as following:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n) \tag{1.5}$$

for every $n = 0, 1, 2, \dots$, where A is α-cocoerceive, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that the sequence $\{x_n\}$ converges strongly to $z \in F(S) \cap VI(C, A)$.

Very recently, Yao and Yao [13], introduced the following iterative process: $x_0 \in C$ and

$$\begin{cases} y_n = P_C(I - \lambda_n A) x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(I - \lambda_n A) y_n, \quad n \ge 0, \end{cases}$$

They also proved the sequence $\{x_n\}$ defined by about iterative process converges strongly to some point $z \in F(S) \cap VI(C, A)$.

In this paper, we introduce a new iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequalities for relaxed (γ, r)-cocoercive mappings in a real Hilbert space. The results are obtained in this paper improve and extend the recent ones announced by Yao and Yao [13], Zeng and Yao [15], Iiduka and Takahashi [2] and some others.

In order to prove our main results, we need the following lemmas.

Lemma 1.3. (Osilike and Igbokwe [5]). *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, we have*

$$\|\alpha_n x + \beta_n y + \gamma_n z\|^2 \leq \alpha_n\|x\|^2 + \beta_n\|y\|^2 + \gamma_n\|z\|^2 - \alpha_n\gamma_n\|x - z\|^2 - \alpha_n\beta_n\|x - y\|^2 - \beta_n\gamma_n\|y - z\|^2.$$

Lemma 1.4 (Suzuki [8]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5 (Xu [12]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where γ_n is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Theorem 2.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $A\colon C \to H$ be relaxed (γ, r)-cocoercive and μ-Lipschitz continuous. Let $S\colon C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C, A) \neq \emptyset$. $\{x_n\}$ is a sequence generated by the following algorithm: $x_0 \in C$ and*

$$\begin{cases} z_n = \omega_n x_n + (1 - \omega_n)P_C(I - t_n A)x_n, \\ y_n = \delta_n x_n + (1 - \delta_n)P_C(I - s_n A)z_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(I - r_n A)y_n, \quad n \geq 0, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *and* $\{\omega_n\}$ *are sequences in* $(0,1)$. *If* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\omega_n\}$, $\{r_n\}$, $\{s_n\}$ *and* $\{t_n\}$ *are chosen such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = \lim_{n\to\infty} |s_{n+1} - s_n| = \lim_{n\to\infty} |t_{n+1} - t_n| = 0$;

(v) $\{r_n\}, \{s_n\}, \{t_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < \frac{2(r-\gamma\mu^2)}{\mu^2}$ *and* $r > \gamma\mu^2$;

(vi) $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \omega_n = 0$.

Then $\{x_n\}$ *converges strongly to* $P_{F(S)\cap VI(C,A)}u$.

Proof. First, we show the mapping $I - s_nA$, $I - r_nA$ and $I - t_nA$ are nonexpansive, respectively. Indeed, from the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T and condition (v), we have

$$\begin{aligned}
&\|(I - s_nA)x - (I - s_nA)y\|^2 \\
&\quad = \|(x-y) - s_n(Ax - Ay)\|^2 \\
&\quad = \|x-y\|^2 - 2s_n\langle x-y, Ax-Ay\rangle + s_n^2\|Ax-Ay\|^2 \\
&\quad \le \|x-y\|^2 - 2s_n[-\gamma\|Ax-Ay\|^2 + r\|x-y\|^2] + s_n^2\|Ax-Ay\|^2 \qquad (2.2) \\
&\quad \le \|x-y\|^2 + 2s_n\mu^2\gamma\|x-y\|^2 - 2s_nr\|x-y\|^2 + \mu^2 s_n^2\|x-y\|^2 \\
&\quad = (1 + 2s_n\mu^2\gamma - 2s_nr + \mu^2 s_n^2)\|x-y\|^2 \\
&\quad \le \|x-y\|^2,
\end{aligned}$$

which implies that the mapping $I - s_nA$ is nonexpansive, and so are $I - t_nA$ and $I - r_nA$. Next, we show the sequence $\{x_n\}$ is bounded. Letting $p \in F(S)\cap VI(C,A)$, we have

$$\begin{aligned}
\|z_n - p\| &= \|\omega_n(x_n - p) + (1-\omega_n)(P_C(I - t_nA)x_n - p)\| \\
&\le \omega_n\|x_n - p\| + (1-\omega_n)\|x_n - p\| = \|x_n - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_n - p\| &= \|\delta_n(x_n - p) + (1-\delta_n)(P_C(I - s_nA)z_n - p)\| \\
&\le \delta_n\|x_n - p\| + (1-\delta_n)\|z_n - p\| \\
&\le \delta_n\|x_n - p\| + (1-\delta_n)\|x_n - p\| = \|x_n - p\|,
\end{aligned}$$

which yields that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(u-p) + \beta_n(x_n - p) + \gamma_n(SP_C(I - r_nA)y_n - p)\| \\
&\le \alpha_n\|u-p\| + \beta_n\|x_n - p\| + \gamma_n\|SP_C(I - r_nA)y_n - p\| \\
&\le \alpha_n\|u-p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\
&\le \alpha_n\|u-p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\
&= (1-\alpha_n)\|x_n - p\| + \alpha_n\|u-p\| \\
&\le \max\{\|x_n - p\|, \|u-p\|\}.
\end{aligned}$$

By simple inductions, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|u - p\|\}\},$$

which gives that the sequence $\{x_n\}$ is bounded, and so are $\{y_n\}$ and $\{z_n\}$. Put $\rho_n = P_C(I - r_nA)y_n$, $\theta_n = P_C(I - s_nA)z_n$ and $\eta_n = P_C(I - t_nA)x_n$. The iterative scheme (2.1) reduces to

$$\begin{cases} z_n = \omega_n x_n + (1 - \omega_n)\eta_n, \\ y_n = \delta_n x_n + (1 - \delta_n)\theta_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S\rho_n. \end{cases}$$

Next, we compute

$$\begin{aligned} \|\eta_n - \eta_{n+1}\| &= \|P_C(I - t_nA)x_n - P_C(I - t_{n+1}A)x_{n+1}\| \\ &\leq \|(I - t_nA)x_n - (I - t_{n+1}A)x_{n+1}\| \\ &= \|(x_n - t_nAx_n) - (x_{n+1} - t_nAx_{n+1}) + (t_{n+1} - t_n)Ax_{n+1}\| \\ &\leq \|x_n - x_{n+1}\| + |t_{n+1} - t_n|\|Ax_{n+1}\|, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \|\theta_n - \theta_{n+1}\| &= \|P_C(I - s_nA)z_n - P_C(I - s_{n+1}A)z_{n+1}\| \\ &\leq \|(I - s_nA)z_n - (I - s_{n+1}A)z_{n+1}\| \\ &= \|(z_n - s_nAz_n) - (z_{n+1} - s_nAz_{n+1}) + (s_{n+1} - s_n)Az_{n+1}\| \\ &\leq \|z_n - z_{n+1}\| + |s_{n+1} - s_n|\|Az_{n+1}\|. \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \|\rho_n - \rho_{n+1}\| &= \|P_C(I - r_nA)y_n - P_C(I - r_{n+1}A)y_{n+1}\| \\ &\leq \|(I - r_nA)y_n - (I - r_{n+1}A)y_{n+1}\| \\ &= \|(y_n - r_nAy_n) - (y_{n+1} - r_nAy_{n+1}) + (r_{n+1} - r_n)Ay_{n+1}\| \\ &\leq \|y_n - y_{n+1}\| + |r_{n+1} - r_n|\|Ay_{n+1}\|. \end{aligned} \tag{2.5}$$

Observing that

$$\begin{cases} z_n = \omega_n x_n + (1 - \omega_n)\eta_n, \\ z_{n+1} = \omega_{n+1} x_{n+1} + (1 - \omega_{n+1})\eta_{n+1}, \end{cases}$$

we have

$$z_n - z_{n+1} = \omega_n(x_n - x_{n+1}) + (1 - \omega_n)(\eta_n - \eta_{n+1}) + (\eta_{n+1} - x_{n+1})(\omega_{n+1} - \omega_n),$$

which yields that

$$\|z_n - z_{n+1}\| \leq \omega_n\|x_n - x_{n+1}\| + (1 - \omega_n)\|\eta_n - \eta_{n+1}\| + \|\eta_{n+1} - x_{n+1}\||\omega_{n+1} - \omega_n|. \tag{2.6}$$

Substitution of (2.3) into (2.6) yields that

$$\|z_n - z_{n+1}\| \leq \|x_n - x_{n+1}\| + M_1(|t_{n+1} - t_n| + |\omega_{n+1} - \omega_n|), \tag{2.7}$$

where M_1 is an appropriate constant such that $M_1 = \max\{\sup_{n\geq 0}\|Ax_n\|,\ \sup_{n\geq 0}\|\eta_n - x_n\|\}$. Observing that

$$\begin{cases} y_n = \delta_n x_n + (1 - \delta_n)\theta_n, \\ y_{n+1} = \delta_{n+1} x_{n+1} + (1 - \delta_{n+1})\theta_{n+1}, \end{cases}$$

we have

$$y_n - y_{n+1} = \delta_n(x_n - x_{n+1}) + (1-\delta_n)(\theta_n - \theta_{n+1}) + (\theta_{n+1} - x_{n+1})(\delta_{n+1} - \delta_n),$$

which yields that,

$$\|y_n - y_{n+1}\| \leq \delta_n\|x_n - x_{n+1}\| + (1-\delta_n)\|\theta_n - \theta_{n+1}\| + \|\theta_{n+1} - x_{n+1}\||\delta_{n+1} - \delta_n|. \quad (2.8)$$

Substituting (2.4) and (2.7) into (2.8), we get

$$\|y_n - y_{n+1}\| \leq \|x_n - x_{n+1}\| + M_2(|t_{n+1} - t_n| + |\omega_{n+1} - \omega_n| + |s_{n+1} - s_n| + |\delta_{n+1} - \delta_n|), \quad (2.9)$$

where M_2 is an appropriate constant such that $M_2 = \max\{M_1, \sup_{n\geq 0}\|Az_n\|, \sup_{n\geq 0}\|\theta_n - x_n\|\}$. Substituting (2.9) into (2.5), we obtain

$$\|\rho_n - \rho_{n+1}\| \leq M_3(|t_{n+1} - t_n| + |\omega_{n+1} - \omega_n| + |s_{n+1} - s_n| + |\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) + \|x_n - x_{n+1}\|, \quad (2.10)$$

where M_3 is an appropriate constant such that $M_3 = \max\{\sup_{n\geq 0}\|Ay_n\|, M_2\}$. Put $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$. That is, $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$. Now, we compute $l_{n+1} - l_n$. Observing that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}S\rho_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n S\rho_n}{1-\beta_n} \\ &= (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(S\rho_{n+1} - S\rho_n) \\ &\quad + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})S\rho_n, \end{aligned} \quad (2.11)$$

we have

$$\|l_{n+1} - l_n\| \leq |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|\rho_{n+1} - \rho_n\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|S\rho_n\|, \quad (2.12)$$

which combined with (2.10) yields that

$$\begin{aligned} \|l_{n+1} - l_n\| &\leq |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}[\|x_n - x_{n+1}\| \\ &\quad + M_3(|t_{n+1} - t_n| + |\omega_{n+1} - \omega_n| + |s_{n+1} - s_n| + |\delta_{n+1} - \delta_n| \\ &\quad + |r_{n+1} - r_n|)] + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|S\rho_n\| \\ &\leq |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|u\| + \|S\rho_n\|) \\ &\quad + \|x_n - x_{n+1}\| + M_3(|t_{n+1} - t_n| + |\omega_{n+1} - \omega_n| + |s_{n+1} - s_n| \\ &\quad + |\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|). \end{aligned} \quad (2.13)$$

It follows that

$$\begin{aligned}
&\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| \\
&\quad\le M_3(|t_{n+1}-t_n|+|\omega_{n+1}-\omega_n|+|s_{n+1}-s_n|+|\delta_{n+1}-\delta_n|+|r_{n+1}-r_n|) \\
&\quad\quad+(\|S\rho_n\|+\|u\|)|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|. \qquad (2.14)
\end{aligned}$$

Observe conditions (ii), (iv) and (vi) and take the limits as $n\to\infty$ to get

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|)\le 0.$$

We can obtain $\lim_{n\to\infty}\|l_n-x_n\|=0$ easily by Lemma 1.4. It follows from condition (iii) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|l_n-x_n\|=0. \qquad (2.15)$$

Observing that $x_{n+1}-x_n=\alpha_n(u-x_n)+\gamma_n(S\rho_n-x_n)$, we can easily get

$$\lim_{n\to\infty}\|S\rho_n-x_n\|=0. \qquad (2.16)$$

For $p\in F(S)\cap VI(C,A)$, we have

$$\begin{aligned}
\|\eta_n-p\|^2&=\|P_C(I-t_nA)x_n-P_C(I-t_nA)p\|^2 \\
&\le\|(x_n-p)-t_n(Ax_n-Ap)\|^2 \\
&=\|x_n-p\|^2-2t_n\langle x_n-p,Ax_n-Ap\rangle+t_n^2\|Ax_n-Ap\|^2 \\
&\le\|x_n-p\|^2-2t_n[-\gamma\|Ax_n-Ap\|^2+r\|x_n-p\|^2]+t_n^2\|Ax_n-Ap\|^2 \\
&\le\|x_n-p\|^2+2t_n\gamma\|Ax_n-Ap\|^2-2t_nr\|x_n-p\|^2+t_n^2\|Ax_n-Ap\|^2 \\
&\le\|x_n-p\|^2+(2t_n\gamma+t_n^2-\frac{2t_nr}{\mu^2})\|Ax_n-Ap\|^2. \qquad (2.17)
\end{aligned}$$

Similarly, we have

$$\|\theta_n-p\|^2\le\|x_n-p\|^2+(2s_n\gamma+s_n^2-\frac{2s_nr}{\mu^2})\|Az_n-Ap\|^2 \qquad (2.18)$$

and

$$\|\rho_n-p\|^2\le\|x_n-p\|^2+(2r_n\gamma+r_n^2-\frac{2r_nr}{\mu^2})\|Ay_n-Ap\|^2. \qquad (2.19)$$

It follows that

$$\begin{aligned}
\|z_n-p\|^2&=\|\omega_n(x_n-p)+(1-\omega_n)(\eta_n-p)\|^2 \\
&\le\omega_n\|x_n-p\|^2+(1-\omega_n)[\|x_n-p\|^2+(2t_n\gamma+t_n^2-\frac{2t_nr}{\mu^2})\|Ax_n-Ap\|^2] \\
&\le\|x_n-p\|^2+(2t_n\gamma+t_n^2-\frac{2t_nr}{\mu^2})\|Ax_n-Ap\|^2 \qquad (2.20)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n-p\|^2&=\|\delta_n(x_n-p)+(1-\delta_n)(\theta_n-p)\|^2 \\
&\le\|x_n-p\|^2+(2s_n\gamma+t_n^2-\frac{2s_nr}{\mu^2})\|Az_n-Ap\|^2. \qquad (2.21)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(u-p)+\beta_n(x_n-p)+\gamma_n(S\rho_n-p)\|^2\\ &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|S\rho_n-p\|^2\\ &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|\rho_n-p\|^2.\end{aligned} \tag{2.22}$$

Substituting (2.19) into (2.22) yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2\\ &\quad+\gamma_n(\|x_n-p\|^2+(2r_n\gamma+r_n^2-\frac{2r_nr}{\mu})\|Ay_n-Ap\|^2)\\ &\le \alpha_n\|u-p\|^2+\|x_n-p\|^2+(2r_n\gamma+r_n^2-\frac{2r_nr}{\mu^2})\|Ay_n-Ap\|^2.\end{aligned} \tag{2.23}$$

It follows from condition (v) that

$$\begin{aligned}&(\frac{2ar}{\mu^2}-2b\gamma-b^2)\|Ay_n-Ap\|^2\\ &\quad\le \alpha_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad= \alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)(\|x_n-p\|-\|x_{n+1}-p\|)\\ &\quad\le \alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.\end{aligned} \tag{2.24}$$

Because of (2.15) and $\lim_{n\to\infty}\alpha_n=0$, we have

$$\lim_{n\to\infty}\|Ay_n-Ap\|=0. \tag{2.25}$$

Using (2.22) again, we have

$$\|x_{n+1}-p\|^2\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2. \tag{2.26}$$

Substituting (2.21) into (2.26) yields that

$$\|x_{n+1}-p\|^2\le \alpha_n\|u-p\|^2+\|x_n-p\|^2+(2s_n\gamma+t_n^2-\frac{2s_nr}{\mu^2})\|Az_n-Ap\|^2). \tag{2.27}$$

It follows from condition (v) that

$$\begin{aligned}&(\frac{2ar}{\mu}-2b\gamma-b^2)\|Az_n-Ap\|^2\\ &\quad\le \alpha_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad= \alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)(\|x_n-p\|-\|x_{n+1}-p\|)\\ &\quad\le \alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.\end{aligned} \tag{2.28}$$

From (2.15) and $\lim_{n\to\infty}\alpha_n=0$, we have

$$\lim_{n\to\infty}\|Az_n-Ap\|=0. \tag{2.29}$$

In a similar way, we can prove

$$\lim_{n\to\infty}\|Ay_n-Ap\|=0. \tag{2.30}$$

Observe that

$$\begin{aligned}\|\rho_n - p\|^2 &= \|P_C(I - r_nA)y_n - P_C(I - r_nA)p\|^2\\ &\le \langle (I - r_nA)y_n - (I - r_nA)p, \rho_n - p\rangle\\ &= \frac{1}{2}\{\|(I - r_nA)y_n - (I - r_nA)p\|^2 + \|\rho_n - p\|^2\\ &\quad - \|(I - r_nA)y_n - (I - r_nA)p - (\rho_n - p)\|^2\}\\ &\le \frac{1}{2}\{\|y_n - p\|^2 + \|\rho_n - p\|^2 - \|(y_n - \rho_n) - r_n(Ay_n - Ap)\|^2\}\\ &= \frac{1}{2}\{\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|y_n - \rho_n\|^2 - r_n^2\|Ay_n - Ap\|^2\\ &\quad + 2r_n\langle y_n - \rho_n, Ay_n - Ap\rangle\},\end{aligned}$$

which yields that

$$\|\rho_n - p\|^2 \le \|x_n - p\|^2 - \|y_n - \rho_n\|^2 + 2r_n\langle y_n - \rho_n, Ay_n - Ap\rangle. \tag{2.31}$$

Similarly, we can prove

$$\|\theta_n - p\|^2 \le \|x_n - p\|^2 - \|z_n - \theta_n\|^2 + 2s_n\langle z_n - \theta_n, Az_n - Ap\rangle \tag{2.32}$$

and

$$\|\eta_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - \eta_n\|^2 + 2t_n\langle x_n - \eta_n, Ax_n - Ap\rangle. \tag{2.33}$$

Substituting (2.31) into (2.22) yields that

$$\begin{aligned}\|x_{n+1} - p\|^2 \le \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \gamma_n\|y_n - \rho_n\|^2\\ + 2\gamma_n r_n\langle y_n - \rho_n, Ay_n - Ap\rangle),\end{aligned}$$

which implies that

$$\begin{aligned}\gamma_n\|y_n - \rho_n\|^2 &\le \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\quad + 2\gamma_n r_n\langle y_n - \rho_n, Ay_n - Ap\rangle\\ &\le \alpha_n\|u - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\|\\ &\quad + 2\gamma_n r_n\|y_n - \rho_n\|\|Ay_n - Ap\|.\end{aligned}$$

Observing that (2.15), (2.25) and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\|y_n - \rho_n\| = 0. \tag{2.34}$$

Using (2.22) again, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|\rho_n - p\|^2\\ &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2\\ &\le \alpha_n\|u - p\|^2 + (\beta_n + \gamma_n\delta_n)\|x_n - p\|^2 + \gamma_n(1 - \delta_n)\|\theta_n - p\|^2.\end{aligned} \tag{2.35}$$

Substituting (2.32) into (2.35) yields that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|u - p\|^2 + \|x_n - p\|^2\\ &\quad - \gamma_n(1 - \delta_n)(\|z_n - \theta_n\|^2 - 2s_n\langle z_n - \theta_n, Az_n - Ap\rangle),\end{aligned}$$

which implies that

$$\begin{aligned}\gamma_n(1-\delta_n)\|z_n-\theta_n\|^2 \le{}& \alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\&+2\gamma_n(1-\delta_n)s_n\|z_n-\theta_n\|\|Az_n-Ap\|\end{aligned}$$

From conditions (ii), (v), (2.15) and (2.29), we obtain

$$\lim_{n\to\infty}\|z_n-\theta_n\|=0. \tag{2.36}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\|x_n-\eta_n\|=0. \tag{2.37}$$

On the other hand, we observe

$$\begin{aligned}\|S\rho_n-\rho_n\| \le{}& \|S\rho_n-x_n\|+\|x_n-\eta_n\|+\|\eta_n-z_n\|\\&+\|z_n-\theta_n\|+\|\theta_n-y_n\|+\|y_n-\rho_n\|.\end{aligned}$$

It follows from condition (vi), (2.16), (2.34), (2.36) and (2.37) that

$$\lim_{n\to\infty}\|S\rho_n-\rho_n\|=0. \tag{2.38}$$

Next, we show

$$\limsup_{n\to\infty}\langle u-q,x_n-q\rangle\le 0,$$

where $q=P_{F(S)\cap VI(C,A)}u$. To show it, we choose a subsequence $\{\rho_{n_i}\}$ of $\{\rho_n\}$ such that

$$\limsup_{n\to\infty}\langle u-q,S\rho_n-q\rangle=\lim_{i\to\infty}\langle u-q,S\rho_{n_i}-q\rangle.$$

As $\{\rho_{n_i}\}$ is bounded, we have that there is a subsequence $\{\rho_{n_{i_j}}\}$ of $\{\rho_{n_i}\}$ converges weakly to p. We may assume that without loss of generality that $\rho_{n_i}\rightharpoonup p$. Observing (2.38), we have $S\rho_{n_j}\rightharpoonup p$. Hence we have $p\in F(S)\cap VI(C,A)$. Indeed, let us first show that $p\in VI(C,A)$. Put

$$Tw_1=\begin{cases}Aw_1+N_Cw_1, & w_1\in C\\ \emptyset, & w_1\notin C.\end{cases}$$

Since A is relaxed (γ,r)-cocoercive and condition (v), we have

$$\langle Ax-Ay,x-y\rangle\ge(-\gamma)\|Ax-Ay\|^2+r\|x-y\|^2\ge(r-\gamma\mu^2)\|x-y\|^2\ge 0,$$

which yields that A is monotone. Thus T is maximal monotone. Let $(w_1,w_2)\in G(T)$. Since $w_2-Aw_1\in N_Cw_1$ and $\rho_n\in C$, we have

$$\langle w_1-\rho_n,w_2-Aw_1\rangle\ge 0.$$

On the other hand, from $\rho_n=P_C(I-r_nA)y_n$, we have

$$\langle w_1-\rho_n,\rho_n-(I-r_nA)y_n\rangle\ge 0$$

and hence

$$\langle w_1-\rho_n,\frac{\rho_n-y_n}{r_n}+Ay_n\rangle\ge 0.$$

It follows that

$$\begin{aligned}\langle w_1-\rho_{n_i}, w_2\rangle &\geq \langle w_1-\rho_{n_i}, Aw_1\rangle \geq \langle w_1-\rho_{n_i}, Aw_1\rangle \\ &\quad - \langle w_1-\rho_{n_i}, \frac{\rho_{n_i}-y_{n_i}}{r_{n_i}}+Ay_{n_i}\rangle \\ &\geq \langle w_1-\rho_{n_i}, Aw_1-\frac{\rho_{n_i}-y_{n_i}}{r_{n_i}}-Ay_{n_i}\rangle \\ &= \langle w_1-\rho_{n_i}, Aw_1-A\rho_{n_i}\rangle + \langle w_1-\rho_{n_i}, A\rho_{n_i}-Ay_{n_i}\rangle \\ &\quad - \langle w_1-\rho_{n_i}, \frac{\rho_{n_i}-y_{n_i}}{r_{n_i}}\rangle \\ &\geq \langle w_1-\rho_{n_i}, A\rho_{n_i}-Ay_{n_i}\rangle - \langle w_1-\rho_{n_i}, \frac{\rho_{n_i}-y_{n_i}}{r_{n_i}}\rangle\end{aligned}$$

which implies that $\langle w_1-p, w_2\rangle \geq 0$ as $i\to\infty$. We have $p\in T^{-1}0$ and hence $p\in VI(C,A)$. Next, let us show $p\in F(S)$. Since Hilbert spaces are Opial's spaces, from (2.38), we have

$$\begin{aligned}\liminf_{i\to\infty}\|\rho_{n_i}-p\| &< \liminf_{i\to\infty}\|\rho_{n_i}-Sp\| = \liminf_{i\to\infty}\|\rho_{n_i}-S\rho_{n_i}+S\rho_{n_i}-Sp\| \\ &\leq \liminf_{i\to\infty}\|S\rho_{n_i}-Sp\| \leq \liminf_{i\to\infty}\|\rho_{n_i}-p\|\end{aligned}$$

which derives a contradiction. Thus, we have $p\in F(S)$. On the other hand, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u-q, x_n-q\rangle &= \limsup_{n\to\infty}\langle u-q, S\rho_n-q\rangle = \lim_{i\to\infty}\langle u-q, S\rho_{n_i}-q\rangle \\ &= \langle u-q, p-q\rangle \leq 0.\end{aligned} \tag{2.39}$$

It follows that

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \langle \alpha_n u+\beta_n x_n+\gamma_n S\rho_n-q, x_{n+1}-q\rangle \\ &= \alpha_n\langle u-q, x_{n+1}-q\rangle + \beta_n\langle x_n-q, x_{n+1}-q\rangle \\ &\quad + \gamma_n\langle S\rho_n-q, x_{n+1}-q\rangle \\ &\leq \alpha_n\langle u-q, x_{n+1}-q\rangle + \frac{1}{2}\beta_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\ &\quad + \frac{1}{2}\gamma_n(\|S\rho_n-q\|^2+\|x_{n+1}-q\|^2) \\ &\leq \alpha_n\langle u-q, x_{n+1}-q\rangle + \frac{1}{2}\beta_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\ &\quad + \frac{1}{2}\gamma_n(\|\rho_n-q\|^2+\|x_{n+1}-q\|^2) \\ &\leq \alpha_n\langle u-q, x_{n+1}-q\rangle + \frac{1}{2}\beta_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\ &\quad + \frac{1}{2}\gamma_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\ &\leq \alpha_n\langle u-q, x_{n+1}-q\rangle + \frac{1}{2}(1-\alpha_n)(\|x_n-q\|^2+\|x_{n+1}-q\|^2),\end{aligned}$$

which yields that

$$\|x_{n+1}-q\|^2 \leq (1-\alpha_n)\|x_n-q\|^2 + 2\alpha_n\langle u-q, x_{n+1}-q\rangle. \tag{2.40}$$

By Lemma 1.5, we can conclude the desired conclusion easily. This completes the proof. ∎

As corollaries of Theorem 2.1, we have the following results immediately.

Corollary 2.2. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $A : C \to H$ be relaxed (γ, r)-cocoercive and μ-Lipschitz continuous. Let $S\colon C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C, A) \neq \emptyset$. $\{x_n\}$ is a sequence generated by the following algorithm: $x_0 \in C$ and*

$$\begin{cases} y_n = P_C(I - s_n A)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(I - r_n A)y_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0, 1)$. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_n\}$ and $\{s_n\}$ are chosen such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = \lim_{n\to\infty} |s_{n+1} - s_n| = 0$;

(v) $\{r_n\}, \{s_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{2(r-\gamma\mu^2)}{\mu^2}$ *and* $r > \gamma\mu^2$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}u$.

Corollary 2.3. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $A : C \to H$ be relaxed (γ, r)-cocoercive and μ-Lipschitz continuous. Let $S\colon C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C, A) \neq \emptyset$. $\{x_n\}$ is a sequence generated by the following algorithm:*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S x_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0, 1)$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are chosen such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $P_{F(S)}u$.

REFERENCES

[**1**] D. Gabay, *Applications of the Method of Multipliers to Variational Inequalities, Augmented Lagrangian Methods*, Edited by M. Fortin and R. Glowinski, North-Holland, Amsterdam, Holland, (1983), 299–331.

[**2**] H. Iiduka, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal. **61** (2005), 341–350.

[**3**] G.M. Korpelevich, *An extragradient method for finding saddle points and for other problems*, Ekonomika i Matematicheskie Metody **12** (1976), 747–756.

[**4**] N. Nadephkina, W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **128** (2006), 191–201.

[**5**] M.O. Osilike, D.I. Igbokwe, *Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations*, Comput. Math. Appl. **40** (2000), 559–567.

[**6**] R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

[**7**] G. Stampacchia, *Formes bilineaires coercivites sur les ensembles convexes*, Comptes rendus Acad. Sci., Paris **258** (1964), 4413–4416.

[**8**] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. **305** (2005), 227–239.

[**9**] W. Takahashi, M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[**10**] R. U. Verma, *Generalized system for relaxed cocoercive variational inequalities and its projection methods*, J. Optim. Theory Appl. **121** (1) (2004), 203–210.

[**11**] R.U. Verma, *General convergence analysis for two-step projection methods and application to variational problems*, Appl. Math. Lett. **18** (11) (2005), 1286–1292.

[**12**] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), 240–256.

[**13**] Y. Yao, J. C. Yao, *On modified iterative method for nonexpansive mappings and monotone mappings*, Appl. Math. Comput. **186** (2007), 1551–1558.

[**14**] J. C. Yao, *Variational inequalities with generalized monotone operators*, Math. Operations Research **19** (1994), 691–705.

[**15**] L.C. Zeng, J. C. Yao, *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems*, Taiwanese J. Math. **10** (2006), 1293–1303.

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