

## A NOTE ON GENERALIZED RECURRENT RIEMANNIAN MANIFOLD

M. Prvanović

**Abstract.** Generalized recurrent Riemannian manifold is a Riemannian manifold whose curvature tensor satisfies the condition (1). In this paper we prove: If the associated 1-form satisfies the condition (3), where  $\gamma \neq 1 \neq 2$ , or, in the case  $\gamma \neq \text{const}$ ,  $\gamma_s A^s \neq 0$ , generalized recurrent Riemannian manifold reduces to a recurrent one.

Let us consider a non-flat Riemannian manifold  $(M, g)$  of dimension  $n \geq 2$ , whose curvature tensor satisfies

$$\nabla_r R_{ijkl} = 2A_r R_{ijkl} + A_i R_{rjkl} + A_j R_{irkl} + A_k R_{ijrl} + A_l R_{ijk r}, \quad (1)$$

where  $A$  is a non-zero 1-form and  $\nabla$  denotes the operator of covariant differentiation with respect to the metric  $g$ . If besides (1) the curvature tensor satisfies

$$A_r R_{ijkl} + A_k R_{ijlr} + A_l R_{ijrk} = 0, \quad (2)$$

the condition (1) reduces to

$$\nabla_r R_{ijkl} = 4A_r R_{ijkl},$$

i.e.  $(M, g)$  is a recurrent manifold. Conversely, every recurrent manifold also satisfies the condition of the form (1). This is the reason for calling the manifold satisfying (1), *generalized recurrent Riemannian manifold*. The 1-form  $A$  will be called its *associated 1-form*.

This type of manifold was introduced by M. C. Chaki [1] and called pseudo-symmetric manifold by him. We avoid this name because it is used for other type of manifolds (see for ex. [3]).

In [4] we proved that if the associated 1-form satisfies

$$\nabla_j A_i = \alpha g_{ij} + \gamma A_i A_j, \quad A_t A^t = 0,$$

where  $\alpha$  and  $\gamma$  are some functions, generalized recurrent Riemannian manifold is a manifold of quasi-constant curvature. We also proved that if  $\alpha = 0$ , i.e. if

$$\nabla_j A_i = \gamma A_i A_j, \quad (3)$$

then  $A_t A^t = 0$ .

The object of this paper is to investigate generalized recurrent Riemannian manifold satisfying (3). In fact, we shall prove

**THEOREM.** *If the associated 1-form satisfies (3) where  $\gamma \neq 1 \neq 2$  or, in the case  $\gamma \neq \text{const}$ ,  $\gamma_s A^s \neq 0$ ,  $\gamma_s = \partial\gamma/\partial x^s$ , generalized recurrent Riemannian manifold reduces to a recurrent one.*

*Proof.* First, we note that for every generalized recurrent Riemannian space, the associated 1-form is a gradient and the following is satisfied [4]

$$\begin{aligned} \nabla_r \nabla_s R_{ijkl} - \nabla_s \nabla_r R_{ijkl} &= A_{ri} R_{sjkl} + A_{rj} R_{iskl} + A_{rk} R_{ijsl} + A_{rl} R_{ijks} \\ &\quad - A_{si} R_{rjkl} - A_{sj} R_{irkl} - A_{sk} R_{ijrl} - A_{sl} R_{ijkr}, \end{aligned} \quad (4)$$

where we have put  $A_{ri} = \nabla_r A_i - A_r A_i$ .

Using the Ricci identity

$$\nabla_r \nabla_s R_{ijkl} - \nabla_s \nabla_r R_{ijkl} = -R_{ajkl} R^a_{isr} - R_{iakl} R^a_{jsr} - R_{ijal} R^a_{ksr} - R_{ijka} R^a_{lsr},$$

the relation (4) can be written in the form

$$\begin{aligned} R_{ajkl} R^a_{isr} + R_{iakl} R^a_{jsr} + R_{ijal} R^a_{ksr} + R_{ijka} R^a_{lsr} \\ = A_{si} R_{rjkl} + A_{sj} R_{irkl} + A_{sk} R_{ijrl} + A_{sl} R_{ijkr} \\ - A_{ri} R_{sjkl} - A_{rj} R_{iskl} - A_{rk} R_{ijsl} - A_{rl} R_{ijks}. \end{aligned} \quad (5)$$

Applying the operator  $\nabla_m$  and using (1) and (5), we get, after some calculations,

$$\begin{aligned} B_{sim} R_{rjkl} + B_{sjm} R_{irkl} + B_{skm} R_{ijrl} + B_{slm} R_{ijkr} \\ - B_{rim} R_{sjkl} - B_{rjm} R_{iskl} - B_{rkm} R_{ijsl} - B_{rlm} R_{ijks} \\ + A_a R^a_{isr} R_{mjkl} + A_a R^a_{jsr} R_{imkl} + A_a R^a_{ksr} R_{ijml} + A_a R^a_{lsr} R_{ijkm} \\ + A_a R^a_{jkl} R_{misr} - A_a R^a_{ikl} R_{mjsr} + A_a R^a_{lij} R_{mkrs} - A_a R^a_{kij} R_{mlsr} = 0, \end{aligned} \quad (6)$$

where we have put  $B_{ijm} = 2A_m A_{ij} + A_i A_{mj} + A_j A_{im} - \nabla_m A_{ij}$ .

But, in view of (3) we have

$$B_{ijm} = -\gamma_m A_i A_j - 2(\gamma - 1)(\gamma - 2) A_i A_j A_m \quad (7)$$

and

$$\nabla_k \nabla_j A_i - \nabla_j \nabla_k A_i = (\gamma_k A_j - \gamma_j A_k) A_i,$$

or, using the Ricci identity,

$$A_a R^a_{ikj} = (\gamma_k A_j - \gamma_j A_k) A_i. \quad (8)$$

In the sequel we shall consider the cases  $\gamma \neq \text{const}$  and  $\gamma = \text{const}$  separately.

The case  $\gamma \neq \text{const}$ ,  $\gamma_s A^s \neq 0$ . First, we note that  $A^i B_{ijm} = A^i B_{jim} = 0$  because of (7) and  $A_t A^t = 0$ . Also,  $A^a A^k R_{aikj} = \gamma_t A^t A_i A_j$ , because of (8). Thus, transvecting (6) with  $A^s$ , we get

$$\begin{aligned} & -B_{rim} A_a R^a_{jkl} + B_{rjm} A_a R^a_{ikl} - B_{rkm} A_a R^a_{lij} + B_{rlm} A_a R^a_{kij} \\ & \quad + \gamma_t A^t A_r [A_i R_{mjkl} + A_j R_{imkl} + A_k R_{ijml} + A_l R_{ijkm}] \\ & + A_a R^a_{jkl} A_s R^s_{rmi} - A_a R^a_{ikl} A_s R^s_{rmj} + A_a R^a_{lij} A_s R^s_{rmk} - A_a R^a_{kij} A_s R^s_{rml} = 0. \end{aligned}$$

Substituting (7) and (8), we have

$$\begin{aligned} & A_r \{ \gamma_t A^t [A_i R_{mjkl} + A_j R_{imkl} + A_k R_{ijml} + A_l R_{ijkm}] \\ & \quad + A_i (\gamma_k \gamma_j A_l A_m - \gamma_l \gamma_j A_k A_m) + A_j (\gamma_l \gamma_i A_k A_m - \gamma_k \gamma_i A_l A_m) \\ & \quad + A_l (\gamma_j \gamma_k A_i A_m - \gamma_i \gamma_k A_j A_m) + A_k (\gamma_i \gamma_l A_j A_m - \gamma_j \gamma_l A_i A_m) \} = 0. \end{aligned}$$

Let  $\vartheta$  be a vector such that  $A_r \vartheta^r = 1$ . Then, transvecting the preceding relation by  $\vartheta^r$ , we get

$$\begin{aligned} & A_i (\gamma_s A^s R_{mjkl} + \gamma_k \gamma_j A_l A_m + \gamma_l \gamma_m A_k A_j - \gamma_l \gamma_j A_k A_m - \gamma_k \gamma_m A_l A_j) \\ & \quad + A_j (\gamma_s A^s R_{imkl} + \gamma_l \gamma_i A_k A_m + \gamma_k \gamma_m A_l A_i - \gamma_k \gamma_i A_l A_m - \gamma_l \gamma_m A_k A_i) \\ & \quad + A_k (\gamma_s A^s R_{ijml} + \gamma_i \gamma_l A_j A_m + \gamma_j \gamma_m A_i A_l - \gamma_j \gamma_l A_i A_m - \gamma_i \gamma_m A_j A_l) \\ & \quad + A_l (\gamma_s A^s R_{ijkm} + \gamma_j \gamma_k A_i A_m + \gamma_i \gamma_m A_j A_k - \gamma_i \gamma_k A_j A_m - \gamma_j \gamma_m A_i A_k) = 0. \end{aligned}$$

This can be written in the form

$$A_i T_{mjkl} + A_j T_{imkl} + A_k T_{ijml} + A_l T_{ijkm} = 0, \quad (9)$$

where we have put

$$T_{mjkl} = \gamma_s A^s R_{mjkl} + \gamma_k \gamma_j A_l A_m + \gamma_l \gamma_m A_k A_j - \gamma_l \gamma_j A_k A_m - \gamma_k \gamma_m A_l A_j.$$

We see that

$$T_{mjkl} = -T_{jmkl}, \quad T_{mjkl} = T_{klmj}, \quad T_{mjkl} + T_{mklj} + T_{mljk} = 0. \quad (10)$$

Thus, we can use the following lemma ([5], lemma 4).

If  $A_i$  and  $T_{mjkl}$  are numbers satisfying (9) and (10), then either each  $A_i$  is zero or each  $T_{mjkl}$  is zero.

According to our assumption,  $A$  is non-zero 1-form. Thus  $T_{mjkl} = 0$ , i.e.

$$\gamma_s A^s R_{mjkl} = \gamma_l \gamma_j A_k A_m + \gamma_k \gamma_m A_l A_j - \gamma_k \gamma_j A_l A_m - \gamma_l \gamma_m A_k A_j,$$

from which we get

$$\gamma_s A^s (A_r R_{ijkl} + A_k R_{ijlr} + A_l R_{ijrk}) = 0.$$

Therefore, if  $\gamma_s A^s \neq 0$ , the condition (2) is satisfied. But this means that generalized recurrent Riemannian manifold reduces to recurrent one.

Case  $\gamma = \text{const}$ ,  $\gamma \neq 1 \neq 2$ . In this case, (7) and (8) reduce to

$$B_{ijm} = -2(\gamma - 1)(\gamma - 2)A_i A_j A_m, \quad A_a R^a{}_{ikj} = 0$$

respectively and (6) becomes

$$\begin{aligned} A_m (A_s A_i R_{rjkl} + A_s A_j R_{irk l} + A_s A_k R_{ijrl} + A_s A_l R_{ijk r} \\ - A_r A_i R_{sjkl} - A_r A_j R_{isk l} - A_r A_k R_{ijs l} - A_r A_l R_{ijk s}) = 0. \end{aligned}$$

Let  $\vartheta$  be a vector such that  $A_i \vartheta^i = 1$ . Then, transvecting the preceding relation with  $\vartheta^m$ , we find

$$\begin{aligned} A_s A_i R_{rjkl} + A_s A_j R_{irk l} + A_s A_k R_{ijrl} + A_s A_l R_{ijk r} \\ - A_r A_i R_{sjkl} - A_r A_j R_{isk l} - A_r A_k R_{ijs l} - A_r A_l R_{ijk s} = 0. \end{aligned} \quad (11)$$

Now, we apply the method used in [2], Proposition 4.1. Let us put

$$\vartheta^i R_{ijkl} = C_{jkl}, \quad \vartheta^a \vartheta^b R_{aijb} = D_{ij}.$$

We see that

$$C_{jkl} = -C_{jlk}, \quad C_{jkl} + C_{klj} + C_{ljk} = 0, \quad D_{ij} = D_{ji}, \quad \vartheta^a D_{aj} = 0.$$

Transvecting (11) with  $\vartheta^i \vartheta^s$ , we get

$$R_{rjkl} + A_j C_{rkl} + A_k C_{jrl} + A_l C_{jkr} + A_r C_{jlk} + A_r A_k D_{jl} - A_r A_l D_{jk} = 0. \quad (12)$$

A cyclic permutation of  $r$ ,  $k$  and  $l$  gives

$$A_k C_{jrl} + A_r C_{jlk} + A_l C_{jkr} = 0,$$

because of which (12) reduces to

$$R_{rjkl} + A_j C_{rkl} + A_r A_k D_{jl} - A_r A_l D_{jk} = 0. \quad (13)$$

Trnsvecting (13) with  $\vartheta^l$  we get  $C_{kjr} = A_r D_{jk} - A_j D_{rk}$ . Substituting this into (13), we obtain

$$R_{rjkl} = A_r A_l D_{jk} - A_r A_k D_{jl} + A_j A_k D_{rl} - A_j A_l D_{rk}.$$

But this relation implies (2).

This completes the proof of the Theorem. ■

#### REFERENCES

- [1] M. C. Chaki, *On pseudo-symmetric manifolds*, Analele Stiintifice ale Univer. "Al.I.Cuza" din Iași, **33**, s.Ia, Matematica, 1987, 53–58
- [2] F. Defever, R. Deszcz, *On semi-Riemannian manifolds satisfying the condition  $R.R = Q(S, R)$* , in: *Geometry and Topology of Submanifolds, III*, Leeds, May 1990, World Sci. Publ., Singapore, 1991
- [3] R. Deszcz, L. Verstraelen, L. Vrancken, *The Symmetry of Warped Product Space-times*, General Relativity and Gravitation, **23**, 6, 1991,
- [4] M. Prvanović, *Generalized recurrent Riemannian manifold*, to be published in Analele Stiintifice ale Univer. "Al.I.Cuza" din Iași
- [5] W. Roter, *On generalized curvature tensors on some Riemannian manifolds*, Colloq. Math. **37**, 2, 1977, 233–240

(received 14 06 1993)

Ljube Stojanovića 14, Beograd