МАТЕМАТИЧКИ ВЕСНИК
51 (1999), 1–8

UDK 517.521
оригинални научни рад
research paper

BAIRE'S SPACE OF PERMUTATIONS OF N AND REARRANGEMENTS OF SERIES

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Abstract. In the first part of the paper we investigate the structure of the space (S, d) of all sequences of positive integers with Baire's metric. In the second part we study properties of the space (E, d) of all permutations of $\mathbf{N}$ in connection with rearrangements of non-absolutely convergent series.

0. Introduction

There are several papers investigating rearrangements of non-absolutely convergent series from the point of view of permutations of the set $\mathbf{N}$ considered as points of a metric space (cf. [1], [3], [4], [5], [7], [8], [9], [10]). The mentioned metric space is endowed with the Fréchet's metric d_1 in [1], [3], [4], [7], [9], [10] and with Baire's metric d in [5]. Let us remark that these two metrics are equivalent on the set S (and also on E) of all sequences of positive integers (of all permutations of $\mathbf{N}$) and therefore many properties of (S, d_1) can be transfered from (S, d_1) to (S, d) (or from (E, d_1) to (E, d)), and conversely.

Several of our considerations will be based on the following classical Riemann's theorem on rearrangements of series with real terms (cf. [5]).

THEOREM A. *Let $\sum_{k=1}^{\infty} a_k$ be a non-absolutely convergent series with real terms, let $-\infty \leqslant t_1 \leqslant t_2 \leqslant +\infty$. Then there exists a permutation $x = (x_j)_1^{\infty} \in E$ such that* $\liminf_{n\to\infty} S_n(x) = t_1$, $\limsup_{n\to\infty} S_n(x) = t_2$, *where* $S_n(x) = \sum_{j=1}^{n} a_{x_j}$ $(n = 1, 2, \dots)$.

Particularly, for every $r \in \mathbf{R}$ there exists a permutation $x = (x_j)_1^{\infty} \in E$ such that

$$\sum_{j=1}^{\infty} a_{x_j} = r. \tag{1}$$

AMS Subject Classification: 40 A 05

Keywords and phrases: Rearrangements of series, Baire's space, set of first category, residual set, σ-porosity, strong porosity

The set of all permutations (of $\mathbf{N}$) $x = (x_j)_1^\infty$ with (1) will be denoted by P_r (in agreement with [5]). The set P_r depends obviously on the sequence $a_1, a_2, \dots$ and therefore we shall write in detail $P_r = P_r(a_1, a_2, \dots)$. Put

$$E_0 = E_0(a_1, a_2, \dots) = \bigcup_{r \in \mathbf{R}} P_r(a_1, a_2, \dots).$$

Although the sets $E_0(a_1, a_2, \dots)$, $P_r(a_1, a_2, \dots)$ depend on $a_1, a_2, \dots$, they have several common properties for all $a_1, a_2, \dots$ such that $\sum_{k=1}^\infty a_k$ is a non-absolutely convergent series.

Recall the concept of metrics d_1 and d (on S or E). Let $x = (x_j)_1^\infty \in S$, $y = (y_j)_1^\infty \in S$. Then we put

$$d_1(x, y) = \sum_{k=1}^\infty \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}.$$

If $x = y$, then $d(x, y) = 0$ and if $x \neq y$, then $d(x, y) = 1/m$, where $m = \min\{\, j : x_j \neq x_i \,\}$ (cf. [1], [2], p. 185).

We shall use the concept of porosity of sets (cf. [12], [13]). Let (Y, ρ) be a metric space, $y \in Y$ and $r > 0$. Then by $K(y, r)$ denote the ball with centre y and radius r, i.e. $K(y, r) = \{\, x \in Y : \rho(x, y) < r \,\}$. Let $M \subseteq Y$. Put

$$\gamma(y, r, M) = \sup\{\, t > 0 : (\exists z \in Y)\, [K(z, t) \subseteq K(y, r)] \wedge [K(z, t) \cap M = \varnothing] \,\},$$

$\overline{p}(y, M) = \limsup_{r \to 0+} \gamma(y, r, M)/r$, $\underline{p}(y, M) = \liminf_{r \to 0+} \gamma(y, r, M)/r$ and if $\overline{p}(y, M) = \underline{p}(y, M)$, then set

$$p(y, M) = \overline{p}(y, M) = \underline{p}(y, M) = \lim_{r \to 0+} \frac{\gamma(y, r, M)}{r}.$$

Obviously, each of the numbers $\overline{p}(y, M)$, $\underline{p}(y, M)$, $p(y, M)$ belongs to the interval $[0, 1]$.

A set $M \subseteq Y$ is said to be porous (c-porous) at y, provided that $\overline{p}(y, M) > 0$ ($\overline{p}(y, M) \geqslant c > 0$). A set $M \subseteq Y$ is said to be σ-porous (σ-c-porous) at y provided that $M = \bigcup_{n=1}^\infty M_n$, M_n $(n = 1, 2, \dots)$ being porous (c-porous) at y.

Let $Y_0 \subseteq Y$. A set $M \subseteq Y$ is said to be porous, c-porous, σ-porous and σ-c-porous in the set Y_0 if M is porous, c-porous, σ-porous and σ-c-porous at every point $y \in Y_0$, respectively.

If a set M is c-porous and σ-c-porous at y, then obviously it is porous and σ-porous at y, respectively.

Every porous set M in Y ia a nowhere dense set in Y and therefore every σ-porous set M in Y is a set of the first Baire category in Y. The converse is not true already in $\mathbf{R}$ (cf. [11]).

From the definition of numbers $\overline{p}(y, M)$, $\underline{p}(y, M)$ we get at once: If $M_1 \subseteq M_2 \subseteq Y$, then for each $y \in Y$ we have $\overline{p}(y, M_1) \geqslant \overline{p}(y, M_2)$, $\underline{p}(y, M_1) \geqslant \underline{p}(y, M_2)$.

Let Y be a metric space again. A set $M \subseteq Y$ is said to be very porous at $y \in Y$ if $\underline{p}(y, M) > 0$ and very strongly porous at y if $p(y, M) = 1$ (cf. [1], p. 327). The set M is said to be very (strongly) porous in $Y_0 \subseteq Y$ if it is very (strongly) porous at each point $y \in Y_0$.

Obviously, if M is very porous at y, it is porous at y, as well. Analogously, if M is very strongly porous at y, it is 1-porous at y.

Further, a set $M \subseteq Y$ is said to be uniformly very porous (in $Y_1 \subseteq Y$) provided that there is a $c > 0$ such that for each $y \in Y_1$ we have $\underline{p}(y, M) \geqslant c > 0$ (cf. [13], p. 327).

In agreement with the previous terminology and in analogy with the notion of σ-porosity we introduce the following concept of uniformly very σ-porous sets.

Definition. 1) A set $M \subseteq Y$ is said to be uniformly very σ-porous (in $Y_0 \subseteq Y$) provided that $M = \bigcup_{n=1}^{\infty} M_n$ and there is a $c > 0$ such that for each $y \in Y_0$ we have $\underline{p}(y, M_n) \geqslant c > 0$ $(n = 1, 2, \dots)$.

2) A set $M \subseteq Y$ is said to be uniformly very strongly σ-porous (in $Y_0 \subseteq Y$) provided that $M = \bigcup_{n=1}^{\infty} M_n$ and for each $y \in Y_0$ we have $p(y, M_n) = 1$ $(n = 1, 2, \dots)$.

If $A \subseteq Y$, then by CA we denote the complement of A, $CA = Y \setminus A$.

In the first part of the paper we shall investigate the structure of the space (S, d). In the second part we shall study properties of the space (E, d) in connection with rearrangements of non-absolutely convergent series.

1. Structure of the space (S, d)

We shall study the structure of (S, d) from the point of view of its subset E and closure $\overline{E} = E_1$ of E in S.

It is well-known (cf. [1], [5], [9], [10]) that the set E is not closed in S. Its closure $E_1 = \overline{E}$ equals to the set of all $x = (x_j)_1^{\infty} \in S$ containing every positive integer at most once (equivalently: E_1 consists of all one-to-one sequences of positive integers).

The metric space (E, d) (subspace of (S, d)) is of the second category at each of its points (cf. [1], [4], [5], [9], [10]). We recall the following well-known result (cf. [1], [4], [5], [7], [9], [10]):

Theorem B. *Let $\sum_{k=1}^{\infty} a_k$ be a non-absolutely convergent series. Denote $H = H(a_1, a_2, \dots)$ the set of all $x = (x_j)_1^{\infty} \in E$ such that $\liminf_{n\to\infty} S_n(x) = -\infty$, $\limsup_{n\to\infty} S_n(x) = +\infty$ $(S_n(x) = \sum_{j=1}^{n} a_{x_j}$, $n = 1, 2, \dots)$. Then the set H is a residual set in E (i.e. $E \setminus H$ is a set of the first Baire category in E).*

This result has been strengthened in [3], where it is proved that the set of all $x = (x_j)_1^{\infty} \in E$ with $(S_n(x))'_n = [-\infty, \infty]$ is residual in E ($(S_n(x))'_n$ denotes the set of all limit points of the sequence $(S_n(x))_{n=1}^{\infty}$).

It is well-known that E is a G_δ-set in S (cf. [1]). We shall complete this result in the following proposition and for the completeness we shall give also the proof of the mentioned result from [1] (see (i) in the following proposition).

PROPOSITION 1.1. *The set $E \subset S$ has the following properties:*

(i) *The set E belongs exactly to the first Borel class, it is a G_δ-set in S.*

(ii) *The set E is dense in itself.*

(iii) *The set E is nowhere dense in S.*

Proof. (i) Put (as in [1]) $G(n,k) = \{x = (x_j)_1^\infty \in S : x_n = k\}$. If $x \in G(n,k)$ then it is easy to see that $K(x, 1/n) \subseteq G(n,k)$, so the set $G(n,k)$ is open in S. But then the set

$$E_2 = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} G(n,k) \tag{2}$$

is a G_δ-set in S. Further we have obviously

$$E = E_1 \cap E_2, \tag{2'}$$

E_1 being closed in S. Hence by (2′) the set E is a G_δ-set in S.

Further, E is neither closed in S (cf. [1], [5]), nor open in S (see (iii)), so the set E belongs exactly to the first Borel class.

(ii) Let $y \in E$, $\varepsilon > 0$. Choose $s \in \mathbf{N}$ such that $1/s < \varepsilon$. Define $z = (z_j)_1^\infty$ as follows: Put $z_j = y_j$ $(j = 1, 2, \dots, s)$, $z_{s+1} = y_{s+1} + 1$ and construct the one-to-one sequence $z_{s+2}, z_{s+3}, \dots$ containing all positive integers different from $z_1, \dots, z_s, z_{s+1}$. Then $z = (z_j)_1^\infty \in E$, $z \in K(y, \varepsilon)$ and $z \neq y$.

(iii) Let $y \in S$, $\varepsilon > 0$. We show that there exists a ball $K_0 \subseteq K(y, \varepsilon)$ such that $K_0 \cap E = \varnothing$. Then on account of a well-known criterion of nowhere density (cf. [6], p. 37), the assertion follows.

Choose $s \in N$ such that $1/s < \varepsilon$. Put $z_j = y_j$ $(j = 1, 2, \dots, s)$, $z_{s+1} = z_s$ and $z_{s+k} = 1$ $(k = 2, 3, \dots)$. Then $z = (z_j)_1^\infty \in K(y, \varepsilon)$, $K(z, 1/(s+2)) \subseteq K(y, \varepsilon)$ and $K(z, 1/(s+2)) \cap E = \varnothing$. ∎

The nowhere density of the set E in S follows also from the following result.

THEOREM 1.1. *The set $E_1 = \overline{E}$ is very strongly porous in S.*

COROLLARY. a) *The set E_1 is nowhere dense in S.*

b) *The set E is nowhere dense in S.*

Proof of Theorem 1.1. Let $q = (q_j)_1^\infty \in S$, $0 < \varepsilon < 1$. Choose $s \in \mathbf{N}$ such that $1/s \leqslant \varepsilon < 1/(s-1)$ $(s \geqslant 2)$. Construct $y = (y_j)_1^\infty \in S$ in this way: Put $y_i = q_i$ $(i = 1, 2, \dots, s)$, $y_{s+1} = q_s$, $y_{s+j} = 1$ $(j = 2, 3, \dots)$. Then $y \in K(q, 1/s)$ and $K(y, 1/(s+1)) \subseteq K(q, 1/s) \subseteq K(q, \varepsilon)$. If $z = (z_j)_1^\infty \in K(y, 1/(s+1))$ then $z_s = z_{s+1}$, so $K(y, 1/(s+1)) \cap E_1 = \varnothing$. But then by the definition of $\gamma(q, \varepsilon, E_1)$ we get

$$\frac{\gamma(q, \varepsilon, E_1)}{\varepsilon} \geqslant \frac{s-1}{s+1}.$$

If $\varepsilon \to 0+$, the $s \to \infty$ and so we get

$$\underline{p}(q, E_1) = \liminf_{\varepsilon \to 0+} \frac{\gamma(q, \varepsilon, E_1)}{\varepsilon} = 1. \quad \blacksquare$$

In the representation $E = E_1 \cap E_2$ of the set E (see $(2')$) the set E_1 is very strongly porous in S (hence it is a nowhere dense set). We now show that E_2 is a residual set in S. This follows from the following result.

THEOREM 1.2. *The set $CE_2 = S \setminus E_2$ is uniformly very strongly σ-porous in S.*

Proof. By (2) we have $E_2 = \bigcap_{k=1}^{\infty} B(k)$, where $B(k) = \bigcup_{n=1}^{\infty} G(n, k)$. Using de Morgan's rule we get

$$CE_2 = \bigcup_{k=1}^{\infty} CB(k), \tag{3}$$

where $CB(k) = \bigcap_{n=1}^{\infty} \{ x = (x_j)_1^{\infty} \in S : x_n \neq k \}$.

We show that $p(y, CB(k)) = 1$ $(k = 1, 2, \dots)$ for each $y \in S$. Then (3) gives the assertion.

Let k be fixed, $y \in S$, $0 < \varepsilon < 1$. Choose an $s \in \mathbf{N}$ such that $1/s \leqslant \varepsilon < 1/(s-1)$. Put $z_j = y_j$ $(j = 1, 2, \dots, s)$ and

$$z_{s+1} = k, \tag{4}$$

$z_{s+j} = 1$ $(j = 2, 3, \dots)$. Then $K(z, 1/(s+1)) \subseteq K(y, 1/s) \subseteq K(y, \varepsilon)$ and by (4) we have $K(z, 1/(s+1)) \cap CB(k) = \varnothing$. Hence

$$\frac{\gamma(y, \varepsilon, CB(k))}{\varepsilon} \geqslant \frac{s-1}{s+1}.$$

If $\varepsilon \to 0+$, then $s \to \infty$ and so we get $p(y, CB(k)) = 1$. $\blacksquare$

2. Structure of the space $(\boldsymbol{E}, \boldsymbol{d})$ and rearrangements of series

If $\sum_{k=1}^{\infty} a_k$ is a non-absolutely convergent series then by Theorem B the set $E_0 = E_0(a_1, a_2, \dots)$ is a set of the first category in E. We shall complete this result by showing that E_0 is a σ-1-porous set at points of a large subset of S.

In the first place we shall investigate to which Borel classes the set $E_0(a_1, a_2, \dots)$ and the sets

$$H^+ = H^+(a_1, a_2, \dots) = \{ x \in E : \limsup_{n \to \infty} S_n(x) = +\infty \},$$

$$H^- = H^-(a_1, a_2, \dots) = \{ x \in E : \liminf_{n \to \infty} S_n(x) = -\infty \}$$

belong.

THEOREM 2.1. *For each non-absolutely convergent series $\sum_{k=1}^{\infty} a_k$ the set $E_0 = E_0(a_1, a_2, \dots)$ is an $F_{\sigma\delta}$-ste in E.*

Remark 2.1. The foregoing theorem does not state exactly the Borel class of the set E_0. According to a result from [10] the set E_0 is dense and simultaneously a boundary set in E. From this it is easy to see that E_0 cannot belong to the zero Borel class. The set E_0 cannot be a G_δ-set in E, since in the opposite case it would be residual in S (cf. [6], p. 49). But this contradicts Theorem B. One can conjecture that E_0 is an F_σ-set in S, but I am not able to prove or disprove this conjecture.

Proof of Theorem 2.1. By the definition of the set E_0 and Cuachy's test for convergence a permutation $x = (x_j)_1^\infty \in E$ belongs to E_0 if and only if the following holds:

$$(\forall k)(\exists m)(\forall j)\, |S_{m+j}(x) - S_m(x)| \leqslant \frac{1}{k}.$$

From this we get

$$E_0 = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \{\, x \in E : |S_{m+j}(x) - S_m(x)| \leqslant 1/k \,\}. \tag{5}$$

Consider that by fixed m and j the function $S_{m+j}(x) - S_m(x) = \sum_{n=m+1}^{m+j} a_{x_n}$ is constant on every ball $K(y, 1/(m+j))$, $y \in S$. Hence it is continuous on S. The assertion follows from (5). ■

It is proved in [9] that the set $B = B(a_1, a_2, \dots) = E \setminus (H^+ \cup H^-)$ belongs to the first Borel class and is an F_σ-set in S. We show that this result is exact.

Theorem 2.2 *If $\sum_{k=1}^\infty a_k$ is a non-absolutely convergent series, then the sets $H^+(a_1, a_2, \dots)$, $H^-(a_1, a_2, \dots)$ and $B(a_1, a_2, \dots)$ belong exactly to the first Borel class, the sets H^+, H^- are G_δ-sets and B is an F_σ-set in S.*

Corollary. *The set $H^* = H^+ \cup H^-$ belongs exactly to the first Borel class and it is a G_δ-set in S.*

Proof of Theorem 2.2. By using Theorem A it is easy to see that each of the sets H^+, H^-, B is dense and simultaneously boundary in S. From this we see that none of these sets belongs to zero Borel class. Further

$$H^+ = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{\, x \in E : S_n(x) > m \,\}. \tag{6}$$

We show that the set $A(n, m) = \{\, x \in E : S_n(x) > m \,\}$ is open in S. Indeed, if $x \in A(n, m)$ then $K(x, 1/n) \subseteq A(n, m)$. But then from (6) we get that H^+ belongs exactly to the first Borel class and it is a G_δ-set in E.

Similarly it can be shown that H^- belongs exactly to the first Borel class and it is a G_δ-set in E. The statement related to B follows from $B = E \setminus H^*$, where $H^* = H^+ \cup H^-$. ■

Put $B_1 = B_1(a_1, a_2, \dots) = \{\, x \in E : (S_n(x))_1^\infty$ is bounded from above$\,\}$, $B_2 = B_2(a_1, a_2, \dots) = \{\, x \in E : (S_n(x))_1^\infty$ is bounded from below$\,\}$. Hence

$B_1 = S \setminus H^+$, $B_2 = S \setminus H^-$. Then $B = B_1 \cap B_2$ and $E_0 \subseteq B$. We shall investigate the porosity of sets B_1, B_2, B, E_0.

LEMMA 2.1. *Let $\sum_{k=1}^{\infty} a_k$ be an arbitrary non-absolutely convergent series. Then the set $B_1 = B_1(a_1, a_2, \dots)$ is σ-1-porous in the set $H^+ = H^+(a_1, a_2, \dots)$.*

COROLLARY. *The sets B, E_0 are σ-1-porous in H^+.*

Proof of Lemma 2.1. Observe that

$$B_1 = \bigcup_{m=1}^{\infty} D(m), \tag{7}$$

where $D(m) = \bigcap_{j=1}^{\infty} \{x \in E : S_j(x) \leqslant m\}$ $(m = 1, 2, \dots)$.

Let $y = (y_j)_1^{\infty} \in H^+$. Then by definition of H^+ there exists a sequence $v_1 < v_2 < \cdots < v_k < \cdots$ of positive integers such that $\lim_{k\to\infty} S_{v_k}(y) = +\infty$. Construct the ball $K(y, 1/v_k)$. Then for all sufficiently large k's (say for $k > k_0$) we have $K(y, 1/v_k) \cap D(m) = \varnothing$, thus $\gamma(y, v_k^{-1}, D(m)) = 1/v_k$ (for $k > k_0$). Since $v_k^{-1} \to 0$ $(k \to \infty)$, we get $v_k \gamma(y, v_k^{-1}, D(m)) = 1$. Thus $\overline{p}(y, D(m)) = 1$ $(m = 1, 2, \dots)$. Lemma 2 follows from (7). ∎

Similarly the following lemma can be proved.

LEMMA 2.2. *Let $\sum_{k=1}^{\infty} a_k$ be a non-absolutely convergent series. Then the set $B_2(a_1, a_2, \dots)$ is σ-1-porous in the set $H^-(a_1, a_2, \dots)$.*

On account of Theorem B, using Lemma 2.1 and Lemma 2.2 we obtain the following result.

THEOREM 2.3. *Let $\sum_{k=1}^{\infty} a_k$ be an arbitrary non-absolutely convergent series. Then each of the sets $B(a_1, a_2, \dots)$, $E_0(a_1, a_2, \dots)$ is σ-1-porous in the residual set $H^+(a_1, a_2, \dots) \cap H^-(a_1, a_2, \dots)$.*

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(received 11.06.1996.)

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