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SEGMENTS OF EXPONENTIAL SERIES AND REGULARLY VARYING SEQUENCES

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Abstract. The task of this paper is to investigate asymptotic behavior of segments of exponential series defined as

$$T_\lambda(x) := \sum_{n<\lambda x} \frac{c_n}{n!} x^n, \qquad \lambda \in \mathbf{R}^+, \quad x \to \infty,$$

where $(c_n)_{n\in\mathbf{N}}$ belongs to the set of regularly varying sequences in Karamata sense of arbitrary index. Precise results are obtained.

Introduction

Karamata's class R_α of regularly varying functions with index $\alpha \in \mathbf{R}$ consists of all functions $a(x)$ representable in the form $a(x) = x^\alpha l(x)$, where $l(x)$ is from the class of so-called slowly varying functions, i.e. defined on positive part of real axis, positive, measurable and satisfying $\lim_{x\to\infty} \frac{l(sx)}{l(x)} = 1$, for each $s > 0$.

According to [3], we could treat regularly varying sequnece (c_n) of index α as generated from some $a \in R_\alpha$, i.e. $c_n = n^\alpha l(n)$, $n \in \mathbf{N}$.

After seventy years, Karamat's theory is very well developed and found applications in different parts of analysis. An excellent survey of results could be found in [1] or [5].

For this article we are motivated by papers [2] and [6]. In [2] the authors proved, by probabilistic methods, the following

PROPOSITION 1. *If a bounded sequence (c_n) behaves regularly with index $-\beta$, $\beta \geqslant 0$, then*

$$\exp(-x) \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n \sim c_{[x]}, \quad x \to \infty.$$

In [6] we extend this proposition to the following

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PROPOSITION 2. *If* $\exp P_p(x) = \sum_{n=0}^{\infty} a_n x^n$, *where* $P_p(x) = b_p x^p + \cdots$; $b_p > 0$, *is a polynomial with (eventually) non-negative coefficients, then*

$$\exp(-P_p(x)) \sum_{n=0}^{\infty} c_n a_n x^n \sim (pb_p)^{\beta} c_{[x^p]}, \qquad x \to \infty; \quad c_0 = 1;$$

for any regularly varying sequence (c_n) *of an arbitrary index* $\beta \in \mathbf{R}$.

Here we are going to show similar (and even more precise) asymptotic relations take place for segments of exponential series cited above.

Results

At the beginning we shall formulate a rather global proposition, showing how the structure of a given power series segment influence the behaviour of another one which involves regularly varying sequences. Namely, let us define

$$S(\lambda, x) := \sum_{n \leqslant \lambda n(x)} a_n x^n, \qquad a_n \geqslant 0, \quad n \in \mathbf{N};$$

where $n(x)$ increases to infinity with x, and an operator T acting on S:

$$TS(\lambda, x) := \sum_{n \leqslant \lambda n(x)} c_n a_n x^n, \qquad n \in \mathbf{N},$$

where $(c_n)_{n \in \mathbf{N}}$ is a regularly varying sequence of index $\alpha \in \mathbf{R}$.

THEOREM A. *If there exist* $f, g_1, g_2 \colon \mathbf{R}^+ \to \mathbf{R}^+$, $b_1 \colon (0,1) \to \mathbf{R}^+$, $b_2 \colon (1, \infty) \to \mathbf{R}^+$, *and*

$$\lim_{x \to \infty} \frac{\ln n(x)}{g_i(x)} = 0, \qquad i = 1, 2, \tag{A1}$$

such that

$$\frac{S(\lambda, x)}{f(x)} = \begin{cases} O(e^{-b_1(\lambda) g_1(x)}), & 0 < \lambda < 1, \\ A + O(e^{-b_2(\lambda) g_2(x)}), & \lambda > 1, \end{cases} \qquad x \to \infty, \quad A \in \mathbf{R}^+, \tag{A2}$$

then

$$\frac{TS(\lambda, x)}{f(x)} = \begin{cases} o(c_{[n(x)]}), & 0 < \lambda < 1, \\ A c_{[n(x)]}(1 + o(1)), & \lambda > 1, \end{cases} \qquad x \to \infty.$$

Proof. We shall use the following well-known properties of regularly varying functions ([5], pp. 19, 20; [1], p. 52):

For $\alpha > 0$,

$$S_1 : \sup_{t \leqslant y} t^{\alpha} L(t) = y^{\alpha} L(y)(1 + o(1)); \quad \inf_{t \geqslant y} t^{\alpha} L(t) = y^{\alpha} L(y)(1 + o(1)), \ y \to \infty;$$

$$S_2 : \inf_{t \leqslant y} t^{-\alpha} L(t) = y^{-\alpha} L(y)(1 + o(1)); \quad \sup_{t \geqslant y} t^{-\alpha} L(t) = y^{-\alpha} L(y)(1 + o(1)), \ y \to \infty;$$

$$S_3 : c_{[\lambda y]} \sim c_{[\lambda [y]]} \sim \lambda^{\alpha} c_{[y]}, \qquad y \to \infty, \quad \lambda \in \mathbf{R}^+, \quad \alpha \in \mathbf{R}.$$

In the sequel we consider a sequence (c_n) generated by some regularly varying function $x^\alpha L(x)$, $\alpha \in \mathbf{R}$, i.e.

$$c_n = n^\alpha L(n), \quad n \in \mathbf{N}; \quad c_0 = 1; \quad c_{[y]} = [y]^\alpha L([y]).$$

Let α and λ ($\alpha \in \mathbf{R}$, $0 < \lambda < 1$) be fixed numbers; then

$$\begin{aligned}
\frac{TS(\lambda, x)}{c_{[n(x)]} f(x)} &= \frac{1}{f(x)} \sum_{n \leqslant \lambda n(x)} \left(\frac{n}{[n(x)]}\right)^{\alpha - 1} \left(\frac{nL(n)}{[n(x)]L([n(x)])}\right) a_n x^n \\
&\leqslant \frac{1}{f(x)} \sup_{n \leqslant \lambda n(x)} \left(\frac{n}{[n(x)]}\right)^{\alpha - 1} \sup_{n \leqslant \lambda n(x)} \frac{nL(n)}{[n(x)]L([n(x)])} \sum_{n \leqslant \lambda n(x)} a_n x^n \\
&= \frac{1}{f(x)} O([n(x)]^{|\alpha|+1}) O\left(\frac{[\lambda n(x)]L([\lambda n(x)])}{[n(x)]L([n(x)])}\right) O(e^{-b_1(\lambda) g_1(x)}) \\
&= O(n(x)^{|\alpha|+1} e^{-b_1(\lambda) g_1(x)}) = O(\exp(-b_1(\lambda) g_1(x)) \left(1 + O\left(\frac{\ln n(x)}{g_1(x)}\right)\right) \\
&= o(1), \qquad x \to \infty
\end{aligned}$$

Hence, the first assertion of Theorem A is proved.

For the second one, let λ and ε ($\lambda > 1$, $0 < \varepsilon < \min(1/2, \lambda - 1)$) be fixed. We get

$$\begin{aligned}
&\frac{TS(\lambda, x)}{c_{[n(x)]} f(x)} \\
&\quad = \frac{1}{c_{[n(x)]} f(x)} \left(\sum_{n \leqslant (1-\varepsilon) n(x)} + \sum_{(1-\varepsilon) n(x) < n \leqslant (1+\varepsilon) n(x)} + \sum_{(1+\varepsilon) n(x) < n \leqslant \lambda n(x)}\right) a_n c_n x^n \\
&\quad = T_1 + T_2 + T_3.
\end{aligned}$$

According to the former argument ($\lambda = 1 - \varepsilon < 1$),

$$T_1 = o(1), \qquad x \to \infty. \tag{A3}$$

Analogously,

$$\begin{aligned}
T_3 &\leqslant \frac{1}{f(x)} \sup_{n \leqslant \lambda n(x)} \left(\frac{c_n}{c_{[n(x)]}}\right) \sum_{(1+\varepsilon) n(x) < n \leqslant \lambda n(x)} a_n x^n \\
&= \frac{1}{f(x)} O(n(x)^{|\alpha|+1}) (S(\lambda, x) - S(1 + \varepsilon, x)) \\
&= O(n(x)^{|\alpha|+1}) O(\exp(-g_2(x) \min(b_2(1+\varepsilon), b_2(\lambda)))) = o(1), \qquad x \to \infty.
\end{aligned} \tag{A4}$$

To estimate T_2, suppose for the moment that index α of (c_n) is positive. Then, since $0 < \varepsilon < 1/2$, using properties S_1 and S_3, we obtain:

$$\begin{aligned}
\sup_{n \leqslant (1+\varepsilon) n(x)} c_n &= c_{[(1+\varepsilon) n(x)]} (1 + o(1)) = c_{[n(x)]} (1 + \varepsilon)^\alpha (1 + o(1)) \\
&= c_{[n(x)]} (1 + \varepsilon O(1) + o(1)), \qquad x \to \infty; \\
\inf_{n > (1-\varepsilon) n(x)} c_n &= c_{[(1-\varepsilon) n(x)]} (1 + o(1)) = c_{[n(x)]} (1 - \varepsilon)^\alpha (1 + o(1)) \\
&= c_{[n(x)]} (1 + \varepsilon O(1) + o(1)), \qquad x \to \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned} T_2 &\leqslant \frac{1}{f(x)c_{n(x)}} \sup_{n\leqslant(1+\varepsilon)n(x)} c_n \sum_{(1-\varepsilon)n(x)<n\leqslant(1+\varepsilon)n(x)} a_n x^n \\ &= \frac{1}{f(x)}(1+\varepsilon O(1)+o(1))(S(1+\varepsilon,x)-S(1-\varepsilon,x)) \\ &= (1+\varepsilon O(1)+o(1))(A+o(1)) = A+\varepsilon O(1)+o(1), \quad x\to\infty, \end{aligned} \tag{A5}$$

and, similarly,

$$\begin{aligned} T_2 &\geqslant \frac{1}{f(x)c_{[n(x)]}} \inf_{n>(1-\varepsilon)n(x)} c_n \cdot (S(1+\varepsilon,x)-S(1-\varepsilon,x)) \\ &= A+\varepsilon O(1)+o(1), \qquad x\to\infty. \end{aligned} \tag{A6}$$

Since the constants in $O(1)$ do not depend on ε and ε can be arbitrarily small, from (A5) and (A6) we conclude that $T_2 \sim A$, $x\to\infty$; this, together with (A3) and (A4), gives the proof of Theorem A for $\alpha>0$.

For $\alpha<0$ we deduce the proof similarly, using properties S_2 and S_3.

If $\alpha=0$, note that $(nL(n))$ is of index 1, hence

$$\begin{aligned} T_2 &= \frac{1}{f(x)L([n(x)])} \sum_{(1-\varepsilon)n(x)<n\leqslant(1+\varepsilon)n(x)} \frac{1}{n}\cdot nL(n)a_n x^n \\ &\leqslant \frac{1}{[(1-\varepsilon)n(x)]+1}[(1+\varepsilon)n(x)](A+o(1)), \end{aligned}$$

and

$$T_2 \geqslant \frac{1}{[(1+\varepsilon)n(x)]}[(1-\varepsilon)n(x)](A+o(1)),$$

which shows that in this case also $T_2\sim A$, $x\to\infty$, and the proof is over. ■

Investigation of possible relationship between $S(\lambda,x)$, $n(x)$ and $f(x)$ satisfying conditions of Theorem A is the subject of our next article. Here we just show that the class of such functions is not empty, i.e. applying results of Theorem A we prove the following

THEOREM B1. *For any regularly varying sequence $(c_n)_{n\in\mathbf{N}}$, $c_0=1$, of arbitrary index $\alpha\in\mathbf{R}$,*

$$e^{-x}\sum_{n\leqslant\lambda x} c_n \frac{x^n}{n!} \sim \begin{cases} o(c_{[x]}), & 0<\lambda<1, \\ \frac{1}{2}c_{[x]}, & \lambda=1, \\ c_{[x]}, & \lambda>1, \end{cases} \qquad x\to\infty.$$

In the neighbourhood of $\lambda=1$ we prove more precisely:

Theorem B2.

$$e^{-x} \sum_{n \leqslant x+h(x)} c_n \frac{x^n}{n!} \sim \left(\frac{1}{2} + \frac{1}{\sqrt{\pi}} \operatorname{Erf}(b/\sqrt{2})\right) c_{[x]}, \quad x \to \infty,$$

where $h(x) := b\sqrt{x}(1+o(1))$, $x \to \infty$; $b \in \mathbf{R}$; $\operatorname{Erf} y := \int_0^y e^{-t^2}\,dt$.

Proof. According to the premises from Theorem A, the proof of cited theorems depends on asymptotic behaviour of the sum $\sum_{k \leqslant n} \frac{x^k}{k!}$, $n = n(x) \to \infty$. Therefore, we derive its integral representation which is more easy to estimate.

$$S(n,x) := \sum_{k \leqslant n} \frac{x^k}{k!} = \frac{x^{n+1}}{n!} \sum_{k \leqslant n} \binom{n}{k} \frac{k!}{x^{k+1}} = \frac{x^{n+1}}{n!} \sum_{k \leqslant n} \binom{n}{k} \int_0^\infty e^{-xt} t^k\,dt,$$

i.e.

$$S(n,x) = \frac{x^{n+1}}{n!} \int_0^\infty e^{-xt}(1+t)^n\,dt. \tag{B0}$$

For $n = [\lambda x]$ we obtain

$$e^{-x}\frac{x^{n+1}}{n!} \sim \frac{x}{\sqrt{2\pi n}} e^{n\ln x - n\ln n + n - x} = \frac{x}{\sqrt{2\pi n}} e^{-x(\frac{n}{x}\ln\frac{n}{x} + 1 - \frac{n}{x})}, \quad x \to \infty. \tag{B1}$$

But

$$\lambda - \frac{1}{x} = \frac{\lambda x - 1}{x} < \frac{n}{x} = \frac{[\lambda x]}{x} \leqslant \frac{\lambda x}{x} = \lambda,$$

i.e. $\frac{n}{x} = \lambda - \frac{\theta}{x}$, $\theta \in [0,1)$. Therefore,

$$\frac{n}{x}\ln\frac{n}{x} + 1 - \frac{n}{x} = \lambda\ln\lambda + 1 - \lambda - \frac{\theta}{x}\ln\lambda + 0\left(\frac{1}{x^2}\right), \quad x \to \infty,$$

i.e. ((B1))

$$e^{-x}\frac{x^{n+1}}{n!} = O(\sqrt{x}\, e^{-(\lambda\ln\lambda + 1 - \lambda)x}), \qquad n = [\lambda x], \quad \lambda \in \mathbf{R}^+, \quad x \to \infty. \tag{B2}$$

Since $\ln(1+t) < t$, $t \in \mathbf{R}^+$, for $0 < \lambda < 1$, $n = [\lambda x]$, we get

$$\int_0^\infty e^{-xt}(1+t)^n\,dt < \int_0^\infty e^{-(x-n)}\,dt = \frac{1}{x-n} \sim \frac{1}{x(1-\lambda)}, \quad x \to \infty.$$

Along with (B2) this gives the estimate

$$e^{-x}S([\lambda x], x) = e^{-x}\sum_{k \leqslant \lambda x} \frac{x^k}{k!} = O(e^{-(\lambda\ln\lambda + 1 - \lambda)x}), \qquad \lambda \in (0,1), \quad x \to \infty. \tag{B3}$$

For $l > 1$, change of variable $1 + t \mapsto t$ gives

$$\int_0^\infty e^{-xt}(1+t)^n\,dt = e^x \int_1^\infty e^{-xt}t^n\,dt = e^x\left(\int_0^\infty - \int_0^1\right) e^{-xt}t^n\,dt = e^x(I_1 + I_2),$$

and, obviously, $I_1 = \dfrac{n!}{x^{n+1}}$, $|I_2| = O\left(\dfrac{e^{-x}}{x}\right)$, which, together with (B0) and (B2) gives

$$e^{-x} S([\lambda x], x) = 1 + O(e^{-x(\lambda \ln \lambda + 1 - \lambda)}), \qquad \lambda \in (1, \infty), \quad x \to \infty. \tag{B4}$$

Note that $b(\lambda) := \lambda \ln \lambda - \lambda + 1$ is non-negative and convex on $\lambda \in (0, +\infty)$, $b(0^+) = 1$, $b(1) = 0$, $b(\lambda) > 0$ for $\lambda \neq 1$, and

$$b(\lambda) > \begin{cases} \frac{1}{2}(1-\lambda)^2, & \lambda \in (0,1), \\ \frac{1}{2}\ln^2 \lambda, & \lambda \in (1, +\infty). \end{cases}$$

Comparing (B3) and (B4) with assertions from Theorem A, we see that conditions (A1), (A2) are satisfied with

$$f(x) = e^x, \;\; n(x) = g_1(x) = g_2(x) = x, \;\; b_1(\lambda) = \frac{(1-\lambda)^2}{2}, \;\; b_2(\lambda) = \frac{\ln^2 \lambda}{2}, \;\; A = 1,$$

from which follows the validity of the first and the third assertion from Theorem B1.

To prove Theorem B2, chnage variable in (B0): $1 + t \mapsto \frac{n}{x}(1 + \frac{t}{\sqrt{n}})$. We get:

$$\begin{aligned} e^{-x} S(n,x) &= \frac{\sqrt{n}}{x} \cdot \frac{x^{n+1}}{n!} \cdot e^{-x} \int_{\sqrt{n}(\frac{x}{n}-1)}^{\infty} e^{-n+x} \left(\frac{n}{x}\right)^n e^{-\sqrt{n}\,t + n \ln(1+\frac{t}{\sqrt{n}})}\, dt \\ &= \frac{\sqrt{n}}{n!} n^n e^{-n} \Big(\int_0^{\infty} + \int_{\sqrt{n}(\frac{x}{n}-1)}^{0} \Big) e^{-(\sqrt{n}\,t - n\ln(1+\frac{t}{\sqrt{n}}))}\, dt. \end{aligned} \tag{B5}$$

Denote the first integral in (B5) by J_1 and the second by J_2 and let

$$g(n,t) := \sqrt{n}\,t - n \ln\left(1 + \frac{t}{\sqrt{n}}\right), \qquad t \geqslant 0, \quad n \in \mathbf{N}.$$

From the facts:

I: $g(n,t)$ is monotone increasing on n.

Proof. $0 \leqslant \int_1^{1+t/\sqrt{n}} \frac{1}{2s}(\sqrt{s} - \frac{1}{\sqrt{s}})^2\, ds = \frac{1}{2}(s - \frac{1}{s}) - \ln s\big|_1^{1+t/\sqrt{n}} = g'_n(n,t)$;

II: $\lim_{n\to\infty} g(n,t) = \frac{t^2}{2}$, $t \in \mathbf{R}^+$;

III: For $n = [x + h(x)]$ follows $\sqrt{n}(\frac{x}{n} - 1) \to -b$, $x \to \infty$;

using Lebesgue's theorem of dominated convergence, we have:

$$J_1 \to \int_0^{\infty} e^{-t^2/2}\, dt = \sqrt{\frac{\pi}{2}}; \tag{B6}$$

$$J_2 \to \int_{-b}^{0} e^{-t^2/2}\, dt = \sqrt{2} \int_0^{b/\sqrt{2}} e^{-t^2}\, dt = \sqrt{2}\, \mathrm{Erf}(b/\sqrt{2}). \tag{B7}$$

Since

$$\frac{\sqrt{n}}{n!} \cdot n^n e^{-n} \to \frac{1}{\sqrt{2\pi}}, \quad n \to \infty,$$

from (B5), (B6), (B7) and Theorem A, the assertion of Theorem B2 follows.

Putting $b = 0$ we obtain the sedond proposition from Theorem B1. ∎

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