New York Journal of Mathematics

New York J. Math. **7** (2001) 267–280.

Positive Radial Solutions of Nonlinear Elliptic Systems

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Abstract. In this article, we are concerned with the existence of positive radial solutions of the problem

$$(S^+)\begin{cases} -\Delta_p u = f(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a ball in R^N and f, g are positive functions satisfying $f(x,0,0) = g(x,0,0) = 0$. Under some growth conditions, we show the existence of a positive radial solution of the problem S^+. We use traditional techniques of the topological degree theory. When $\Omega = R^N$, we give some sufficient conditions of nonexistence.

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1. Introduction and main result

In this work, we are concerned with the existence of positive radial solutions of the problem

$$(S^+)\begin{cases} -\Delta_p u = a(x)u|u|^{\alpha-1} + b(x)v|v|^{\beta-1} & \text{in } \Omega, \\ -\Delta_q v = c(x)u|u|^{\gamma-1} + d(x)v|v|^{\delta-1} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

Received August 25, 2000.
Mathematics Subject Classification. 35J25, 35J60.
Key words and phrases. Blow up argument, degree theory, Leray-Schauder theorem, excision property.

ISSN 1076-9803/01

where $\Omega := B_R$ is the ball centered in zero and radius $R > 0$ in R^N, a, b, c and d are given positive continuous functions. Our motivation for studying the system S^+ is based essentially from the fact that the problem has not necessarily a variational structure. We shall make recourse to topological degree methods by using the blow-up technique introduced by Gidas and Spruck [10] in the scalar case. This method explores the different exponents $(\alpha, \beta, \delta, \gamma)$. In the scalar case the interested reader may refer to [5], [6] and [16]. In the case of systems, many authors have extended this method to different situations (see [4], [3] and [15]).

In recent years, for the scalar case the problems of existence and nonexistence have been studied by several authors by using different approaches (see[5], [6] and [16]). For the systems case, we mention the recent results of Boccardo, Fleckinger and de Thelin [2] where the authors prove the existence of the weak solutions of the following problem:

$$(1.1)\qquad \begin{cases} -\Delta_p u = a(x)u|u|^{\alpha-1} + b(x)v|v|^{\beta-1} + h_1(x) & \text{in } \Omega, \\ -\Delta_q v = c(x)u|u|^{\gamma-1} + d(x)v|v|^{\delta-1} + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

under the following assumptions:

$$\text{(H1)}\qquad \max(p, q) < N.$$

$$\text{(H2)}\qquad (p-1)(q-1) > \beta\gamma.$$

(H3) One of the following conditions holds:

(i) $p - 1 > \alpha, \quad q - 1 > \delta.$

(ii) $\begin{cases} p - 1 = \alpha, \quad q - 1 = \delta, \\ \| a \| < \lambda_{(1,p)} \text{ and } \| d \| < \lambda_{(1,q)}. \end{cases}$

(iii) $\begin{cases} p - 1 = \alpha, \quad q - 1 < \delta, \\ \text{and } \| a \| < \lambda_{(1,p)}. \end{cases}$

Here, Ω is smooth and bounded in R^N, $\lambda_{(1,m)}$ $(m = p, q)$ is the first eigenvalue of the operator Δ_m $(m = p, q)$ on Ω and $h_1 \in L^{p'}(\Omega)$, $h_2 \in L^{q'}(\Omega)$. We observe that, with the same approach in [2], if h_1 and h_2 are identically zero, the solution (u, v) would be a trivial solution. Always in the system case, the interested reader may refer to [1], [4], [7], [8], [9], [11] and [12].

Now, we state our main result.

Theorem 1.1. *We assume that the hypotheses* (H1), (H2) *and* (H3) *hold. We also suppose that*

$$\text{(H4)}\qquad a, b, c, d \in C^0([0, +\infty[) \quad \text{with} \quad \inf_{s\in[0,+\infty[}(a(s), b(s), c(s), d(s)) > 0.$$

Then the problem (S^+) *possesses a solution* (u, v) *in* $C^1(B_R) \cap C^2(B_R\backslash\{0\})$, *such that* $u > 0$, $v > 0$ *in* B_R.

The paper is organized as follows. At first, we consider the operator of solution S_1 associated to the problem (S^+) which allows us to seek solutions of the problem (S^+) as a fixed points of S_1. In Section 2 we introduce two families of operators, $(S_\lambda)_\lambda$ and $(T_\mu)_\mu$, linked to the problem (S^+), acting in a suitable functional space and we give a fundamental lemma. In Section 3, we prove that for any positive solution (u, v) of the problem, it is bounded. By using the theory of degree, we show that there exists a positive number $\rho_1 > 0$ sufficiently large such that $\deg(S_1, B(0, \rho_1)) = 1$. On the other hand, in Section 4 by means of the argument blow-up, we show that there exists a number $\rho_2 > 0$ sufficiently small such that $\deg(S_1, B(0, \rho_2)) = 0$. In Section 5 by the excision property we deduce the existence of the nontrivial positive solutions of (S^+) stated in Theorem 1.1. Finally, in Section 6 we give sufficient conditions for the nonexistence of positive radial solutions of the problem (S^+) on $\Omega = R^N$.

2. Preliminaries

We now consider χ the space

$$\chi = \{(u, v) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega}) \mid u = v = 0 \,\text{on}\, \partial\Omega\}$$

equipped with the norm $\|(u, v)\| = \|u\|_\infty + \|v\|_\infty$, which makes it a Banach space. Let S_λ and $T_\tau : \chi \to \chi$ be the operators defined by $S_\lambda(u, v) = (S^1(u, v); S^2(u, v))$ and $T_\tau(u, v) = (T^1(u, v); T^2(u, v))$ such that

$$\begin{aligned} S^1(u,v)(r) &= \lambda^{\frac{1}{p-1}} \int_r^R \left[t^{1-N} \int_0^t s^{N-1}(a(s)|u(s)|^\alpha + b(s)|v(s)|^\beta) ds \right]^{\frac{1}{p-1}} dt, \\ S^2(u,v)(r) &= \lambda^{\frac{1}{q-1}} \int_r^R \left[t^{1-N} \int_0^t s^{N-1}(c(s)|u(s)|^\gamma + d(s)|v(s)|^\delta) ds \right]^{\frac{1}{q-1}} dt, \end{aligned}$$

and

$$\begin{aligned} T^1(u,v)(r) &= \int_r^R \left[t^{1-N} \int_0^t s^{N-1}(a(s)|u(s)|^\alpha + b(s)|(v(s)+\tau)|^\beta) ds \right]^{\frac{1}{p-1}} dt, \\ T^2(u,v)(r) &= \int_r^R \left[t^{1-N} \int_0^t s^{N-1}(c(s)|u(s)|^\gamma + d(s)|v(s)|^\delta) ds \right]^{\frac{1}{q-1}} dt. \end{aligned}$$

It is well know that, for all $\lambda \in [0, 1]$ and for all $\tau \in [0, \infty[$, S_λ and T_τ are completely continuous operators on χ. From the Maximum principle this implies that $S_\lambda(\chi) \subset \chi$ and that the problem (S^+) is equivalent to find some non trivial fixed point $(u, v) \in \chi$ of the operator S_1 (by taking $\lambda = 1$) such that $u'(0) = v'(0) = 0$.

We make use in a fundamental way of the following lemma (cf. [3, Lemma 2.1, p. 2076]):

Lemma 2.1. *Let* $u \in C^1([0.R]) \cap C^2(]0, R])$, $u \geq 0$, *satisfying*

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq 0 \text{ on } \quad [0, R]. \tag{2.1}$$

Then, for any $r \in]0, \frac{R}{2}[$ *we have :*

$$u(r) \geq C_{N,p}\, r|u'(r)| \tag{2.2}$$

where

$$C_{N,p} = \frac{p-1}{N-p}\left(1 - 2^{\frac{p-N}{p-1}}\right). \tag{2.3}$$

Proof. Integrating (2.1) from r to $s \in [r, \frac{R}{2}[$ we have:

$$s^{N-1}|u'(s)|^{p-1} \geq r^{N-1}|u'(r)|^{p-1} \tag{2.4}$$

and therefore:

$$-u'(s) \geq r^{\frac{N-1}{p-1}} |u'(r)| s^{-\frac{N-1}{p-1}}. \tag{2.5}$$

Integrating again from r to 2r with respect to s, we obtain:

$$u(r) \geq u(r) - u(2r) \geq r^{\frac{N-1}{p-1}} |u'(r)| \int_r^{2r} s^{-\frac{N-1}{p-1}} ds. \tag{2.6}$$

Since $\int_r^{2r} s^{-\frac{N-1}{p-1}} ds = C_{N,p} r^{-\frac{N-p}{p-1}}$, we obtain the Lemma. □

In the following sections, we do not distinguish notationally between a sequence and one of its subsequences, to keep the notation simple.

3. A priori bounds for positive solutions of (S^+)

Proposition 3.1. *Under the hypotheses* (H1), (H2), (H3) *and* (H4) *there exists some* $C_0 > 0$ *such that* $\forall \lambda \in [0,1]$ *if* $(u,v) \in \chi$ *is a fixed point of the operator* S_λ *then*

$$\|(u,v)\| \leq C_0.$$

This implies that $\forall \rho_1 > C_0$, $\forall \lambda \in]0,1[$ *we have*

$$\deg(I - S_\lambda, B(0,\rho_1), 0) = \text{const} = 1, \tag{3.1}$$

where $B(0,\rho_1) = \{(u,v) \in \chi \mid \|(u,v)\| \leq \rho_1\}$.

Proof. We suppose by contradiction that there exist $\lambda \in [0,1]$ and $(u,v) \in \chi$ such that

$$(u,v) = S_\lambda(u,v) \tag{3.2}$$

with $\|(u,v)\| = c > 0$. Notice that by definition of S_λ we get $u' \leq 0$, $v' \leq 0$ in $[0,R]$. Hence $\|(u,v)\| = u(0) + v(0)$. Thus, since

$$u(0) = \lambda^{\frac{1}{p-1}} \int_0^R \left[t^{1-N} \int_0^t s^{N-1}(a(s)|u(s)|^\alpha + b(s)|v(s)|^\beta) ds \right]^{\frac{1}{p-1}} dt, \tag{3.3}$$

$$v(0) = \lambda^{\frac{1}{q-1}} \int_0^R \left[t^{1-N} \int_0^t s^{N-1}(c(s)|u(s)|^\gamma + d(s)|v(s)|^\delta) ds \right]^{\frac{1}{q-1}} dt,$$

we have

$$u(0) \leq C\lambda^{\frac{1}{p-1}} \left[(u(0))^\alpha + (v(0))^\beta \right]^{\frac{1}{p-1}} \tag{3.4}$$

$$v(0) \leq C\lambda^{\frac{1}{q-1}} \left[(u(0))^\gamma + (v(0))^\delta \right]^{\frac{1}{q-1}}. \tag{3.5}$$

Moreover, from (H3), there exist two numbers $\ell > 0$ and $k > 0$ such that

$$\frac{\beta}{p-1} < \frac{\ell}{k} < \frac{q-1}{\gamma}. \tag{3.6}$$

Denote

$$\sigma = (u(0))^{\frac{1}{\ell}} + (v(0))^{\frac{1}{k}}, \tag{3.7}$$

Hence, from (3.4) and (3.5), we get

$$(u(0))^{\frac{1}{\ell}} \leq C\lambda^{\frac{1}{\ell(p-1)}} \left[\sigma^{\ell\alpha} + \sigma^{k\beta} \right]^{\frac{1}{\ell(p-1)}} \tag{3.8}$$

$$(v(0))^{\frac{1}{k}} \leq C\lambda^{\frac{1}{k(q-1)}} \left[\sigma^{\ell\gamma} + \sigma^{k\delta}\right]^{\frac{1}{k(q-1)}} . \tag{3.9}$$

Summing (3.8) and (3.9), we deduce that σ satisfies

$$\begin{aligned} 1 \leq\ & C\lambda^{\frac{1}{\ell(p-1)}} \left[\sigma^{\ell(\alpha-p+1)} + \sigma^{k\beta-\ell(p-1)}\right]^{\frac{1}{\ell(p-1)}} \\ & + C\lambda^{\frac{1}{k(q-1)}} \left[\sigma^{\ell\gamma-k(q-1)} + \sigma^{k(\delta-q+1)}\right]^{\frac{1}{k(q-1)}} . \end{aligned} \tag{3.10}$$

First Case: (H3)(i) is satisfied.

Here, (3.10) leads us to a contradiction for σ sufficiently large.

Second Case: (H3)(ii) or (H3)(iii) is satisfied.

In this case we suppose that there exist some sequences $\{\lambda_n\}$ and $\{(u_n, v_n)\}$ satisfy (3.2), this implies that

$$\begin{array}{ll} -\Delta_p u_n = \lambda_n a(x) u_n |u_n|^{\alpha-1} + \lambda_n b(x) v_n |v_n|^{\beta-1} & \text{in } B(0,R), \\ -\Delta_q v_n = \lambda_n c(x) u_n |u_n|^{\gamma-1} + \lambda_n d(x) v_n |v_n|^{\delta-1} & \text{in } B(0,R), \\ u_n = v_n = 0 & \text{on } \partial B(0,R), \end{array} \tag{3.11}$$

and we suppose that $c_n = \|(u_n, v_n)\| \to +\infty$ as $n \to +\infty$. Then, from (3.10), we deduce easily that $\lambda_n \to \lambda > 0$ as $n \to +\infty$. We introduce new functions $\tilde{u}_n$ and $\tilde{v}_n$ in the following way:

$$\tilde{u}_n(r) = \frac{u_n(r)}{{\sigma_n}^{\ell}}, \quad \tilde{v}_n(r) = \frac{v_n(r)}{{\sigma_n}^{k}}$$

where,

$$\sigma_n = (u_n(0))^{\frac{1}{\ell}} + (v_n(0))^{\frac{1}{k}}.$$

Taking $(\tilde{u}_n, \tilde{v}_n)$ in (3.11) we get, in $B(0,R)$

$$-\Delta_p \tilde{u}_n(x) = {\sigma_n}^{\ell(\alpha+1-p)} \lambda_n a(x) |\tilde{u}_n(x)|^{\alpha} + {\sigma_n}^{-\ell(p-1)+k\beta} \lambda_n b(x) |\tilde{v}_n(x)|^{\beta} \tag{3.12}$$

$$-\Delta_q \tilde{v}_n(x) = {\sigma_n}^{-k(q-1)+\ell\gamma} \lambda_n c(x) |\tilde{u}_n(x)|^{\gamma} + {\sigma_n}^{k(\delta+1-q)} \lambda_n d(x) |\tilde{v}_n(x)|^{\delta}, \tag{3.13}$$

$$\tilde{u}_n = \tilde{v}_n = 0 \qquad \text{on} \quad \partial B(0,R),$$

Multiplying (3.12) by $\tilde{u}_n$, (3.13) by $\tilde{v}_n$ and by integrating, we infer

$$\begin{aligned} \int_B |\bigtriangledown \tilde{u}_n(x)|^p &= {\sigma_n}^{\ell(\delta+1-p)} \lambda_n \int_B a(x) |\tilde{u}_n(x)|^{\alpha+1} dx \\ &\quad + {\sigma_n}^{-\ell(p-1)+k\beta} \lambda_n \int_B b(x) |\tilde{v}_n(x)|^{\delta} \tilde{u}_n(x) dx \\ \int_B |\bigtriangledown \tilde{v}_n(x)|^q &= {\sigma_n}^{-k(q-1)+\ell\gamma} \lambda_n \int_B c(x) |\tilde{u}_n(x)|^{\gamma} \tilde{v}_n(x) dx \\ &\quad + {\sigma_n}^{k(\delta+1-q)} \lambda_n \int_B d(x) |\tilde{v}_n(x)|^{\delta+1} dx. \end{aligned}$$

Observe that

$$(\tilde{u}_n(0))^{\frac{1}{\ell}} + (\tilde{u}_n(0))^{\frac{1}{k}} = 1.$$

Consequently, from (H3)(ii) or (H3)(iii), (H4) and (3.6) we deduce that $(\tilde{u}_n, \tilde{v}_n)$ is bounded in ${W_0}^{1,p}(B(0,R)) \times {W_0}^{1,q}(B(0,R))$.

Thus $(\tilde{u}_n, \tilde{v}_n)$ converges weakly to some $(\tilde{u}, \tilde{v}) \in {W_0}^{1,p}(B(0,R)) \times {W_0}^{1,q}(B(0,R))$. On the other hand, it easy to see that

$$\|\Delta_p \tilde{u}_n\| \le C, \qquad \forall n \in N,$$

$$\|\Delta_q \tilde{v}_n\| \le C, \qquad \forall n \in N$$

with some positive constant $C > 0$ depending on (N, p, q, a, b, c, d). Therefore, for all n we have $(\tilde{u}_n, \tilde{v}_n) \in C^1(\overline{B}(0,R)) \times C^1(\overline{B}(0,R))$ and $\| \bigtriangledown \tilde{u}_n\| \le K$ and $\| \bigtriangledown \tilde{v}_n\| \le K$. Now since $\|(\tilde{u}_n, \tilde{v}_n)\| = 1$ for all n, the Arzelà-Ascoli theorem together with the weak convergence of $(\tilde{u}_n, \tilde{v}_n)$ to $(\tilde{u}, \tilde{v})$ ensure that $(\tilde{u}_n, \tilde{v}_n)$ converges uniformly to $(\tilde{u}, \tilde{v})$ and that $(\tilde{u}, \tilde{v})$ is not identically zero. Consequently, by passing to the limit it follows that:

1. If (H3)(ii) is satisfied

$$-\Delta_p \tilde{u}(x) = \lambda a(x) |\tilde{u}(x)|^{p-2} \tilde{u}(x) \quad \text{in} \quad B(0,R),$$

$$-\Delta_q \tilde{v}(x) = \lambda d(x) |\tilde{v}(x)|^{q-2} \tilde{v}(x) \quad \text{in} \quad B(0,R).$$

   But from $\|a\| < \lambda_{(1,p)}$ and $\|d\| < \lambda_{(1,q)}$ we get the contradiction.

2. If (H3)(iii) is satisfied, we obtain

$$\begin{aligned} -\Delta_p \tilde{u}(x) &= \lambda a(x) |\tilde{u}(x)|^{p-2} \tilde{u}(x) \quad \text{in} \quad B(0,R), \\ -\Delta_q \tilde{v}(x) &= 0 \quad \text{in} \quad B(0,R), \\ \tilde{u} = \tilde{v} &= 0 \quad \text{on} \quad \partial B(0,R). \end{aligned}$$

   Then from $\|a\| < \lambda_{(1,p)}$, we deduce the contradiction.

So, in the different cases there exists $C_0 > 0$ sufficiently large such that $\forall \rho_1 > C_0$ we have

$$\deg(I - S_\lambda, B(0, \rho_1), 0) = \text{const} \quad \forall \lambda \in [0,1].$$

Hence

$$\deg(I - S_1, B(0, \rho_1), 0) = \deg(I - S_0, B(0, \rho_1), 0) = 1 \quad \forall \rho_1 > C_0. \tag{3.14}$$

The proof of Proposition 3.1 is complete. □

4. The blow up to isolate the trivial solution

We shall prove, under (H1), (H2), and (H4), that there exists some $\rho_2 > 0$ such that

$$\deg(I - T_\tau, B(0, \rho_2), 0) = 0 \quad \forall \tau \in [0, \infty[.$$

Proposition 4.1. *Under the assumptions* (H1), (H2) *and* (H4) *there exists some* $\rho > 0$ *such that for all* $\tau \in [0, \infty[$ *and for all fixed points* $(u, v) \in \chi \backslash \{(0,0)\}$ *of* T_τ *we have* $\|(u,v)\| > \rho$. *This implies that, for* ρ_2 *sufficiently small,*

$$\deg(I - T_\tau, B(0, \rho), 0) = \text{const} = 0 \quad \forall \tau \in [0, \infty[.$$

Proof. Firstly, from the maximum principle, it follows that the problem

$$(u, v) = T_\tau((u, v)) \tag{4.1}$$

is equivalent to find solutions u, v of

$$-\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)' = r^{N-1}\left[a(r)|u(r)|^{\alpha} + b(r)|v(r)+\tau|^{\beta}\right], \tag{4.2}$$

$$-\left(r^{N-1}|v'(r)|^{q-2}v'(r)\right)' = r^{N-1}\left[c(r)|u(r)|^{\gamma} + d(r)|v(r)|^{\delta}\right], \tag{4.3}$$

$$u'(0) = v'(0) = u(R) = v(R) = 0. \tag{4.4}$$

By integrating on $[0, r]$ we get

$$-u'(r) \geq C\, r^{\frac{1}{p-1}}(v(r)+\tau)^{\frac{\beta}{p-1}}, \tag{4.5}$$

$$-v'(r) \geq C\, r^{\frac{1}{q-1}}(u(r))^{\frac{\delta}{q-1}}. \tag{4.6}$$

Hence, $u' < 0$ and $v' < 0$ and it follows that $0 \leq u(r)$, $0 \leq v(r)$.
Thus, from (4.5), we have

$$-u'(r) \geq C\, r^{\frac{1}{p-1}}\tau^{\frac{\beta}{p-1}}. \tag{4.7}$$

By integrating (4.7) from 0 to R, we obtain that

$$u(0) \geq C\, R^{\frac{p}{p-1}}\tau^{\frac{\beta}{p-1}}. \tag{4.8}$$

Now, we introduce new functions $\tilde{u}$ and $\tilde{v}$ in the following way:

$$\tilde{u}(r) = \frac{u(r)}{\sigma^{\ell}} \tag{4.9}$$

$$\tilde{v}(r) = \frac{v(r)}{\sigma^{k}},$$

and make the change of variables

$$y = \frac{r}{\sigma}, \qquad \text{on } [0, R] \tag{4.10}$$

where

$$\sigma = (u(0))^{\frac{1}{\ell}} + (v(0))^{\frac{1}{k}} \tag{4.11}$$

and ℓ, k are positive numbers to be chosen below.

In this way we obtain the following equations for $\tilde{u}(y)$ and $\tilde{v}(y)$ defined on interval $[0, \frac{R}{\sigma}]$:

$$-\frac{d}{dy}\left(y^{N-1}\left|\frac{d\tilde{u}}{dy}(y)\right|^{p-2}\frac{d\tilde{u}}{dy}(y)\right) = y^{N-1}F(\tilde{u}(y), \tilde{v}(y)), \tag{4.12}$$

$$-\frac{d}{dy}\left(y^{N-1}\left|\frac{d\tilde{v}}{dy}(y)\right|^{q-2}\frac{d\tilde{v}}{dy}(y)\right) = y^{N-1}G(\tilde{u}(y), \tilde{v}(y)), \tag{4.13}$$

$$\frac{d\tilde{u}}{dy}(0) = \frac{d\tilde{v}}{dy}(0) = \tilde{u}(R_{\sigma}) = \tilde{v}(R_{\sigma}) = 0, \tag{4.14}$$

where

$$F(\tilde{u}(y), \tilde{v}(y)) = \left[a(\sigma y)A|\tilde{u}(y)|^{\alpha} + b(\sigma y)B\left|\tilde{v}(y) + \frac{\tau}{\sigma^{k}}\right|^{\beta}\right], \tag{4.15}$$

$$G(\tilde{u}(y), \tilde{v}(y)) = \left[c(\sigma y))C|\tilde{u}(y)|^{\gamma} + d(\sigma y))D|\tilde{v}(y))|^{\delta}\right], \tag{4.16}$$

and

$$A = \sigma^{p+\ell(\alpha-p+1)} \qquad B = \sigma^{p-\ell(p-1)+k\beta}, \tag{4.17}$$
$$C = \sigma^{q+k(q-1)+\ell\gamma} \qquad D = \sigma^{q+k(\delta-q+1)},$$
$$R_\sigma = \frac{R}{\sigma}.$$

By choosing

$$\ell = \frac{p(q-1)+\beta q}{(p-1)(q-1)-\beta\gamma} \quad \text{and} \quad k = \frac{q(p-1)+p\gamma}{(p-1)(q-1)-\beta\gamma}, \tag{4.18}$$

we obtain

$$A = \sigma^{\ell\alpha-k\beta}, \quad B = 1, \quad C = 1, \quad D = \sigma^{k\delta-\ell\gamma}. \tag{4.19}$$

Note that $(\tilde{u}, \tilde{v})$ satisfies

$$\frac{d\tilde{u}}{dy}(y) \leq 0, \quad \tilde{u}(y) \leq 1 \quad \forall y \in [0, R_\sigma], \tag{4.20}$$

$$\frac{d\tilde{v}}{dy}(y) \leq 0, \quad \tilde{v}(y) \leq 1 \quad \forall y \in [0, R_\sigma] \tag{4.21}$$

and

$$(\tilde{u}(0))^{\frac{1}{\ell}} + (\tilde{v}(0))^{\frac{1}{k}} = 1. \tag{4.22}$$

Thus, we have

$$\begin{array}{l} -(y^{N-1}|\tilde{u}'(y)|^{p-2}\tilde{u}'(y))' \geq y^{N-1}b(\sigma y)|\tilde{v}(y)|^\beta, \quad \text{on } [0, R_\sigma] \\ -(y^{N-1}|\tilde{u}'(y)|^{q-2}\tilde{u}'(y))' \geq y^{N-1}c(\sigma y)|\tilde{u}(y)|^\gamma, \quad \text{on } [0, R_\sigma] \\ \tilde{u}'(0) = \tilde{v}'(0) = 0. \end{array} \tag{4.23}$$

Integrating (4.23) on $(0, y)$ and taking into account that (H4) holds, we have $\forall y \in [0, R_\sigma]$

$$|\tilde{u}'(y)| \geq \left(\frac{y}{N}\right)^{\frac{1}{p-1}} b_1(\tilde{v}(y))^{\frac{\beta}{p-1}}, \tag{4.24}$$

$$|\tilde{v}'(y)| \geq \left(\frac{y}{N}\right)^{\frac{1}{q-1}} c_1(\tilde{u}(y))^{\frac{\gamma}{q-1}}. \tag{4.25}$$

From Lemma 2.1, we have for $\forall y \in \left]0, \frac{R_\sigma}{2}\right]$

$$\tilde{u}(y) \geq C_{N,p}y|\tilde{u}'(y)| \geq C_{N,p}\left(\frac{1}{N}\right)^{\frac{1}{p-1}} y^{\frac{p}{p-1}} b_1|\tilde{v}(y)|^{\frac{\beta}{p-1}}, \tag{4.26}$$

$$\tilde{v}(y)) \geq C_{N,q}y|\tilde{v}'(y)| \geq C_{N,q}\left(\frac{1}{N}\right)^{\frac{1}{q-1}} y^{\frac{q}{q-1}} c_1|\tilde{u}(y)|^{\frac{\gamma}{q-1}}. \tag{4.27}$$

Thus, from (4.26) and (4.27), we obtain

$$(\tilde{v}(y))^{\frac{(p-1)(q-1)-\beta\gamma}{q(p-1)+p\gamma}} \geq C\, y, \quad \forall y \in \left]0, \frac{R_\sigma}{2}\right], \tag{4.28}$$

$$(\tilde{u}(y))^{\frac{(p-1)(q-1)-\beta\gamma}{p(q-1)+q\beta}} \geq C\, y, \quad \forall y \in \left]0, \frac{R_\sigma}{2}\right], \tag{4.29}$$

where here and henceforth $C > 0$ denotes a positive constant depending only of (a, b, c, d, N, p, q). Taking into account (4.20), (4.21) and since $(\tilde{u}, \tilde{v})$ are non increasing functions on $[0, R_\sigma]$, we obtain

$$y \le C, \quad \forall y \in \left[0, \frac{R_\sigma}{2}\right], \tag{4.30}$$

where $C := C(a, b, c, d, N, p, q)$. Then, as $R_\sigma \to \infty$ when $\sigma \to 0$, (4.30) it is not true for σ sufficiently small. Consequently, since

$$\sigma \le \rho^{\frac{1}{\ell}} + \rho^{\frac{1}{k}}$$

where $\|(u, v)\| = \rho$, it follows, according the above argument, that for ρ sufficiently small the equation $(u, v) = T_\tau((u, v))$ has no solution on $\partial B(0, \rho)$ for $\tau \in [0, +\infty[$. Then, $\deg(I - T_\tau, B(0, \rho), 0)$ is well-defined and by properties of topological degree, we get that

$$\deg(I - T_\tau, B(0, \rho), 0) = \text{const}, \quad \forall \tau \ge 0. \tag{4.31}$$

Moreover, from (4.8), T_{τ_1} has no solution in $B(0, \rho)$ when τ_1 it is sufficiently large than ρ, then we get

$$\deg(I - T_{\tau_1}, B(0, \rho), 0) = 0.$$

Consequently, from of the Leray-Schauder degree properties, we deduce that

$$\deg(I - T_\tau, B(0, \rho), 0) = \deg(I - T_{\tau_1}, B(0, \rho), 0) = 0.$$

□

5. Proof of Theorem 1.1

The proof is an immediate consequence of Proposition 3.1 and Proposition 4.1. By taking ρ_2 sufficiently small, we may assume, from Proposition 4.1 and Leray-Schauder degree properties, that

$$\deg(I - T_\tau, B(0, \rho), 0) = \deg(I - T_0, B(0, \rho), 0) = 0. \tag{5.1}$$

Thus, from Proposition 3.1, for $\rho_1 > 0$ sufficiently large we have

$$\deg(I - S_1, B(0, \rho_1), 0) = 1. \tag{5.2}$$

Then, since

$$S_1 = T_0,$$

by excision property we obtain

$$\deg(I - S_1, B(0, \rho_1) \backslash B(0, \rho_2), 0) = +1. \tag{5.3}$$

Consequently S_1 admits at least one fixed point $(u, v) \neq (0, 0)$. Hence, we obtain the results of Theorem 1.1.

6. Nonexistence

In this section we study some nonexistence result for positive radial solutions for quasilinear system of the form

$$(S_{p,q})\begin{cases} -\Delta_p u \geq a(x)u|u|^{\alpha-1} + b(x)v|v|^{\beta-1} & \text{in } R^N, \\ -\Delta_q v \geq c(x)u|u|^{\gamma-1} + d(x)v|v|^{\delta-1} & \text{in } R^N, \end{cases}$$

First consider the semilinear case, i.e., $p = q = 2$. When, $b = c = 0$, the system $(S_{p,q})$ reduced simply to the case of two single equations

$$-\Delta u \geq u^\alpha, \quad -\Delta v \geq v^\delta \quad \text{on} \quad R^N.$$

This prototype model has been studied quite extensively. For example, we survey some results on a single equation, namely

$$-\Delta u = u^\alpha \qquad \text{on} \quad R^N.$$

In this case we give the results of Gidas and Spruck [10] where the authors prove that if

$$0 < \alpha < \frac{N+2}{N-2}$$

then $u = 0$. A very elementary proof valid for

$$0 < \alpha < \frac{N}{N-2}$$

was given by Souto [15]. In fact his proof is valid for the case of u being a nonnegative supersolution, i.e.,

$$-\Delta u \geq u^\alpha \qquad \text{on} \quad R^N.$$

Always in the semilinear case, if $a = d = 0$ the system $(S_{p,q})$ becomes

$$-\Delta u \geq v^\beta, \quad -\Delta v \geq u^\gamma,$$

which is natural extension of the well known Lane-Emden equation and thus is referred to as the Lane-Emden system. This case is studied by Serrin and Zou [13]; the authors give a nonexistence of positive solutions for system $(S_{2,2})$ when the exponents β and γ are subcritical in the sense

$$\frac{1}{\beta+1} + \frac{1}{\gamma+1} > \frac{N-2}{N}.$$

Moreover, in [14] the same authors prove the existence of positive (radial) solution (u, v) on R^N for the system under the following assumption

$$\frac{1}{\beta+1} + \frac{1}{\gamma+1} \leq \frac{N-2}{N}.$$

Let us now mention the key of our result concerning radial solutions of the quasilinear problem $(S_{p,q})$ in R^N.

Lemma 6.1. *Let* $r_0 \geq 0$, $N > m$ *and* $w \in C^1([r_0, +\infty[) \cap C^2([r_0, +\infty[)$ *is a positive supersolution of*

$$-(r^{N-1}|w'(r)|^{m-2}w'(r))' \geq 0 \qquad \text{on} \quad [r_0, +\infty[. \tag{6.1}$$

Assume

$$w(r) > 0 \quad \text{and} \quad w'(r) < 0 \qquad \forall r \in [r_0, +\infty[.$$

Then there exists a nonnegative number $C > 0$ such that

$$r^{\frac{N-m}{m-1}} w(r) > C.$$

Proof. Since u satisfies (6.1) and $w'(r) < 0$, we deduce that $r^{N-1}|w'(r)|^{p-1}$ is an increasing function on $[r_0, \infty[$. Hence there exists a non negative number C_0 such that

$$r^{N-1}|w'(r)|^{m-1} > C_0 \qquad \forall r \in [r_0, +\infty[. \tag{6.2}$$

Thus, from Lemma 2.1, there exists a nonnegative number $C_{N,m}$ such that

$$w(r) \geq C_{N,m}\, r|w'(r)| \qquad \forall r \in [r_0, +\infty[. \tag{6.3}$$

Consequently, multiplying (6.3) by $r^{\frac{N-m}{m-1}}$ we obtain

$$r^{\frac{N-m}{m-1}} u(r) \geq C_{N,m}\, r^{\frac{N-1}{m-1}} |w'(r)| \qquad \forall r \in [r_0, +\infty[. \tag{6.4}$$

Then, from (6.2)and (6.4), we deduce that

$$r^{\frac{N-m}{m-1}} w(r) \geq C_{N,m}\, r^{\frac{N-1}{m-1}} |w'(r)| \geq C_{N,m} C_0^{\frac{1}{m-1}} \qquad \forall r \in [r_0, +\infty[.$$

Hence the proof of the lemma. □

Our main result is the following:

Theorem 6.1. *Let $u, v \in C^1(R^N) \cap C^2(R^N \backslash 0)$ be nonnegative radial solutions of*

$$\begin{cases} -\Delta_p u \geq b_1 v^\beta, \\ -\Delta_q v \geq c_1 u^\gamma, \end{cases}$$

where $b_1 > 0$ and $c_1 > 0$. Assume

$$\max\{p, q\} < N, \quad \beta > q - 1, \quad \text{and} \quad \gamma > p - 1, \tag{H5}$$

$$\frac{1}{\beta} + \frac{1}{\gamma} > \frac{N-p}{N(p-1)} + \frac{N-q}{N(q-1)}. \tag{H6}$$

Then $u = v = 0$.

Proof. Since (u, v) is supposed to be radial positive solution, then (u, v) satisfies

$$\begin{aligned} &-(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq r^{N-1} b_1 |v(r)|^\beta, \\ &-(r^{N-1}|v'(r)|^{q-2}v'(r))' \geq r^{N-1} c_1 |u(r)|^\gamma, \\ &u'(0) = v'(0) = 0. \end{aligned} \tag{6.5}$$

Integrating (6.5) on $(0, r)$ and taking into account that $u' < 0$, $v' < 0$, we get

$$|u'(r)| \geq \left(\frac{r}{N}\right)^{\frac{1}{p-1}} \left[b_1 v^\beta(r)\right]^{\frac{1}{p-1}}, \quad r > 0 \tag{6.6}$$

$$|v'(r)| \geq \left(\frac{r}{N}\right)^{\frac{1}{q-1}} \left[c_1 u^\gamma(r)\right]^{\frac{1}{q-1}}, \quad r > 0. \tag{6.7}$$

Thus, from Lemma 2.1, we have

$$u(r) \geq C_{N,p} r|u'(r)| \geq C_{N,p} \left(\frac{1}{N}\right)^{\frac{1}{p-1}} r^{\frac{p}{p-1}} \left[b_1 v^\beta(r)\right]^{\frac{1}{p-1}}, \quad r > 0 \tag{6.8}$$

$$v(r) \geq C_{N,p} r|v'(r)| \geq C_{N,p} \left(\frac{1}{N}\right)^{\frac{1}{q-1}} r^{\frac{q}{q-1}} \left[c_1 u^\gamma(r)\right]^{\frac{1}{q-1}}, \quad r > 0. \tag{6.9}$$

Then, from (6.8) and (6.9), we deduce

$$|u(r)|^{p-1} \geq Cr^p b_1 v^\beta(r), \quad \forall r > 0 \tag{6.10}$$

$$|v(r)|^{q-1} \geq Cr^q c_1 u^\gamma(r), \quad \forall r > 0. \tag{6.11}$$

Hence, easily we obtain

$$r^{\frac{-N}{\beta}+\frac{N-q}{q-1}} \left| r^{\frac{N-p}{p-1}} u(r) \right|^{\frac{p-1}{\beta}} \geq C r^{\frac{N-q}{q-1}} v(r), \quad \forall r > 0 \tag{6.12}$$

$$r^{\frac{-N}{\gamma}+\frac{N-p}{p-1}} \left| r^{\frac{N-q}{q-1}} v(r) \right|^{\frac{q-1}{\gamma}} \geq C r^{\frac{N-p}{p-1}} u(r), \quad \forall r > 0. \tag{6.13}$$

Multiplying (6.12) by (6.13), we get

$$r^{\frac{-N}{\beta}+\frac{N-q}{q-1}\,\frac{-N}{\gamma}+\frac{N-p}{p-1}} \geq C \left| r^{\frac{N-q}{q-1}} v(r) \right|^{\frac{\gamma-q+1}{\gamma}} \left| r^{\frac{N-p}{p-1}} u(r) \right|^{\frac{\beta-p+1}{\beta}}, \quad \forall r > 0. \tag{6.14}$$

Consequently, from (H5) and Lemma 6.1, there exists a number $C > 0$ such that for all $r > r_0 > 0$ we have

$$r^{\frac{-N}{\beta}+\frac{N-q}{q-1}\,\frac{-N}{\gamma}+\frac{N-p}{p-1}} \geq C.$$

Then, from (H6), we obtain a contradiction. This concludes the proof of the Theorem 6.1. □

Theorem 6.2. *We make the following assumptions:*

(j) $\max(p, q) < N.$

(jj) $\begin{cases} p-1 \geq \alpha, \quad q-1 \geq \delta \quad \textit{or} \\ (p-1)(q-1) \geq \beta\gamma. \end{cases}$

(jjj) $a, b, c, d : [0, +\infty[\to [0, +\infty[$ *are continuous functions such that*

$$\inf_{s \in [0,+\infty[} (a(s), b(s), c(s)d(s)) > 0.$$

Under these assumptions, the problem

$$(S_{p,q}) \begin{cases} -\Delta_p u \geq a(x)u|u|^{\alpha-1} + b(x)v|v|^{\beta-1} & \textit{in } R^N, \\ -\Delta_q v \geq c(x)u|u|^{\gamma-1} + d(x)v|v|^{\delta-1} & \textit{in } R^N, \end{cases}$$

has no radial positive solutions in $C^1(R^N) \cap C^2(R^N \backslash 0)$.

Proof. By contradiction, let (u, v) be radial positive solution of $(S_{p,q})$. Then (u, v) satisfies

$$\begin{array}{l} -(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq r^{N-1}\left[a(r)|u(r)|^\alpha + b(r)|v(r)|^\beta\right], \\ -(r^{N-1}|v'(r)|^{q-2}v'(r))' \geq r^{N-1}\left[c(r)|u(r)|^\gamma + d(r)|v(r)|^\delta\right], \\ u'(0) = v'(0) = 0. \end{array} \tag{6.15}$$

Arguing as in proof of Theorem 6.1, we deduce from (jjj) that there exits a non-negative number C such that

$$|u(r)|^{p-1} \geq Cr^p \left[a_1 u^\alpha(r) + b_1 v^\beta(r)\right], \quad \forall r > 0 \tag{6.16}$$

$$|v(r)|^{q-1} \geq Cr^q \left[c_1 u^\gamma(r) + d_1 v^\delta(r)\right], \quad \forall r > 0. \tag{6.17}$$

Consequently:

Case 1. $\alpha \le p-1$ and $\delta \le q-1$.

From (6.16) and (6.17) we obtain

$$|u(0)|^{p-1-\alpha} \ge |u(r)|^{p-1-\alpha} \ge Cr^p, \quad \forall r > 0, \tag{6.18}$$

$$|v(0)|^{q-1-\delta} \ge |v(r)|^{q-1-\delta} \ge Cr^q, \quad \forall r > 0. \tag{6.19}$$

Since u and v are nonincreasing, (6.18) and (6.19) lead us to a contradiction.

Case 2. $(p-1)(q-1) > \beta\gamma$.

$$|u(r)|^{p-1} \ge C\ r^p b_1 v^\beta(r), \quad \forall r > 0, \tag{6.20}$$

$$|v(r)|^{q-1} \ge C\ r^q c_1 u^\gamma(r), \quad \forall r > 0. \tag{6.21}$$

Thus, from (6.20) and (6.21)

$$(v(r))^{\frac{(p-1)(q-1)-\beta\gamma}{q(p-1)+p\gamma}} \ge C\ r, \quad \forall r > 0, \tag{6.22}$$

$$(u(r))^{\frac{(p-1)(q-1)-\beta\gamma}{p(q-1)+q\beta}} \ge C\ r, \quad \forall r > 0. \tag{6.23}$$

By an argument like that in Case 1, (6.22) and (6.23), provide a contradiction. This concludes the proof of Theorem 6.2. □

Acknowledgment. The author is most grateful to a referee for careful and constructive comments on an earlier version of this paper.

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This paper is available via http://nyjm.albany.edu:8000/j/2001/7-17.html.