New York Journal of Mathematics

New York J. Math. **7** (2001) 71–86.

On Commutation Relations for 3-Graded Lie Algebras

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Abstract. We prove some commutation relations for a 3-graded Lie algebra, i.e., a $\mathbb{Z}$-graded Lie algebra whose nonzero homogeneous elements have degrees -1, 0 or 1, over a field K. In particular, we examine the free 3-graded Lie algebra generated by an element of degree -1 and another of degree 1. We show that if K has characteristic zero, such a Lie algebra can be realized as a Lie algebra of matrices over polynomials in one indeterminate. In the end, we apply the results obtained to derive the classical commutation relations for elements in the universal enveloping algebra of $\mathfrak{sl}_2(K)$.

Contents

1. Introduction

Many expressions obtained for semisimple Lie algebras of hermitian type indicate that there is an algebraic pattern in the commutation relations for 3-graded Lie

Received March 27, 2001.
Mathematics Subject Classification. 17B70, 16W10.
Key words and phrases. 3-graded Lie algebras, kernel functions.

ISSN 1076-9803/01

algebras, i.e., $\mathbb{Z}$-graded Lie algebras of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1}$$

over a field K of characteristic zero, in particular between elements of the universal enveloping algebras of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$. In order to study these relations, we introduce the free 3-graded Lie algebra $\mathfrak{g}(x, y)$ generated by elements x of degree 1 and y of degree -1. In fact, one can manipulate formal series in the elements of $\mathfrak{g}(x, y)$ and exponentiate them to get a group, which can be treated similarly to the analytic case. This way we obtain a sort of formal Harish-Chandra decomposition which allows us to compute the commutator of two elements in the universal enveloping algebras of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ respectively. The crucial step to prove these relations is Theorem 8 which shows that the center of $\mathfrak{g}(x, y)$ is zero, if $\operatorname{char} K = 0$. As a consequence, $\mathfrak{g}(x, y)$ can be realized as a subalgebra of $\mathfrak{sl}_2(tK[t])$, where t is an indeterminate. This embedding allows one to perform computations inside $SL_2(K[[t]])$, essentially as one does analysis on $SL(2, \mathbb{C})$, to obtain results about $\mathfrak{g}(x, y)$.

According to Corollary 7, Theorem 8 shows that $\mathfrak{g}(x, y)$ is isomorphic to $\mathfrak{g}^{\#}(x, y)$, the free KKT algebra in a pair of variables, if $\operatorname{char} K = 0$. It is known from certain identities in the theory of Jordan pairs that, for $\operatorname{char} K \neq 2, 3$, $\mathfrak{g}^{\#}(x, y)$ can be embedded in $\mathfrak{sl}_2(tK[t])$ as well (see Theorem 6(a)). In contrast, we don't know the situation about $\mathfrak{g}(x, y)$ if K has nonzero characteristic.

We would like to thank the referee of this article for some useful comments which made the text more readable.

2. Basic facts on graded algebras

(i) Let S be a nonempty set. Recall that module M over a ring R is said to be *S-graded* if

$$M = \bigoplus_{s \in S} M_s \qquad \text{with } M_s \text{ a subspace.}$$

Such a decomposition is called *S-gradation* of M and the elements of M_s are said to be *homogeneous* of degree s. If $\Delta \subset \mathbb{Z}$ and $S = \Delta^2$, the term bigradation is synonymous with gradation. Given two S-graded modules M and N over R, a *homomorphism* or *linear map of graded modules* from M into N is a homomorphism $h : M \to N$ of modules such that $h(M_s) \subset N_s$, $s \in S$.

Therefore, in this sense, a gradation of a module is just the choice of a direct sum decomposition. One advantage of such a choice is to allow the introduction of a topology on M. In fact, suppose throughout this article that R and M_s, $s \in S$, are endowed with their discrete topologies. Then each term M_s can be seen as a topological module. The product module with the product topology becomes a topological module and M, with the relative topology, a topological submodule.

The relative topology on a subset X of the product module (including M) is called the *formal topology* of X.

Now, let $G = \mathbb{N}^m$ or $\mathbb{Z}^m$, $m \geqslant 1$. Recall that an (possibly non-associative) algebra $\mathfrak{A}$ over a commutative ring R is said to be *G-graded* if $\mathfrak{A}$ is G-graded as a module and

$$\mathfrak{A}_g \mathfrak{A}_h \subset \mathfrak{A}_{g+h}, \quad g, h \in G.$$

In this case, the G-*gradation* of $\mathfrak{A}$ is the same as the gradation of its underlying module, as well as its *homogeneous* elements and so on.

(ii) If $\rho : G \to G'$ is a homomorphism of monoids then $\mathfrak{A}$ is G'-*graded* with respect to the decomposition

$$\mathfrak{A} = \bigoplus_{h \in G'} \mathfrak{A}_h \quad \text{with} \quad \mathfrak{A}_h = \bigoplus_{\substack{g \in G \\ \rho(g) = h}} \mathfrak{A}_g.$$

That gradation is said to be *derived* from the G-gradation *by means of* ρ. As a special case, we have the *total degree map* $\rho_t(x) = x_1 + \cdots + x_m$, where $x \in G = \mathbb{N}^m$ $(m = 1, 2, \dots)$ and $G' = \mathbb{N}$, and the derived gradation is called *total gradation* of $\mathfrak{A}$.

(iii) Let us remark the connection between a graded Lie algebra and its universal enveloping algebra. If $\mathfrak{A}$ is a graded associative algebra over a commutative ring R then $\mathfrak{A}_L$, the Lie algebra with the same elements as $\mathfrak{A}$ and bracket equal its commutator, is graded with the same gradation. In an opposite direction, let $\mathfrak{L}$ be a graded Lie algebra over R. Its universal enveloping algebra $\mathfrak{U}(\mathfrak{L})$ acquires a unique gradation such that the universal enveloping algebra homomorphism $\mathfrak{i} : \mathfrak{L} \to \mathfrak{U}_L(\mathfrak{L})$ preserves degrees.

Now, if $\mathfrak{A}$ is a G-graded unital associative algebra over R and $\varphi : \mathfrak{L} \to \mathfrak{A}_L$ a Lie algebra homomorphism of graded Lie algebras, the unique homomorphism $\psi : \mathfrak{U}(\mathfrak{L}) \to \mathfrak{A}$ of associative algebras with identity such that $\psi \circ \mathfrak{i} = \varphi$ also preserves degrees.

(iv) A *completion* of a Hausdorff topological module M over a topological ring R is a pair $(f, \widehat{M})$ where $\widehat{M}$ is a complete Hausdorff topological module over R and $f : M \to \widehat{M}$ is a topological module isomorphism from M onto a dense submodule of $\widehat{M}$. Such a completion exists and it is unique up to a diagram-commuting topological module isomorphism. Analogously, one defines the completion of a Hausdorff topological algebra over a commutative topological ring.

The product of the factors of a graded module, as considered in (i), is a complete Hausdorff topological module since each factor is complete and Hausdorff as well and therefore it is, with the natural inclusion of M, a completion of M.

However, it is not always true that the *formal topology* of a graded algebra, that is, that of its underlying graded module, makes it a topological algebra. It occurs if $G = \mathbb{N}^m$, where m is some positive integer.

If $h : \mathfrak{A} \to \mathfrak{B}$ is a continuous linear map between Hausdorff topological algebras then it can be extended uniquely to a continuous linear map $\widehat{h} : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{B}}$. Thus, given a topological subalgebra $\mathfrak{A}$ of a complete Hausdorff topological algebra $\mathfrak{B}$, the inclusion of $\mathfrak{A}$ inside its closure $\overline{\mathfrak{A}}$ in $\mathfrak{B}$ is a completion of $\mathfrak{A}$.

(v) Let $\mathfrak{A}$ be a $\mathbb{N}^m$-graded algebra over R. Then it is a topological algebra with its formal topology, R endowed with the discrete topology. From this point it is easy to describe abstractly a completion of $\mathfrak{A}$. Since $\widehat{\mathfrak{A}}$ is also a completion of the underlying topological module of $\mathfrak{A}$, each element is uniquely written as a series of homogeneous elements in $\mathfrak{A}$ and by continuity, the product between a and b in $\widehat{\mathfrak{A}}$

$$a = \sum_{g\in\mathbb{N}^m} a_g, \qquad b = \sum_{g\in\mathbb{N}^m} b_g \qquad a_g, b_g \in \mathfrak{A}_g$$

is given by

$$ab = \sum_{g\in\mathbb{N}^m} (ab)_g \qquad \text{where} \qquad (ab)_g = \sum_{g_1+g_2=g} a_{g_1} b_{g_2} \quad \in \mathfrak{A}_g.$$

The proof of next lemma is immediate but the conclusion could be totally false if, for instance, the homomorphism between the graded algebras were just an ordinary homomorphism of algebras.

Lemma 1. *Let $\mathfrak{A}$, $\mathfrak{B}$ be $\mathbb{N}^m$-graded algebras, both over R. An injective continuous homomorphism of graded algebras $\psi : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism of topological algebras from $\mathfrak{A}$ onto its image $\psi(\mathfrak{A})$ and its extension $\widehat{\psi} : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{B}}$ is a isomorphism of topological algebras from $\widehat{\mathfrak{A}}$ onto $\overline{\psi(\mathfrak{A})} \subset \widehat{\mathfrak{B}}$.*

(vi) We now fix $R = K$ a field of characteristic zero and $\mathfrak{A}$ a $\mathbb{N}^m$-graded unital associative algebra over K, $m \in \mathbb{N}^*$. The total gradation of $\mathfrak{A}$ produces the same topology as the $\mathbb{N}^m$-gradation, so the respective completions are isomorphic as topological algebras. This way, although the results involving topological aspects like continuity or convergence of series are stated here for $G = \mathbb{N}^m$, it suffices to consider in the proofs the case $m = 1$.

$\widehat{\mathfrak{A}}$ can be decomposed as $\widehat{\mathfrak{A}}^* \bigoplus \mathfrak{A}_0$, where $\widehat{\mathfrak{A}}^*$ is the ideal of $\widehat{\mathfrak{A}}$ consisting of the elements with no component of degree zero. We write $\mathfrak{A}^* = \mathfrak{A} \cap \widehat{\mathfrak{A}}^*$.

The maps $\exp : \widehat{\mathfrak{A}}^* \to 1 + \widehat{\mathfrak{A}}^*$ and $\log : 1 + \widehat{\mathfrak{A}}^* \to \widehat{\mathfrak{A}}^*$ defined by their classical power series expansions converge in $\widehat{\mathfrak{A}}$ and satisfy the usual properties

$$\exp\log(1+a) = 1+a \qquad \log(\exp a) = a, \qquad a \in \widehat{\mathfrak{A}}^*. \tag{2}$$

$$\exp a \, \exp b = \exp \mathfrak{Z}(a,b), \qquad a, b \in \widehat{\mathfrak{A}}^* \qquad \text{with} \tag{3}$$

$$\mathfrak{Z}(a,b) = \sum_{m\geqslant 1} \sum_{p_i+q_i>0} \frac{(-1)^{m-1} [\cdots \overbrace{[aa] \cdots a]}^{p_1} \overbrace{b] \cdots b]}^{q_1} \cdots \overbrace{a] \cdots a]}^{p_m} \overbrace{b] \cdots b]}^{q_m}}{m\,(p_1+q_1+\cdots+p_m+q_m)\; p_1!\, q_1! \cdots p_m!\, q_m!} \in \widehat{\mathfrak{A}}^*.$$

It follows from the above identities that $1 + \widehat{\mathfrak{A}}^*$ is a topological group with respect to the multiplication and topology induced by $\widehat{\mathfrak{A}}$ and the maps exp and log are homeomorphisms. From this point, the proof of the next lemma is immediate.

Lemma 2. (i) *Let $\mathfrak{A}$ be a $\mathbb{N}^m$-graded unital associative algebra over a field K of characteristic zero and $\mathfrak{L}$ a graded Lie subalgebra of $\mathfrak{A}_L$ such that $\mathfrak{L} \subset \mathfrak{A}^*$. Then the set $G = \exp(\widehat{\mathfrak{L}})$, $\widehat{\mathfrak{L}} \subset \widehat{\mathfrak{A}}^*$, with its formal topology (that is, the topology induced by $\widehat{\mathfrak{A}}$), is a topological subgroup of $1 + \widehat{\mathfrak{A}}^*$, called the group obtained exponentiating $\widehat{\mathfrak{L}}$ inside $\widehat{\mathfrak{A}}$, and the map $\exp : \widehat{\mathfrak{L}} \to G$ is a homeomorphism.*

(ii) *Let $\mathfrak{A}_i$ be $\mathbb{N}^m$-graded unital associative algebras over a field K of characteristic zero, $\mathfrak{L}_i$ a graded Lie subalgebra of $(\mathfrak{A}_i)_L$ such that $\mathfrak{L}_i \subset {\mathfrak{A}_i}^*$ and $G_i = \exp(\widehat{\mathfrak{L}}_i)$, $i = 1, 2$. Then for any continuous homomorphism of Lie algebras $h : \mathfrak{L}_1 \to \mathfrak{L}_2$, there exists a unique homomorphism $H : G_1 \to G_2$ of groups such that the diagram*

$$\begin{array}{ccc} \widehat{\mathfrak{L}}_1 & \xrightarrow{\widehat{h}} & \widehat{\mathfrak{L}}_2 \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ G_1 & \xrightarrow[H]{} & G_2 \end{array}$$

is commutative. Furthermore, H is continuous with respect to the formal topologies of G_1 and G_2.

3. On 3-graded Lie algebras

3.1. **Introduction.** A $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ of the form

(4) $$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

over a field K is called a 3-*graded Lie algebra.*

A *Kantor-Koecher-Tits algebra*, or *KKT algebra* for short, is a 3-graded Lie algebra satisfying

(5) $$\begin{aligned} &\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]; \\ &\{x \in \mathfrak{g}_0 \,|\, [x, \mathfrak{g}_{-1}] = [x, \mathfrak{g}_1] = 0\} = 0\,. \end{aligned}$$

Suppose char $K \neq 2, 3$. A *Jordan pair* is a pair of K-modules along with trilinear maps

(6) $$\{\ ,\ ,\ \}_\sigma : V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$$

$$(x, y, z) \mapsto \{x, y, z\}_\sigma,\ \sigma = \pm$$

satisfying

(7) $$\{x, y, z\}_\sigma = \{z, y, x\}_\sigma \quad \text{and}$$

(8) $$[D_\sigma(x, y), D_\sigma(z, w)] = D_\sigma(\{x, y, z\}_\sigma, w) - D_\sigma(z, \{y, x, w\}_{-\sigma})$$

for $x, z \in V^\sigma$ and $y, w \in V^{-\sigma}$, where $D_\sigma : V^\sigma \times V^{-\sigma} \to \mathrm{End}(V^\sigma)$ is defined by

(9) $$D_\sigma(x, y)z = \{x, y, z\}_\sigma\ .$$

Roughly speaking, the concepts of Jordan pairs and KKT algebras are the same. Given a KKT algebra $\mathfrak{g}$, the pair $(V^+, V^-) = (\mathfrak{g}_1, \mathfrak{g}_{-1})$ is a Jordan pair for the trilinear product

$$\{x, y, z\}_\sigma = [[x, y], z]$$

$x, z \in V^\sigma$ and $y \in V^{-\sigma}$. Conversely, given a Jordan pair (V^+, V^-), considering $V^\pm$ as subspaces of $V^+ \oplus V^-$ one has the following KKT algebra:

$$\mathfrak{g}_1 = V^+,\ \mathfrak{g}_{-1} = V^-,$$

$\mathfrak{g}_0 =$ linear combinations of $\{\psi(z, w) : V^+ \oplus V^- \longrightarrow V^+ \oplus V^-\ ,\ z \in V^+, w \in V^-\}$

where

$$\psi(z,w) = \begin{pmatrix} D_+(z,w) & 0 \\ 0 & -D_-(w,z) \end{pmatrix}.$$

with bracket defined as follows:

$$[z+\psi+w, z'+\psi'+w'] = \psi z' - \psi' z + \psi(z,w') + [\psi,\psi'] - \psi(z',w) + \psi w' - \psi' w\,,$$
$$z,z' \in V^+,\ w,w' \in V^-,\ \psi,\psi' \in \mathfrak{g}_0.$$

We have characterized the Jordan pair in terms of graded Lie algebras or, in other words, such a correspondence is an equivalence between the category of Jordan Pairs and the category of KKT algebras.

From a 3-graded Lie algebra $\mathfrak{g}$ one obtains a KKT algebra $\mathfrak{g}^{\#}$, defining

$$\mathfrak{g}^{\#} = \mathfrak{g}'/(\mathfrak{g}'_0 \cap Z_{\mathfrak{g}'}) \tag{10}$$

where $\mathfrak{g}'$ is the 3-graded Lie subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, and $Z_{\mathfrak{g}'}$ denotes its center. Therefore one has

$$\mathfrak{g}_i^{\#} \cong \mathfrak{g}_i,\ i=-1,1, \quad \mathfrak{g}_0^{\#} \cong \mathfrak{g}'_0/(\mathfrak{g}'_0 \cap Z_{\mathfrak{g}'}) \cong \mathrm{ad}_{\mathfrak{g}'}\,\mathfrak{g}'_0, \quad \mathfrak{g}'_0 = [\mathfrak{g}_{-1},\mathfrak{g}_1]. \tag{11}$$

3.2. **The free case.** We say that a Lie algebra $\mathfrak{g}(x,y)$ over K is the *free* 3*-graded Lie algebra generated by variables x of degree* 1 *and y of degree* -1 if for any 3-graded Lie algebra $\mathfrak{h}$ over K and elements $\overline{x} \in \mathfrak{h}_1$ and $\overline{y} \in \mathfrak{h}_{-1}$ there is a unique homomorphism of graded Lie algebras $\psi : \mathfrak{g}(x,y) \to \mathfrak{h}$ such that $\psi(x) = \overline{x}$ and $\psi(y) = \overline{y}$. We alternatively say that $\mathfrak{g}(x,y)$ is the *free* 3*-graded Lie algebra generated by the pair* (x,y). Clearly $\mathfrak{g}(x,y)$ is unique provided it exists.

The existence proof is standard but since it is constructive and introduces some concepts to be used ahead, we give here a sketch. First, let $\mathfrak{F}\mathfrak{L}(x,y)$ be the free Lie algebra over K in two variables x and y with the natural bigradation given by the number n_x of occurrences of x and n_y of y in a Lie monomial and let $\mathfrak{I}$ be the ideal of $\mathfrak{F}\mathfrak{L}(x,y)$ spanned by the homogeneous elements of $\mathfrak{F}\mathfrak{L}(x,y)$ such that $|n_x - n_y| \geq 2$.

Recall that a criterion for an ideal in a graded algebra to be graded is that it be generated by homogeneous elements of the algebra. Thus $\mathfrak{I}$ is a graded ideal, $\mathfrak{g}(x,y) = \mathfrak{F}\mathfrak{L}(x,y)/\mathfrak{I}$ inherits the natural bigradation of $\mathfrak{F}\mathfrak{L}(x,y)$ and the natural projection $\tau : \mathfrak{F}\mathfrak{L}(x,y) \to \mathfrak{g}(x,y)$ becomes a homomorphism of bigraded algebras. By (iii) in Section 2, this bigradation can be extended to a bigradation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}(x,y))$, making it a bigraded algebra. The image by τ of a Lie monomial in $\mathfrak{F}\mathfrak{L}(x,y)$ is called a *Lie monomial* of $\mathfrak{g}(x,y)$.

From its *natural bigradation*, let us produce another for $\mathfrak{g}(x,y)$ given by $n_x - n_y$. This correspond to the gradation obtained from the other one by means of the homomorphism of monoids $\rho_d : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ given by $\rho_d(m,n) = m - n$. With this gradation, we have finally

$$\mathfrak{g}(x,y) = \mathfrak{g}(x,y)_{-1} \oplus \mathfrak{g}(x,y)_0 \oplus \mathfrak{g}(x,y)_1\ .$$

Now, for an arbitrary 3-graded Lie algebra $\mathfrak{h}$ over K and elements $\overline{x} \in \mathfrak{h}_1$ and $\overline{y} \in \mathfrak{h}_{-1}$, let $\phi : \mathfrak{F}\mathfrak{L}(x,y) \to \mathfrak{h}$ be the homomorphism defined by $\phi(x) = \overline{x}$ and $\phi(y) = \overline{y}$. Since $\phi(\mathfrak{I})$ is contained in the ideal of $\mathfrak{h}$ generated by the image of the homogeneous elements of $\mathfrak{F}\mathfrak{L}(x,y)$ such that $|n_x - n_y| \geq 2$, it is zero and ϕ descends to a homomorphism $\psi : \mathfrak{g}(x,y) \to \mathfrak{h}$ such that $\psi \circ \tau = \phi$. From the fact that x, y are generators for $\mathfrak{g}(x,y)$, it is easy to see that ψ is a homomorphism of graded Lie algebras with respect to the 3-gradations of $\mathfrak{g}(x,y)$ and $\mathfrak{h}$ and that such a homomorphism is unique.

3.3. **Commutativity of iterated brackets.** Let $\mathfrak{g}$ be a 3-graded Lie algebra. For $a, b \in \mathfrak{g}$ define

$$[[a,b]]^1 = [a,b], \qquad [[a,b]]^i = [a,[b,[[a,b]]^{i-1}\ ]\ ], \quad i = 2,3,\ldots$$

Now let $z \in \mathfrak{g}_1, w \in \mathfrak{g}_{-1}$. For the sake of simplicity, we sometimes use the following notation: let $I : \mathfrak{g} \to \mathfrak{g}$ be the identity map, $A_i : \mathfrak{g} \to \mathfrak{g}$, $i = 0,1,\ldots$ such that $A_0 = I$ on $\mathfrak{g}_1$, $-I$ on $\mathfrak{g}_{-1}$ and zero on $\mathfrak{g}_0$, $A_i = \mathrm{ad}[[z,w]]^i$, $i = 1,2,\ldots,$ and write for short $A = A_1$.

The next lemma and its corollaries are intended to prove Theorem 6.

Lemma 3. *If* char $K \neq 3$, $[\ [\ [z,w],[z,v]\ ],z] = 0, \quad z \in \mathfrak{g}_1,\ v,w \in \mathfrak{g}_{-1}.$

Proof. Here we write $A = \mathrm{ad}[z,w]$, i.e., A relative to the elements z and w. The left hand side is

$$\begin{aligned}
&= [\ [[z,[z,v]],w],z] + [\ [z,[w,[z,v]]],z]\\
&= A[z,[z,v]] - [\ [z,Av],z]\\
&= [Az,[z,v]] + [z,A[z,v]] - [\ [z,Av],z]\\
&= [Az,[z,v]] + [z,[Az,v]] + [z,[z,Av]] - [\ [z,Av],z]\\
&= [z,[Az,v]] + [z,[Az,v]] + 2\,[z,[z,Av]]\\
&= 2\,[z,[Az,v]] + 2\,[z,[z,Av]].
\end{aligned}$$

On the other hand, the left hand side is also equal to

$$[\ A[z,v],z] = [[\ Az,v],z] + [[z,\ Av],z]$$

Therefore

$$3[\ [[z,w],[z,v]],z] = 0,$$

and since $3 \neq 0$,

$$[\ [[z,w],[z,v]],z] = 0.$$

□

Notice that from $\mathfrak{g}$ one obtains a 3-graded Lie algebra $\mathfrak{g}^{op}$ having the same underlying Lie algebra structure as $\mathfrak{g}$ but with $\mathbb{Z}$-gradation given by

$$\mathfrak{g}_i^{op} = \mathfrak{g}_{-i}, \quad i \in \mathbb{Z}.$$

Hence one can interchange z with w, $\mathfrak{g}_i$ with $\mathfrak{g}_{-i}$, etc in the statements proved for the general 3-graded Lie algebra $\mathfrak{g}$. Such a procedure will be referred simply as *duality* in what follows.

Corollary 4. *Under the hypothesis of Lemma* 3*,*

(1) $[\,[[z,w]]^i,z] = -[\,[[w,z]]^i,z],$
(2) $[\,[[z,w]]^i,w] = -[\,[[w,z]]^i,w].$

Proof. (1) follows from Lemma 3. (2) follows from (1) and duality. □

Corollary 5. *Under the hypothesis of Lemma* 3*,*

(1) $[[[z,w],[[z,w]]^{i}],z]=0$,
(2) $[[[z,w],[[z,w]]^{i}],w]=0$.

Proof. (1) The case where $i>1$ follows from Lemma 3 for $v=[w,[[z,w]]^{i-1}]$.
(2) From (1) and duality, we have

$$\begin{aligned} 0 &= [[[w,z],[[w,z]]^{i}],w] \\ &= -[[[z,w],[[w,z]]^{i}],w] \\ &= -[[[z,[[w,z]]^{i}],w],w]-[[z,[w,[[w,z]]^{i}]],w] \\ &= [[[z,[[z,w]]^{i}],w],w]+[[z,[w,[[z,w]]^{i}]],w] \end{aligned}$$

(by Corollary 4)

$$= [[[z,w],[[z,w]]^{i}],w].$$

□

Theorem 6. *Suppose* $\operatorname{char} K \neq 2,3$*. Let* $\pi:\mathfrak{g}(x,y)\to\mathfrak{g}^{\#}(x,y)$ *be the natural projection and* $\mathfrak{i}:\mathfrak{g}(x,y)\to\mathfrak{sl}_2(tK[t])$ *the homomorphism of graded Lie algebras defined by*

$$\mathfrak{i}(x)=\begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix} \qquad \mathfrak{i}(y)=\begin{pmatrix}0 & 0\\ t & 0\end{pmatrix}.$$

(a) *There exists a monomorphism*

$$\mathfrak{i}^{\#}:\mathfrak{g}^{\#}(x,y)\to\mathfrak{sl}_2(tK[t])$$

of graded Lie algebras such that

$$\mathfrak{i}=\mathfrak{i}^{\#}\circ\pi.$$

(b) $Z_{\mathfrak{g}(x,y)}=\ker\mathfrak{i}=\ker\pi=\operatorname{span}\{[[x,y],[[x,y]]^{i}]\in\mathfrak{g}(x,y)\mid i>1\}$.

Proof. Recall that $\mathfrak{sl}_2(tK[t])$ is also a graded Lie algebra with degrees between -1 and 1 over K with respect to the gradation

$$\mathfrak{sl}_2(tK[t])_{-1}=\left\{\begin{pmatrix}0 & 0\\ p(t) & 0\end{pmatrix} \mid p(t)\in tK[t]\right\},$$

$$\mathfrak{sl}_2(tK[t])_{0}=\left\{\begin{pmatrix}p(t) & 0\\ 0 & -p(t)\end{pmatrix} \mid p(t)\in tK[t]\right\},$$

$$\mathfrak{sl}_2(tK[t])_{1}=\left\{\begin{pmatrix}0 & p(t)\\ 0 & 0\end{pmatrix} \mid p(t)\in tK[t]\right\},$$

$$\mathfrak{sl}_2(tK[t])_{i}=\{0\}\subset\mathfrak{sl}_2(tK[t]),\quad i\in\mathbb{Z}\setminus\{-1,0,1\}.$$

Let $\mathfrak{i}:\mathfrak{g}(x,y)\to\mathfrak{sl}_2(tK[t])$ be the homomorphism defined by

$$\mathfrak{i}(x)=\begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix}, \qquad \mathfrak{i}(y)=\begin{pmatrix}0 & 0\\ t & 0\end{pmatrix}.$$

It is easy to see that this homomorphism is a homomorphism of graded Lie algebras with respect to the total gradation of $\mathfrak{g}(x,y)$ and the natural $\mathbb{N}$-gradation of $\mathfrak{s}l_2(tK[t])$ induced by $tK[t]$. The set

$$\{x, y, [x,y], [x,[y,[[x,y]]^i]], [y,[x,[[x,y]]^i]], [[[x,y]]^i, x], [[[x,y]]^i, y]\,;\, i>0\}$$

spans $\mathfrak{g}(x,y)$ by Corollary 5. Analyzing the image of such elements by $\mathfrak{i}$ in $\mathfrak{s}l_2(tK[t])$ we conclude that

$$\ker \mathfrak{i} = \operatorname{span}\{[\,[x,y],\,[[x,y]]^i] \in \mathfrak{g}(x,y) \,|\, i=2,\dots\}.$$

From Corollary 5 we have

$$\{[\,[x,y],\,[[x,y]]^i] \in \mathfrak{g}(x,y) \,|\, i=2,\dots\} \subset Z_{\mathfrak{g}(x,y)}$$

and hence

$$\ker \mathfrak{i} \subset Z_{\mathfrak{g}(x,y)}.$$

On the other hand, since $\mathfrak{i}$ is a homomorphism, the image of the center of $\mathfrak{g}(x,y)$ by $\mathfrak{i}$ is contained in the center of the image $\mathfrak{i}(\mathfrak{g}(x,y))$, which is zero. Therefore

$$Z_{\mathfrak{g}(x,y)} \subset \ker \mathfrak{i}$$

and

$$Z_{\mathfrak{g}(x,y)} = \ker \mathfrak{i}.$$

Now,

$$\ker \pi = Z_{\mathfrak{g}(x,y)} \cap \mathfrak{g}(x,y)_0 = Z_{\mathfrak{g}(x,y)} = \ker \mathfrak{i}$$

and hence $\mathfrak{i}$ descends to a monomorphism

$$\mathfrak{i}^{\#} : \mathfrak{g}^{\#}(x,y) \to \mathfrak{s}l_2(tK[t])$$

of graded Lie algebras, concluding the proof of the theorem. □

Corollary 7. *Let* $\operatorname{char} K \neq 2,3$. *Then the following statements are equivalent:*

(a) $\mathfrak{g}(x,y) = \mathfrak{g}^{\#}(x,y)$ ($\mathfrak{g}(x,y)$ *is a* KKT *algebra*).
(b) *The center of* $\mathfrak{g}(x,y)$ *is zero.*
(c) *The homomorphism of graded Lie algebras* $\mathfrak{i} : \mathfrak{g}(x,y) \to \mathfrak{s}l_2(tK[t])$ *defined by* $\mathfrak{i}(x) = tE,\ \mathfrak{i}(y) = tF$ *is a monomorphism.*

The main result in this section is the next theorem, which shows that if $\mathfrak{g}(x,y)$ is defined over a field of characteristic zero then the above statements are satisfied.

Theorem 8. *If* $\operatorname{char} K = 0$, *the center of* $\mathfrak{g}(x,y)$ *is zero.*

Proof. It suffices to show that for an any 3-graded Lie algebra $\mathfrak{g}$ over K

$$[\,[z,w],\ [[z,w]]^m\,] = 0,\ z \in \mathfrak{g}_1, w \in \mathfrak{g}_{-1},\ m \text{ a positive integer},$$

by Theorem 6(b).

We proceed by induction. For $m=1$ it is trivial. Suppose valid for $m \le p$, for some positive natural p. First, let us prove some relations

(i) $A_{i+1}z = AA_iz, \quad p \ge i \ge 0.$

The case $i=0$ is trivial. Suppose $p \ge i \ge 1$. Then:

$$A_{i+1}z = -[[z, A_iw], z] = -[z,[A_iw,z]] = -[A_iz,[z,w]] = AA_iz.$$

(ii) $A_{i+1}w = -AA_iw, \quad i \ge 0.$

$$A_{i+1}w = -[[z,A_iw],w] = -[[z,w],A_iw] = -AA_iw.$$

(iii) If $i+j=p$ for integers $i,j\geq 0$ then $[A_{i+1}z, A_jw] = A[z, A_pw]+[A_iz, A_{j+1}w]$.
In fact, suppose $p\geq 2$.

$$[A_{i+1}z, A_jw] = [AA_iz, A_jw] = A[A_iz, A_jw] + [A_iz, A_{j+1}w].$$

But

$$\begin{aligned} A[A_iz, A_jw] &= A[A_iA_jw, z] + AA_i[z, A_jw] = A[A_iA_jw, z] \\ &= (-1)^{j-1}A[A_iA^jw, z] = (-1)^{j-1}A[A^jA_iw, z] \\ &= (-1)^{i+j}A[A^jA^iw, z] = -A[A_pw, z] = A[z, A_pw]. \end{aligned}$$

Therefore

$$[A_{i+1}z, A_jw] = A[z, A_pw] + [A_iz, A_{j+1}w],$$

which is obviously valid if $p=1$.

Returning to the proof of the theorem, we have for $m=p+1$:

$$\begin{aligned} A_{p+1}[z,w] &= [A_{p+1}z, w] + [z, A_{p+1}w] = -[A_{p+1}z, A_0w] + [z, A_{p+1}w] \\ &= -(p+1)A[z, A_pw] - [A_0z, A_{p+1}w] + [z, A_{p+1}w] \quad \text{(by (iii))}. \end{aligned}$$

Thus,

$$(p+2)A_{p+1}[z,w] = -[z, A_{p+1}w] + [z, A_{p+1}w] = 0,$$

concluding the proof of the theorem. □

Let $\mathfrak{g}$ be a 3-graded Lie algebra over K, char $K=0$.

Lemma 9. *If* $z\in\mathfrak{g}_1, w\in\mathfrak{g}_{-1}$,
(i) $A_{i+1}z = AA_iz$,
(ii) $A_{i+1}w = -AA_iw$,
(iii) $[A_{i+1}z, A_jw] = [A_iz, A_{j+1}w]$,
(iv) $[A_iz, A_jw] = [A_jz, A_iw]$ for integers $i,j\geq 0$.

Proof. They follow from the proof of Theorem 8. □

Proposition 10. $[\ [[z,w]]^m\ ,\ [[z,w]]^n\ ] = 0, \quad n,m\geq 1, \quad z\in\mathfrak{g}_1, w\in\mathfrak{g}_{-1}$.

Proof.

$$\begin{aligned} [\ [[z,w]]^m\ ,\ [[z,w]]^n\ ] &= A_m[[z,w]]^n = -A_m[z, A_{n-1}w] \\ &= -[A_mz, A_{n-1}w] - [z, A_mA_{n-1}w] \\ &= -[A_0z, A_{m+n-1}w] - [z, A_mA_{n-1}w] \\ &= -[z, A_{m+n+-1}w] - [z, A_mA_{n-1}w] \\ &= [z, A_{m+n-1}w] - [z, A_{m+n-1}w] \\ &= 0. \end{aligned}$$

□

3.4. **Embedding results.** As before, let K be a field, $K[t]$ the algebra of polynomials over K in t and $tK[t]$ its ideal consisting of multiples of t; analogously, let $K[[t]]$ denote the algebra of formal power series in t with coefficients in K and $tK[[t]]$ its ideal whose elements have independent term equal to zero.

If A denotes one of the above algebras or ideals then let $M_2(A)$ be the algebra of 2×2 matrices over A and $\mathfrak{sl}_2(A)$ the Lie algebra over K consisting of the elements of $M_2(A)$ with trace zero.

The natural gradation of $K[t]$ induces another in $M_2(K[t])$ where the homogeneous elements of a certain degree are exactly the matrices whose entries are monomials in $K[t]$ with the same degree. These *natural* gradations will be used implicitly in what follows.

$M_2(tK[t])$ is a graded ideal (in the obvious sense) of $M_2(K[t])$ and

$$\mathfrak{sl}_2(K[t]) \quad \text{and} \quad \mathfrak{sl}_2(tK[t])$$

are graded Lie subalgebras of $(M_2(K[t]))_L$.

Replacing $K[t]$ with $K[[t]]$ one obtains the respective completions.

Let

$$SL_2(K[[t]])$$

be the group of matrices in $M_2(K[[t]])$ with determinant 1 and

$$SL_2^*(K[[t]]) = \exp(\mathfrak{sl}_2(tK[[t]]))$$

the group obtained exponentiating $\mathfrak{sl}_2(tK[[t]])$ inside $M_2(K[[t]])$.

One has

$$SL_2^*(K[[t]]) = (1 + M_2(tK[[t]])) \cap SL_2(K[[t]]).$$

The following theorem could be called the *embedding theorem* for $\mathfrak{g}(x,y)$. It follows promptly from the previous results.

Theorem 11. *Let $\mathfrak{g}(x,y)$ be the free 3-graded Lie algebra over K, $\operatorname{char} K = 0$, generated by variables x of degree 1 and y of degree -1, endowed with the total gradation given by the sum of occurrences of x and y in each Lie monomial.*

(a) *The homomorphism $\mathfrak{i} : \mathfrak{g}(x,y) \to \mathfrak{sl}_2(tK[t])$ defined by*

$$\mathfrak{i}(x) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \qquad \mathfrak{i}(y) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

is a monomorphism of graded Lie algebras.

(b) *This monomorphism extends to a monomorphism*

$$\widehat{\mathfrak{i}} : \widehat{\mathfrak{g}}(x,y) \to \mathfrak{sl}_2(tK[[t]])$$

of Lie algebras, where $\widehat{\mathfrak{g}}(x,y)$ is the completion of $\mathfrak{g}(x,y)$ with respect to the total gradation.

(c) *Let $\mathfrak{G}(x,y)$ and $SL_2^*(K[[t]])$ be the groups obtained exponentiating $\widehat{\mathfrak{g}}(x,y)$ and $\mathfrak{sl}_2(tK[[t]])$ inside $\widehat{\mathfrak{U}}(\mathfrak{g}(x,y))$ and $M_2(K[[t]])$ respectively. Then we have the* *commutative diagram*

$$\begin{array}{ccc} \widehat{\mathfrak{g}}(x,y) & \xrightarrow{\ \widehat{\mathfrak{i}}\ } & \mathfrak{s}l_2(tK[[t]]) \\ {\scriptstyle\exp}\big\downarrow & & \big\downarrow{\scriptstyle\exp} \\ \mathfrak{G}(x,y) & \xrightarrow[\ \mathfrak{I}\ ]{} & SL_2^*(K[[t]]) \end{array}$$

where the vertical maps are bijective and the map at the bottom is a monomorphism of groups.

(d) *The horizontal monomorphisms in the previous diagram extend to a homomorphism* $I : \widehat{\mathfrak{U}}(\mathfrak{g}(x,y)) \to M_2(K[[t]])$ *of associative algebras with unit.*

(e) *With respect to the formal topology on the subsets of* $\widehat{\mathfrak{U}}(\mathfrak{g}(x,y))$ *and* $M_2(K[[t]])$ (*i.e., the relative topology*)*, the map in* (d) *is continuous and all other maps above are homeomorphisms onto their respective images.*

Proof. (a) It follows from Corollary 7 and Theorem 8.

(b) $\mathfrak{i}$ is continuous with respect to the formal topology and hence it extends to $\widehat{\mathfrak{i}} : \widehat{\mathfrak{g}}(x,y) \to \mathfrak{s}l_2(tK[[t]])$, an isomorphism of topological Lie algebras onto its image by Lemma 1.

(c) The existence of the homomorphism of groups $\mathfrak{I}$ follows immediately from Lemma 2. Since exp and log are homeomorphisms, it follows from the diagram that $\mathfrak{I}$ is, in fact, an isomorphism of topological groups onto its image.

(d) The monomorphism $\mathfrak{i}$ of graded Lie algebras extends to a homomorphism of graded associative algebras with unit $i' : \mathfrak{U}(\mathfrak{g}(x,y)) \to M_2(K[t])$ by the results of (iii) in Section 2. Since i' is continuous, it extends to a continuous homomorphism $I : \widehat{\mathfrak{U}}(\mathfrak{g}(x,y)) \to M_2(K[[t]])$ of associative algebras with unit. From the commutative diagram in (c), we conclude that I also extends $\mathfrak{I}$.

(e) It has already been proved.

□

4. Commutation relations

4.1. **Exponential relation.** Suppose $\operatorname{char} K = 0$.

Corollary 12. *For* $\mathfrak{g}(x,y)$ *one has*

$$\exp x \exp y = \exp\big((1 + \operatorname{ad} y \operatorname{ad} x/2)^{-1} y\big)\, L(x,y)\, \exp\big((1 + \operatorname{ad} x \operatorname{ad} y/2)^{-1} x\big),$$

where

$$\text{(i)} \qquad L(x,y) = \exp\Big(\frac{\log(1 + \operatorname{ad} x \operatorname{ad} y/2)}{\operatorname{ad} x \operatorname{ad} y/2}[x,y]\Big),$$

which can be expanded as

$$\text{(ii)} \qquad L(x,y) = \sum_{m \geqslant 0} \frac{1}{m!} \prod_{i=1}^{m} \big([x,y] - (m-i) \operatorname{ad} x \operatorname{ad} y/2\big).$$

In (ii)*, for* $u, v \in \mathcal{U}(\mathfrak{g}(x,y))$ *and* $U, V \in End_K\big(\mathcal{U}(\mathfrak{g}(x,y))\big)$*, we define the product in* $\mathcal{U}(\mathfrak{g}(x,y)) \oplus End_K\big(\mathcal{U}(\mathfrak{g}(x,y))\big)$ *by*

$$(u \oplus U)(v \oplus V) = (uv + U(v)) \oplus (uV + U \circ V), \quad \text{and}$$

$$\prod_{i=1}^{n} a_i = (\cdots((a_1 a_2) a_3) \cdots a_n) \qquad a_i \in \mathcal{U}(\mathfrak{g}(x,y)) \oplus End\big(\mathcal{U}(\mathfrak{g}(x,y))\big)$$

(*for* $n < 1$ *the product is defined to be* 1).

Proof. (i) It follows from Theorem 11 and

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t(1+t^2)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1+t^2 & 0 \\ 0 & (1+t^2)^{-1} \end{pmatrix} \begin{pmatrix} 1 & t(1+t^2)^{-1} \\ 0 & 1 \end{pmatrix}.$$

(ii) Let us calculate the coefficient of

$$([[x,y]]^{i_1})^{n_1}([[x,y]]^{i_2})^{n_2} \cdots ([[x,y]]^{i_r})^{n_r}, \quad i_j \geqslant 1, n_j \geqslant 0,$$

in formula (ii). Let $m = n_1 i_1 + \cdots + n_r i_r$ and $n = n_1 + \cdots + n_r$. We write $\mathbf{s}$ for the sequence of n elements obtained by first listing n_1 copies of i_1, then n_2 copies of i_2 and so on until n_r copies of i_r. Denote by $\mathfrak{S}_n$ the symmetric group on $1, \ldots, n$ and $\mathcal{P}(\mathbf{s}) = \{\sigma(\mathbf{s}) : \sigma \in \mathfrak{S}(n)\}$; that is, all the permutations of $\mathbf{s}$. The coefficient equals

$$\frac{(-1)^{m-n}}{2^{m-n}\, m!} \sum_{k \in \mathcal{P}(\mathbf{s})} \frac{(m-1)!}{(m-k_1)!} \frac{(m-k_1-1)!}{(m-k_1-k_2)!} \cdots \frac{(m-k_1-k_2-\cdots-k_{n-1}-1)!}{0!}$$

$$= \frac{(-1)^{m-n}}{2^{m-n}} \sum_{k \in \mathcal{P}(\mathbf{s})} \frac{1}{m(m-k_1)\cdots(m-k_1-k_2-\cdots-k_{n-1})}$$

$$= \frac{(-1)^{m-n}}{2^{m-n}} \frac{1}{n_1! \cdots n_r!} \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{m(m-j_{\sigma(1)})\cdots(m-j_{\sigma(1)}-j_{\sigma(2)}\cdots-j_{\sigma(n-1)})}$$

$$= \frac{(-1)^{m-n}}{2^{m-n}} \frac{1}{n_1! \cdots n_r!} \frac{1}{i_1^{n_1} \cdots i_r^{n_r}},$$

which coincides with the corresponding coefficient in formula (i). □

Remark 1. The series $K(x,y) = L(-x,y)$ is called the *canonical kernel function* of $\mathfrak{g}(x,y)$ and it is closely related to the geometry of a bounded symmetric domain in $\mathbb{C}^n$. See [3], [4], [10] for details.

Example 1. In the notation of Theorem 11, we have

$$\mathrm{i}([[x,y]]^i) = 2^{i-1} \begin{pmatrix} t^{2i} & 0 \\ 0 & -t^{2i} \end{pmatrix} \quad i = 1, 2, \ldots$$

and therefore $\{[[x,y]]^i, i = 1, 2, \ldots\}$ is a commutative family of elements in $\mathfrak{g}(x,y)$ as well as $\{[[z,w]]^i, i = 1, 2, \ldots\}$ in $\mathfrak{g}$ for $z \in \mathfrak{g}_1$ and $w \in \mathfrak{g}_{-1}$. Hence, the Embedding Theorem 11 provides an alternative proof to Proposition 10.

4.2. **Commutation relations for powers of elements in $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$.** From this point to the end of the article we fix once and for all a 3-graded Lie algebra $\mathfrak{g}$ over a field K of characteristic zero. Let $z \in \mathfrak{g}_1$ and $w \in \mathfrak{g}_{-1}$. Corollary 12 implies that

$$\frac{z^i}{i!}\frac{w^j}{j!} = \sum_{m=0}^{\min(i,j)} \sum_{\substack{m_1+m_2+m_3=m \\ m_1,\, m_2,\, m_3 \geqslant 0}} A_{m_1}^{j-m} B_{m_2} C_{m_3}^{i-m} \tag{12}$$

where

$$A_0^0 = 1\,, \quad A_m^0 = 0, \quad m > 0\,.$$

$$A_m^k = \frac{1}{k!} \sum_{\substack{n_1+\cdots+n_k=m \\ n_1,\dots,n_k \geqslant 0}} ((-\operatorname{ad} w \operatorname{ad} z/2)^{n_1} w) \cdots ((-\operatorname{ad} w \operatorname{ad} z/2)^{n_k} w),\ k > 0\,, m \geqslant 0.$$

$$B_m = \frac{1}{m!} \prod_{i=1}^{m} ([z,w] - (m-i) \operatorname{ad} z \operatorname{ad} w/2)\,, \quad m \geqslant 0.$$

$$C_0^0 = 1\,, \quad C_m^0 = 0, \quad m > 0\,.$$

$$C_m^k = \frac{1}{k!} \sum_{\substack{n_1+\cdots+n_k=m \\ n_1,\dots,n_k \geqslant 0}} ((-\operatorname{ad} z \operatorname{ad} w/2)^{n_1} z) \cdots ((-\operatorname{ad} z \operatorname{ad} w/2)^{n_k} z), \quad k > 0\,, m \geqslant 0.$$

Given $u \in \mathcal{U}(\mathfrak{g})$ write $u = u_0 + u_*$ for unique u_0, u_* such that $u_0 \in \mathcal{U}(\mathfrak{g}_0)$ and $u_* \in \mathfrak{g}_{-1}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{g}_1$. If $i = j = n$ we have

$$\left(\frac{z^n}{n!}\frac{w^n}{n!}\right)_0 = \frac{1}{n!} \prod_{i=1}^{n} ([z,w] - (n-i) \operatorname{ad} z \operatorname{ad} w/2)\,. \tag{13}$$

For instance, for $\mathfrak{g} = \mathfrak{sl}_2(K)$ and

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z = E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad w = F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\begin{aligned}
\frac{E^i}{i!}\frac{F^j}{j!} &= \sum_{m=0}^{\min(i,j)} \frac{F^{j-m}}{(j-m)!} \left(\sum_{m_2=0}^{m} (-1)^{m-m_2} \binom{i+j-m-m_2-1}{m-m_2} \binom{H}{m_2} \right) \frac{E^{i-m}}{(i-m)!} \\
&= \sum_{m=0}^{\min(i,j)} \frac{F^{j-m}}{(j-m)!} \left(\sum_{m_2=0}^{m} \binom{2m-i-j}{m-m_2} \binom{H}{m_2} \right) \frac{E^{i-m}}{(i-m)!} \\
&= \sum_{m=0}^{\min(i,j)} \frac{F^{j-m}}{(j-m)!} \binom{H-i-j+2m}{m} \frac{E^{i-m}}{(i-m)!} \quad .
\end{aligned}$$

Example 2. From identity (12) we have

$$\begin{aligned}
z^2 w^2 = 2\,[z,w]^2 &- [z,[w,[z,w]]] \\
&+ 4\,w\,[z,w]\,z - 2\,[w,[z,w]]\,z - 2\,w\,[z,[w,z]] + w^2 z^2.
\end{aligned} \tag{14}$$

4.3. **Polarizing multilinear identities.** Let V, W be vector spaces over K and $F : V^n \to W$ a n-linear map, $n \geqslant 1$. Then

$$(15)\quad \sum_{\sigma\in\mathfrak{S}_n} F(X_{\sigma(1)},\cdots,X_{\sigma(n)}) = \sum_{k=1}^{n}\sum_{i_1<\cdots<i_k}(-1)^{n-k}F(X_{i_1}+\cdots+X_{i_k},\dots,X_{i_1}+\cdots+X_{i_k}),$$

where $X_1,\cdots,X_n \in V$ and $\mathfrak{S}_n$ is the symmetric group on $1,\dots,n$.

The above expression can be used to polarize the previous relations. For example, the commutator of two monomials in $\mathcal{U}(\mathfrak{g}_1)$ and $\mathcal{U}(\mathfrak{g}_{-1})$ has $\mathcal{U}(\mathfrak{g}_0)$-component in the "$\mathcal{U}(\mathfrak{g}_{-1})\mathcal{U}(\mathfrak{g}_0)\mathcal{U}(\mathfrak{g}_1)$" decomposition given by:

$$(16)\quad (z_1\cdots z_n\, w_1\cdots w_n)_0 = \sum_{\sigma,\rho\in\mathfrak{S}_n}\frac{1}{n!}\Big(\prod_{i=1}^{n}\big([z_{\sigma(i)},w_{\rho(i)}]-(n-i)\,\mathrm{ad}\,z_{\sigma(i)}\,\mathrm{ad}\,w_{\rho(i)}/2\big)\Big)$$

where $z_i \in \mathfrak{g}_1,\ w_i \in \mathfrak{g}_{-1}$ and

$$\prod_{i=1}^{n} a_i = (\cdots((a_1a_2)a_3)\cdots a_n)$$

$a_i \in \mathcal{U}(\mathfrak{g}) \oplus End\big(\mathcal{U}(\mathfrak{g})\big),\quad 1 \leqslant i \leqslant n.$

Remark 2. Expression (16) has been used in association with the contravariant form on highest weight modules over the complexification of a semisimple Lie algebra of hermitian type. See [3], [4] for details.

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