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The average length of a trajectory in a certain billiard in a flat two-torus

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Abstract. We remove a small disc of radius $\varepsilon > 0$ from the flat torus $\mathbb{T}^2$ and consider a point-like particle that starts moving from the center of the disk with linear trajectory under angle ω. Let $\widetilde{\tau}_\varepsilon(\omega)$ denote the first exit time of the particle. For any interval $I \subseteq [0, 2\pi)$, any $r > 0$, and any $\delta > 0$, we estimate the moments of $\widetilde{\tau}_\varepsilon$ on I and prove the asymptotic formula

$$\int_I \widetilde{\tau}_\varepsilon^r(\omega)\, d\omega = c_r |I| \varepsilon^{-r} + O_\delta(\varepsilon^{-r+\frac{1}{8}-\delta}) \qquad \text{as } \varepsilon \to 0^+,$$

where c_r is the constant

$$\frac{12}{\pi^2} \int_0^{1/2} \left(x(x^{r-1} + (1-x)^{r-1}) + \frac{1-(1-x)^r}{rx(1-x)} - \frac{1-(1-x)^{r+1}}{(r+1)x(1-x)} \right) dx.$$

A similar estimate is obtained for the moments of the number of reflections in the side cushions when $\mathbb{T}^2$ is identified with $[0,1)^2$.

Contents

1. Introduction and main results

For each $0 < \varepsilon < \frac{1}{2}$ we consider the region

$$Z_\varepsilon = \{z \in \mathbb{R}^2 \,;\, \operatorname{dist}(z, \mathbb{Z}^2) > \varepsilon\}$$

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and the *first exit time* (also called *free path length* by some authors)

$$\tau_\varepsilon(z,\omega) = \inf\{\tau > 0\,;\, z + \tau\omega \in \partial Z_\varepsilon\}, \quad z \in Z_\varepsilon,\ \omega \in \mathbb{T},$$

of a point-like particle which starts moving from the point z with linear trajectory, velocity ω, and constant speed equal to 1. This is the model of the periodic two-dimensional Lorentz gas, intensively studied during the last decades (see [2], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [20], [21], [29], [31], [32] for a non-exhaustive list of references). The phase space of the system consists in the range of the initial position and velocity and is one of the spaces $Y_\varepsilon \times \mathbb{T}$ with the normalized Lebesgue measure, or $\Sigma_\varepsilon^+ = \{(x,y) \in \partial Y_\varepsilon \times \mathbb{T}\,;\, \omega \cdot n_x > 0\}$ with the normalized Liouville measure.

Equivalently, one can consider the billiard table $Y_\varepsilon = Z_\varepsilon/\mathbb{Z}^2$ obtained by removing pockets of the form of quarters of a circle of radius ε from the corners. The reflections in the side cushions are specular and the motion ends when the point-like particle reaches one of the pockets at the corners. In this setting $\tau_\varepsilon(z,\omega)$ coincides with the exit time from the table (see Figure 1).

This paper considers the situation where the trajectory starts at the origin $O = (0,0)$. In this case the phase space only consists in the range of the initial velocity of the particle. It is given by the one-dimensional torus $\mathbb{T}$ and can be reduced, for obvious symmetry reasons, to the interval $\left[0, \frac{\pi}{4}\right]$. From the point of view of Diophantine approximation this corresponds to a homogeneous problem. We shall be concerned with estimating the moments of the first exit time $\widetilde{\tau}_\varepsilon(\omega) = \tau_\varepsilon(O,\omega)$ as $\varepsilon \to 0^+$ when the phase space is the range $[0, \frac{\pi}{4}]$ of the velocity ω. This question was raised by Ya. G. Sinai in a seminar at the Moskow University in 1981. We answer the question by supplying asymptotic formulas with explicit main term and error for all the moments of $\widetilde{\tau}_\varepsilon$ in short intervals as follows:

Theorem 1.1. *For any interval $I \subseteq [0, \frac{\pi}{4}]$ and any $r, \delta > 0$, one has*

$$\varepsilon^r \int\limits_I \widetilde{\tau}_\varepsilon^r(\omega)\, d\omega = c_r |I| + \begin{cases} O_{r,\delta}(\varepsilon^{\frac{1}{8}-\delta}) & \text{if } r \neq 2 \\ O_{r,\delta}(\varepsilon^{\frac{1}{4}-\delta}) & \text{if } r = 2 \end{cases} \qquad \text{as} \ \ \varepsilon \to 0^+,$$

where

$$c_r = \frac{12}{\pi^2} \int\limits_0^{1/2} \left(x(x^{r-1} + (1-x)^{r-1}) + \frac{1-(1-x)^r}{rx(1-x)} - \frac{1-(1-x)^{r+1}}{(r+1)x(1-x)} \right) dx.$$

The mean free path length is in this case

$$\frac{4}{\pi} \int\limits_0^{\pi/4} \widetilde{\tau}_\varepsilon(\omega)\, d\omega \sim \frac{c_1}{\varepsilon} = \frac{12}{\pi^2} \cdot \frac{\ln 2}{2\varepsilon} \approx \frac{0.421383}{\varepsilon}.$$

Note also that

$$\lim_{r \to 0^+} c_r = -\frac{12}{\pi^2} \int\limits_0^{1/2} \frac{\ln(1-x)}{x(1-x)}\, dx = 1.$$

To prove Theorem 1.1 we first replace the circular scatterers by cross-like scatterers $[m-\varepsilon, m+\varepsilon] \times \{n\} \cup \{m\} \times [n-\varepsilon, n+\varepsilon]$, $m, n \in \mathbb{Z}^2 \setminus \{(0,0)\}$.[1] We denote by $l_\varepsilon(\omega)$ the free path length in this situation, and first prove:

Theorem 1.2. *For any interval $I \subseteq [0, \frac{\pi}{4}]$ and any $r, \alpha, \delta > 0$, one has*

$$\varepsilon^r \int_I l_\varepsilon^r(\omega)\, d\omega = c_r \int_I \frac{dx}{\cos^r x} + \begin{cases} O_{r,\delta}(\varepsilon^{\frac{1}{2}-2\alpha-\delta} + |I|\varepsilon^\alpha) & \text{if } r \neq 2 \\ O_\delta(\varepsilon^{\frac{1}{2}-\delta}) & \text{if } r = 2 \end{cases} \qquad \text{as } \varepsilon \to 0^+.$$

We consider the probability measures $\widetilde{\mu}_\varepsilon^I$ and μ_ε^I on $[0,\infty)$, defined by

$$\widetilde{\mu}_\varepsilon^I(f) = \frac{1}{|I|} \int_I f\big(\varepsilon \widetilde{\tau}_\varepsilon(\omega)\big)\, d\omega, \quad \mu_\varepsilon^I(f) = \frac{1}{|I|} \int_I f\big(\varepsilon l_\varepsilon(\omega)\big)\, d\omega, \qquad f \in C_c([0,\infty)).$$

Their supports are all contained in $[0, \sqrt{2}]$ as a result of Lemma 3.1. Moreover, we infer from Theorems 1.1 and 1.2 that their moments of order $n \in \mathbb{N}^*$ are of the form[2]

$$\widetilde{\mu}_\varepsilon^I(X^n) = \frac{\varepsilon^n}{|I|} \int_I \widetilde{\tau}_\varepsilon^n(\omega)\, d\omega = c_n + \frac{1}{|I|} O_{n,\delta}(\varepsilon^{\frac{1}{8}-\delta});$$

$$\mu_\varepsilon^I(X^n) = \frac{\varepsilon^n}{|I|} \int_I l_\varepsilon^n(\omega)\, d\omega = \frac{c_n}{|I|} \int_I \frac{dx}{\cos^n x} + \frac{1}{|I|} O_{n,\delta}(\varepsilon^{\frac{1}{6}-\delta}).$$

These asymptotic formulas show in particular that $\widetilde{\mu}_\varepsilon^I(X^n)$ and $\mu_\varepsilon^I(X^n)$ converge to the main terms as $\varepsilon \to 0^+$. The Banach-Alaoglu and Stone-Weierstrass theorems now lead to:

Corollary 1.3. *There exist probability measures $\widetilde{\mu}$ and $\widetilde{\mu}^I$ on $[0, \sqrt{2}]$ such that*

$$\widetilde{\mu}_\varepsilon^I \to \widetilde{\mu} \quad \text{and} \quad \mu_\varepsilon^I \to \mu^I \quad \text{weakly as } \varepsilon \to 0^+.$$

Moreover, the moments of $\widetilde{\mu}$ and μ^I are

$$\int_0^\infty t^n\, d\widetilde{\mu}(t) = c_n$$

and respectively

$$\int_0^\infty t^n\, d\mu^I(t) = \frac{c_n}{|I|} \int_I \frac{dx}{\cos^n x}.$$

Besides, we estimate the average of the number of reflections $\widetilde{R}_\varepsilon(\omega)$ in the side cushions of the billiard table in the case of circular scatterers and prove:

Theorem 1.4. *For any interval $I \subseteq [0, \frac{\pi}{4}]$ and any $r, \delta > 0$, one has*

$$\varepsilon^r \int_I \widetilde{R}_\varepsilon^r(\omega)\, d\omega = c_r \int_I (\sin x + \cos x)^r\, dx + O_{r,\delta}(\varepsilon^{\frac{1}{8}-\delta}) \qquad \text{as } \varepsilon \to 0^+.$$

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[1] Actually it is not hard to see that for $\omega \in [0, \frac{\pi}{4}]$ the result for cross-like scatterers is asymptotically the same as when using vertical scatterers $\{m\} \times [n-\varepsilon, n+\varepsilon]$.

[2] We denote $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, 3, \dots\}$.

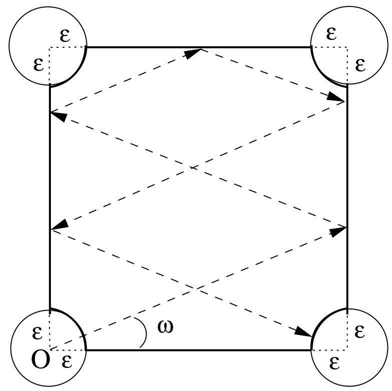

Figure 1. The trajectory of the billiard

Again, we first consider the case of cross-like (or vertical) scatterers, let $R_\varepsilon(\omega)$ denote the number of reflections in the side cushions of the billiard table in this case, and prove:

Theorem 1.5. *For any interval $I \subseteq [0, \frac{\pi}{4}]$ and any $r, \alpha, \delta > 0$, one has*

$$\varepsilon^r \int_I R_\varepsilon^r(\omega)\, d\omega = c_r \int_I (1+\tan x)^r\, dx + O_{r,\delta}(\varepsilon^{\frac{1}{2}-2\alpha-\delta} + |I|\varepsilon^\alpha) \qquad \text{as } \varepsilon \to 0^+.$$

We may also consider the probability measures $\widetilde{\nu}_\varepsilon^I$ and ν_ε^I on $[0,\infty)$ associated with the random variables $\varepsilon \widetilde{R}_\varepsilon$ and $\varepsilon R_\varepsilon$, and defined by

$$\widetilde{\nu}_\varepsilon^I(f) = \frac{1}{|I|}\int_I f\big(\varepsilon \widetilde{R}_\varepsilon(\omega)\big)\, d\omega, \quad \nu_\varepsilon^I(f) = \frac{1}{|I|}\int_I f\big(\varepsilon R_\varepsilon(\omega)\big)\, d\omega, \qquad f \in C_c([0,\infty)).$$

From Theorems 1.4 and 1.5 we derive:

Corollary 1.6. *There exist probability measures $\widetilde{\nu}^I$ and $\widetilde{\nu}^I$ on $[0,\sqrt{2}]$ such that*

$$\widetilde{\nu}_\varepsilon^I \to \widetilde{\nu}^I \quad \text{and} \quad \nu_\varepsilon^I \to \nu^I \quad \text{as} \ \ \varepsilon \to 0^+.$$

Moreover, the moments of $\widetilde{\nu}^I$ and ν^I are

$$\int_0^\infty t^n\, d\widetilde{\nu}^I(t) = \frac{c_n}{|I|}\int_I (\sin x + \cos x)^n\, dx,$$

and respectively

$$\int_0^\infty t^n\, d\nu^I(t) = \frac{c_n}{|I|}\int_I (1+\tan x)^n\, dx.$$

In the case $I \subseteq [\frac{\pi}{4}, \frac{\pi}{2}]$ one gets formulas similar to the ones in Theorems 1.1, 1.2, 1.4 and 1.5 after performing a symmetry with respect to a diagonal of the square, i.e., replacing (α, β) by $(\frac{\pi}{2}-\beta, \frac{\pi}{2}-\alpha)$.

The proofs make use of techniques employed in the study of the spacings between Farey fractions, pioneered in [23], [24], [25], and furthered recently in [3], [4], [1],

[28] where estimates for Kloosterman sums are being used. The first step consists in proving Theorems 1.2 and 1.5, which refer to the case of cross-like or vertical scatterers. In this case one can directly take advantage of the fact that the intervals $I_{a/q} = [\frac{a-\varepsilon}{q}, \frac{a+\varepsilon}{q}]$, with $\frac{a}{q}$ Farey fraction of order $Q = [\frac{1}{\varepsilon}]$, provide a covering of $[0,1]$ such that two intervals $I_{a/q}$ and $I_{a'/q'}$ overlap if and only if $\frac{a}{q}$ and $\frac{a'}{q'}$ are consecutive Farey fractions of order Q.

Finally, the case of circular scatterers is settled by partitioning the range I into $[\varepsilon^{-\theta}]$ intervals of equal size for a convenient value of the exponent θ, and replacing the small circles of radius ε first by vertical scatterers of type $\{m\}\times[n-\varepsilon_-(m,n), n+\varepsilon_+(m,n)]$, and finally by scatterers of type $\{m\}\times[n-\widetilde{\varepsilon}, n+\widetilde{\varepsilon}]$ for appropriate choices of $\varepsilon_\pm(m,n)$ and $\widetilde{\varepsilon}$.

It should be possible in theory to compute the densities of the limit measures from their moments using either the Cauchy transform or the inverse Mellin transform. An attempt of this kind does not seem to easily lead however to a tractable formula for these densities. The convergence of the measures $\widetilde{\mu}_\varepsilon^I$ and $\widetilde{\nu}_\varepsilon^I$ was proved in a different way and the limit measures were explicitly computed in [5].

Techniques using Farey fractions and Kloosterman sums were recently used in [6] to establish the existence, and compute the distribution, of the free path length for the periodic two-dimensional Lorentz gas in the small-scatterer limit in the case where the trajectory does not necessary start from the origin, and one averages over both initial position and initial velocity.

This is the final version of the paper with the same title, circulated as preprint arXiv math.NT/0110208.

2. Farey fractions and Kloosterman sums

For each integer $Q \geq 1$, let $\mathcal{F}_Q$ denote the set of Farey fractions of order Q, i.e., irreducible fractions in the interval $(0,1]$ with denominator $\leq Q$. The number of Farey fractions of order Q in an interval $J \subseteq [0,1]$ can be expressed as

$$\#(J \cap \mathcal{F}_Q) = \frac{Q^2|J|}{2\zeta(2)} + O(Q \ln Q).$$

Recall that if $\frac{a}{q} < \frac{a'}{q'}$ are two consecutive elements in $\mathcal{F}_Q$, then

$$a'q - aq' = 1 \qquad \text{and} \qquad q + q' > Q.$$

Conversely, if $q, q' \in \{1, \ldots, Q\}$ and $q + q' > Q$, then there are $a \in \{1, \ldots, q-1\}$ and $a' \in \{1, \ldots, q'-1\}$ such that $\frac{a}{q} < \frac{a'}{q'}$ are consecutive elements in $\mathcal{F}_Q$. Proofs of these well-known properties of Farey fractions can be found for instance in [26], [23], [30].

Throughout the paper we shall denote by $\mathcal{F}_Q^<$, and respectively by $\mathcal{F}_Q^>$, the set of pairs $(\frac{a}{q}, \frac{a'}{q'})$ of consecutive elements in $\mathcal{F}_Q$ with $q < q'$, and respectively with

$q > q'$. We also set

$$\mathbb{Z}^2_{\mathrm{pr}} = \{(a,b) \in \mathbb{Z}^2 \,;\, \gcd(a,b)=1\};$$

$$\sideset{^J}{}\sum_{a/q} = \sum_{\substack{(a/q,a'/q') \in \mathcal{F}_Q^{<} \\ a/q \in J}} \qquad \text{and} \qquad \sideset{}{^J}\sum_{a/q} = \sum_{\substack{(a/q,a'/q') \in \mathcal{F}_Q^{>} \\ a/q \in J}};$$

$$\Delta_Q = \{(x,y) \in \mathbb{Z}^2_{\mathrm{pr}} \,;\, 0 < x,y \le Q,\ x+y > Q\};$$

$$\mathcal{R}_{m,n} = [m,m+1] \times [n,n+1], \qquad m,n \in \mathbb{R}.$$

For each region $\mathcal{R}$ in $\mathbb{R}^2$ and each C^1 function $f : \mathcal{R} \to \mathbb{C}$, we denote

$$\|f\|_{\infty,\mathcal{R}} = \sup_{(x,y)\in\mathcal{R}} |f(x,y)|, \quad \|Df\|_{\infty,\mathcal{R}} = \sup_{(x,y)\in\mathcal{R}} \left(\left| \frac{\partial f}{\partial x}(x,y) \right| + \left| \frac{\partial f}{\partial y}(x,y) \right| \right).$$

The notation $f \ll g$ means the same thing as $f = O(g)$; that is, there exists an absolute constant $c > 0$ such that $|f| \le cg$ for all values of the variable under consideration. When the constant depends on a parameter δ, this dependence will be indicated by writing $f \ll_\delta g$. The notation $f \asymp g$ will mean that $f \ll g$ and $g \ll f$ simultaneously.

We shall be mainly interested in consecutive Farey fractions $\frac{a}{q} < \frac{a'}{q'}$ in $\mathcal{F}_Q$ with the property that, say, $\frac{a}{q}$ belongs to a prescribed interval $J \subseteq [0,1]$. The equality $a'q - aq' = 1$ yields $a = q - \bar{q}'$, where $\bar{x}$ denotes the unique integer in $\{1,2,\dots,q-1\}$ for which $x\bar{x} = 1 \pmod{q}$. Thus $\frac{a}{q} \in J = [t_1,t_2]$ is equivalent to $\bar{q}' \in J_q^{(1)} := [(1-t_2)q, (1-t_1)q]$. Moreover, $\frac{a'}{q'} \in J$ is equivalent to $\bar{q} \in J_{q'}^{(2)} := [t_1 q', t_2 q']$, where this time $\bar{q}$ denotes the multiplicative inverse of $q \pmod{q'}$.

An important device employed in [3], [4], [1] to estimate sums over primitive lattice points is the Weil type [33] estimate

$$|S(m,n;q)| \ll \tau(q) \gcd(m,n,q)^{\frac{1}{2}} q^{\frac{1}{2}} \tag{2.1}$$

on complete Kloosterman sums

$$S(m,n;q) = \sum_{\substack{x \in [1,q] \\ \gcd(x,q)=1}} e\left(\frac{mx + n\bar{x}}{q} \right),$$

in the presence of an integer albeit not necessarily prime modulus q, proved in [27] (see also [19]). The bound from (2.1) can be used (see [4, Lemma 1.7]) to prove the estimate

$$N_q(\mathcal{I},\mathcal{J}) = \frac{\varphi(q)}{q^2} |\mathcal{I}|\,|\mathcal{J}| + O_\delta(q^{\frac{1}{2}+\delta}) \tag{2.2}$$

for the number $N_q(\mathcal{I},\mathcal{J})$ of pairs of integers $(x,y) \in \mathcal{I} \times \mathcal{J}$ for which $xy = 1 \pmod{q}$, whenever $\mathcal{I}$ and $\mathcal{J}$ are intervals which contain at most q integers.

We shall use the following slight improvement of Corollary 1 and Lemma 8 in [4]. The proof follows literally the reasoning from Lemmas 2, 3 and 8 in [4].

Lemma 2.1. *Let $\Omega \subseteq [1,R]\times[1,R]$ be a convex region and let f be a C^1 function on Ω. Then:*

(i)
$$\sum_{(a,b)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}}} f(a,b) = \frac{1}{\zeta(2)} \iint\limits_{\Omega} f(x,y)\,dx\,dy + \mathcal{E}_{R,\Omega,f},$$

where

$$\mathcal{E}_{R,\Omega,f} \ll \|f\|_{\infty,\Omega} R\ln R + \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \mathcal{R}_{m,n}\subset\overline{\Omega}}} \|Df\|_{\infty,\mathcal{R}_{m,n}} \ln R.$$

(ii) *For any interval $J\subseteq[0,1]$ one has*

$$\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}}\\ \bar b\in J_a}} f(a,b) = \frac{|J|}{\zeta(2)} \iint\limits_{\Omega} f(x,y)\,dx\,dy + \mathcal{F}_{R,\Omega,f,J},$$

where

$$\mathcal{F}_{R,\Omega,f,J} \ll_\delta \|f\|_{\infty,\Omega} m_f R^{\frac{3}{2}+\delta} + \|f\|_{\infty,\Omega}\mathrm{length}(\partial\Omega)\ln R + \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \mathcal{R}_{m,n}\in\overline{\Omega}}} \|Df\|_{\infty,\mathcal{R}_{m,n}} \ln R$$

for any $\delta>0$, where $\bar b$ denotes[3] *the multiplicative inverse of $b \pmod a$, J_a is either $J_a^{(1)}$ or $J_a^{(2)}$, and m_f is an upper bound for the number of intervals of monotonicity of each of the functions $y\mapsto f(x,y)$.*

The proof of (ii) relies on (2.2). We also note the following important corollary of (2.2), which will be often employed in this paper and in the subsequent work from [5] and [6].

Lemma 2.2. *Assume that $q\geq 1$ is an integer, $\mathcal{I}$ and $\mathcal{J}$ are intervals which contain at most q integers, and $f:\mathcal{I}\times\mathcal{J}\to\mathbb{R}$ is a C^1 function. Then for any integer $T>1$ one has*

$$\sum_{\substack{a\in\mathcal{I},\, b\in\mathcal{J}\\ ab=1 (\mathrm{mod}\ q)}} f(a,b) = \frac{\varphi(q)}{q^2} \iint\limits_{\mathcal{I}\times\mathcal{J}} f(x,y)dxdy + E_{q,\mathcal{I},\mathcal{J},f,T},$$

where

$$E_{q,\mathcal{I},\mathcal{J},f,T} \ll_\delta T^2 q^{\frac12+\delta}\|f\|_\infty + Tq^{\frac32+\delta}\|Df\|_\infty + \frac{|\mathcal{I}|\,|\mathcal{J}|\,\|Df\|_\infty}{T}$$

for all $\delta>0$. Here $\|\cdot\|_\infty$ denotes the L^∞-norm on $\mathcal{I}\times\mathcal{J}$.

Proof. If $T\geq q$, then the error is larger than the sum to estimate and there is nothing to prove.

If $T<q$, we partition the intervals $\mathcal{I}$ and $\mathcal{J}$ respectively into T intervals $\mathcal{I}_1,\dots,\mathcal{I}_T$ and $\mathcal{J}_1,\dots,\mathcal{J}_T$ of equal size $|\mathcal{I}_i| = \frac{|\mathcal{I}|}{T}$ and $|\mathcal{J}_j| = \frac{|\mathcal{J}|}{T}$. The idea is to approximate $f(x,y)$ by a constant whenever $(x,y)\in\mathcal{I}_i\times\mathcal{J}_j$. For, we choose for each pair of indices (i,j) a point $(x_{ij},y_{ij})\in\mathcal{I}_i\times\mathcal{J}_j$ for which

$$\iint\limits_{\mathcal{I}_i\times\mathcal{J}_j} f = |\mathcal{I}_i|\,|\mathcal{J}_j| f(x_{ij},y_{ij}). \tag{2.3}$$

[3] When writing $\bar b\in J_a$ we implicitly assume that $\gcd(a,b)=1$.

For $(x,y) \in \mathcal{I}_i \times \mathcal{J}_j$ the mean value theorem gives

$$\begin{aligned} f(x,y) &= f(x_{ij},y_{ij}) + O\big((|\mathcal{I}_i| + |\mathcal{J}_j|)\|Df\|_\infty\big) \\ &= f(x_{ij},y_{ij}) + O\bigg(\frac{q}{T}\,\|Df\|_\infty\bigg). \end{aligned} \tag{2.4}$$

This gives in turn

$$\begin{aligned} \sum_{\substack{a\in\mathcal{I},b\in\mathcal{J}\\ ab=1 (\mathrm{mod}\ q)}} f(a,b) &= \sum_{i,j=1}^{T} \sum_{\substack{(x,y)\in\mathcal{I}_i\times\mathcal{J}_j\\ xy=1 (\mathrm{mod}\ q)}} f(x,y) \\ &= \sum_{i,j=1}^{T} N_q(\mathcal{I}_i,\mathcal{J}_j)\left(f(x_{ij},y_{ij}) + O\Big(\frac{q\|Df\|_\infty}{T}\Big)\right). \end{aligned} \tag{2.5}$$

Since each interval $\mathcal{I}_i$ and $\mathcal{J}_j$ contains at most q integers, estimate (2.2) applies to them and gives

$$N_q(\mathcal{I}_i,\mathcal{J}_j) = \frac{\varphi(q)}{q^2}\,|\mathcal{I}_i|\,|\mathcal{J}_j| + O_\delta(q^{\frac{1}{2}+\delta}). \tag{2.6}$$

As a result of (2.6) and (2.3), the main term in (2.5) becomes

$$\begin{aligned} &\frac{\varphi(q)}{q^2} \sum_{i,j=1}^{T} |\mathcal{I}_i|\,|\mathcal{J}_j| f(x_{ij},y_{ij}) + O_\delta(T^2 q^{\frac{1}{2}+\delta}\|f\|_\infty) \\ &\quad = \frac{\varphi(q)}{q^2} \int\limits_{\mathcal{I}\times\mathcal{J}} f \;+ O_\delta(T^2 q^{\frac{1}{2}+\delta}\|f\|_\infty), \end{aligned}$$

while the error term in (2.5) will be

$$\ll \frac{q\|Df\|_\infty}{T}\left(\frac{\varphi(q)}{q^2}\,|\mathcal{I}|\,|\mathcal{J}| + T^2 q^{\frac{1}{2}+\delta}\right) \leq \|Df\|_\infty\left(\frac{|\mathcal{I}|\,|\mathcal{J}|}{T} + Tq^{\frac{3}{2}+\delta}\right).$$

□

3. The second moment of the first exit time for cross-like scatterers

Throughout this section we keep $0 < \varepsilon < \frac{1}{2}$ fixed, and take

$$Q = Q_\varepsilon = \left[\frac{1}{\varepsilon}\right] = \text{the integer part of } \frac{1}{\varepsilon}\,.$$

We also denote

$$\begin{aligned} \mathbb{Z}^{2*} &= \mathbb{Z}^2 \setminus \{(0,0)\}, \\ C_\varepsilon &= \{0\}\times[-\varepsilon,\varepsilon] \cup [-\varepsilon,\varepsilon]\times\{0\}, \\ V_\varepsilon &= \{0\}\times[-\varepsilon,\varepsilon], \\ l_\varepsilon(\omega) &= \inf\{\tau > 0\,;\, (\tau\cos\omega,\tau\sin\omega) \in C_\varepsilon + \mathbb{Z}^{2*}\} \\ t_P &= \text{the slope of the line } OP, \\ \|(x,y)\| &= \sqrt{x^2+y^2}\,, \qquad x,y\in\mathbb{R}. \end{aligned}$$

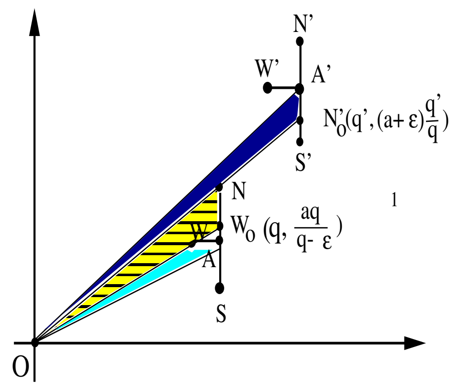

FIGURE 2. The case $q < q'$

Let

$$\mathfrak{C}_\varepsilon = C_\varepsilon + \{(q,a)\,;\, a/q \in \mathcal{F}_Q\}$$

denote the translates of C_ε at all integer points with slope in $\mathcal{F}_Q$.

For each point $A(q,a)$ with $\frac{a}{q} \in \mathcal{F}_Q$ we construct a vertical segment NS of length 2ε and a horizontal segment WE of length 2ε, both centered at A.

Performing symmetries with respect to the integer vertical and horizontal lines, the problem translates into a covering version in $\mathbb{R}^2$. It is clear that one can discard the points (q',a') with $\gcd(q',a') = d > 1$, which are already hidden by $\left(\frac{q'}{d}, \frac{a'}{d}\right)$.

The trajectory will now originate at $O = (0,0)$ and end when it reaches one of the components $(q,a) + C_\varepsilon$ of $\mathfrak{C}_\varepsilon$, $\frac{a}{q} \in \mathcal{F}_Q$, as seen in the next elementary but useful lemma.

Lemma 3.1. *Any ray of direction $\omega \in [0, \frac{\pi}{4}]$ which originates at O inevitably intersects $\mathfrak{C}_\varepsilon$. Moreover, if $\gamma = \frac{a}{q} < \gamma' = \frac{a'}{q'}$ are two consecutive Farey fractions in $\mathcal{F}_Q$ and $\tan\omega \in [\gamma,\gamma']$, then the ray of direction ω intersects either $(q,a) + C_\varepsilon$ or $(q',a') + C_\varepsilon$ and does not intersect any other component of $\mathfrak{C}_\varepsilon$.*[4]

Proof. We shall utilize the inequalities $q + q' \geq Q + 1 > \frac{1}{\varepsilon} \geq Q \geq \max\{q,q'\}$, getting

$$t_A = \frac{a}{q} \leq t_{S'} = \frac{a'-\varepsilon}{q'} < t_N = \frac{a+\varepsilon}{q} \leq t_{A'} = \frac{a'}{q'}\,.$$

In the case $q < q'$, we set $\{W_0\} = OW \cap NS$ and $\{N_0'\} = ON \cap N'S'$ (see Figure 2 and note that $a < a'$), inferring that

[4]Equivalently, the intervals $I_{a/q} = [\frac{a-\varepsilon}{q}, \frac{a+\varepsilon}{q}]$, $\frac{a}{q} \in \mathcal{F}_Q$, cover $[0,1]$ and two such intervals $I_{a/q}$ and $I_{a'/q'}$ overlap if and only if $\frac{a}{q}$ and $\frac{a'}{q'}$ are consecutive elements in $\mathcal{F}_Q$.

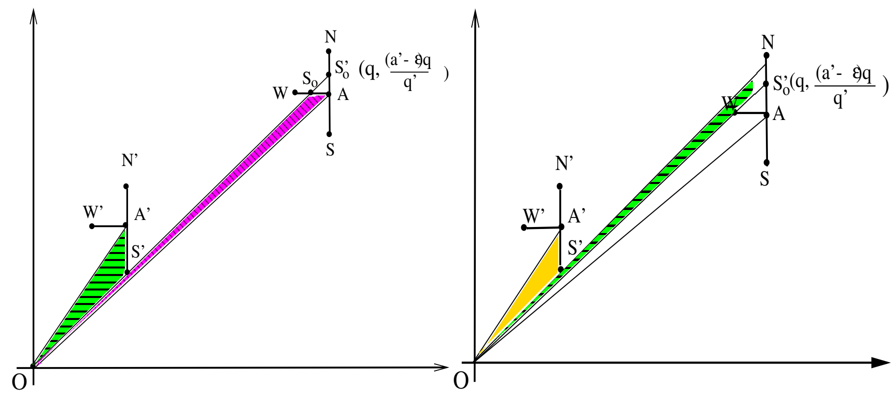

FIGURE 3. The case $q' < q$ and $t_{S'} \le t_W$, respectively $q' < q$ and $t_{S'} > t_W$.

$$(3.1)\qquad \int_{\arctan\gamma}^{\arctan\gamma'} l_\varepsilon^2(\omega)\,d\omega = 2\,\mathrm{area}(\triangle OAN) + 2\,\mathrm{area}(\triangle ON_0'A') - 2\,\mathrm{area}(\triangle AW_0W)$$
$$= \varepsilon q + q'\left(a' - \frac{(a+\varepsilon)q'}{q}\right) + O(\varepsilon^2)$$
$$= \frac{q' - \varepsilon(q'^2 - q^2)}{q} + O(\varepsilon^2).$$

In the case $q > q'$ one has $a' < a$. Moreover,

$$t_A = \gamma \le \min(t_W, t_{S'}) = \min\left(\frac{a}{q-\varepsilon}, \frac{a'-\varepsilon}{q'}\right) \le \max\{t_W, t_{S'}\} \le \gamma' = t_{A'}.$$

This shows that any ray of slope $\tan\omega \in [\gamma, \gamma']$ intersects either $(q,a) + C_\varepsilon$ or $(q', a') + C_\varepsilon$ and no other component of $\mathfrak{C}_\varepsilon$ (see Figures 2 and 3).

Besides, we estimate the average of the second moment of the length $l_\varepsilon(\omega)$ of the trajectory when $\tan\omega \in [\gamma, \gamma']$.

When $t_{S'} \le t_W$ (i.e., $a' + q > \varepsilon + \frac{1}{\varepsilon}$), we set $\{S_0\} = OS' \cap AW$, $\{S_0'\} = OS' \cap NS$, and note (see the first picture in Figure 3) that

$$(3.2)\qquad \int_{\arctan\gamma}^{\arctan\gamma'} l_\varepsilon^2(\omega)\,d\omega = 2\,\mathrm{area}(\triangle OA'S') + 2\,\mathrm{area}(\triangle OAS_0') - 2\,\mathrm{area}(\triangle AS_0S_0')$$
$$= \varepsilon q' + q\left(\frac{(a'-\varepsilon)q}{q'} - a\right) + O(\varepsilon^2)$$
$$= \frac{q - \varepsilon(q^2 - q'^2)}{q'} + O(\varepsilon^2).$$

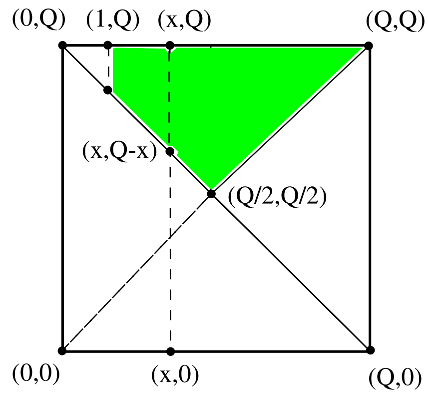

FIGURE 4. The region Ω_Q

When $t_{S'} > t_W$, we set $\{W_0\} = OW \cap NS$, $\{S_0'\} = OS' \cap NS$, and get (see the second picture in Figure 3)

$$\int_{\arctan\gamma}^{\arctan\gamma'} l_\varepsilon^2(\omega)\,d\omega = 2\operatorname{area}(\triangle OA'S') + 2\operatorname{area}(\triangle OAS_0') - 2\operatorname{area}(\triangle AWW_0) \tag{3.3}$$
$$= \frac{q-\varepsilon(q^2-q'^2)}{q'} + O(\varepsilon^2).$$

□

We consider the region

$$\Omega_Q = \{(x,y)\in\mathbb{R}^2\,;\,1\le x\le y\le Q,\ x+y>Q\}$$

and the function

$$f(x,y) = \frac{y+\varepsilon(x^2-y^2)}{x} = \frac{y(1-\varepsilon y)}{x} + \varepsilon x, \qquad (x,y)\in\Omega_Q.$$

Consider also $I = [\alpha,\beta] \subseteq [0,\frac{\pi}{4}]$, take $t_1 = \tan\alpha$, $t_2 = \tan\beta$, and let $J = [t_1,t_2] \subseteq [0,1]$. For $(x,y)\in\Omega_Q$ one has $x > Q-y > \frac{1}{\varepsilon}-1-y$, which gives $1-\varepsilon y < \varepsilon(x+1) \le 2\varepsilon x$. It is also seen that $1-\varepsilon y \ge 1-\frac{y}{Q} \ge 0$. As a result we find that $\|f\|_{\infty,\Omega_Q} \le 3$. Since $\varepsilon^2\#\mathcal{F}_Q < 1$, formulas (3.1), (3.2), (3.3) provide

$$\int_I l_\varepsilon^2(\omega)\,d\omega = 2\sum_{a/q}^{J}\int_{\arctan\frac{a}{q}}^{\arctan\frac{a'}{q'}} l_\varepsilon^2(\omega)\,d\omega + O(1) = 2\sum_{a/q}^{J} f(q,q') + O(1). \tag{3.4}$$

To master the latest sum, we aim to apply Lemma 2.1 to Ω_Q. With the notation from Section 2, relation (3.4) yields

$$\int_I l_\varepsilon^2(\omega)\,d\omega = 2\sum_{\substack{(a,b)\in\Omega_Q\\ \bar b\in J_a^{(1)}}} f(a,b) + O(1). \tag{3.5}$$

We also see that for $(x,y)\in\Omega_Q$ one has $\left|\frac{\partial f}{\partial x}\right| = \left|\varepsilon - \frac{y(1-\varepsilon y)}{x^2}\right| \le \varepsilon + \frac{2\varepsilon y}{x} \le \frac{3}{x}$ and $\left|\frac{\partial f}{\partial y}\right| = \frac{|1-2\varepsilon y|}{x} \le \frac{1}{x}$; hence

$$\sum_{\substack{(a,b)\in\mathbb{Z}^2\\ \mathcal{R}_{a,b}\subset\bar{\Omega}_Q}} \|Df\|_{\infty,\mathcal{R}_{a,b}} \ll \sum_{x=1}^{Q}\ \sum_{y=\max\{Q-x,x\}}^{Q} \frac{1}{x} \ll \sum_{x=1}^{Q} 1 = Q. \tag{3.6}$$

Now we can apply Lemma 2.1 (ii) to the sum from (3.5), and employ (3.6) and $m_f \le 2$, to infer that

$$\int_I l_\varepsilon^2(\omega)\,d\omega = \frac{2(t_2-t_1)}{\zeta(2)} \iint_{\Omega_Q} f(x,y)\,dx\,dy + O_\delta(Q^{\frac{3}{2}+\delta}). \tag{3.7}$$

When $\alpha=0$ and $\beta=\frac{\pi}{4}$, Lemma 2.1 (i) improves upon the error in (3.7) to

$$\int_0^{\pi/4} l_\varepsilon^2(\omega)\,d\omega = \frac{2}{\zeta(2)} \iint_{\Omega_Q} f(x,y)\,dx\,dy + O(Q\ln Q). \tag{3.8}$$

In summary, (3.7), (3.8) and the equality

$$\iint_{\Omega_Q} f(x,y)\,dx\,dy = \frac{1+2\ln 2}{12}\,Q^2$$

lead to:

Theorem 3.2. (i) $\displaystyle \varepsilon^2 \int_0^{\pi/4} l_\varepsilon^2(\omega)\,d\omega = \frac{1+2\ln 2}{\pi^2} + O(\varepsilon|\ln\varepsilon|)$ *as* $\varepsilon \to 0^+$.

(ii) *For any* $0\le\alpha<\beta\le\frac{\pi}{4}$ *and* $\delta>0$, *one has*

$$\varepsilon^2 \int_\alpha^\beta l_\varepsilon^2(\omega)\,d\omega = \frac{(1+2\ln 2)(\tan\beta-\tan\alpha)}{\pi^2} + O_\delta(\varepsilon^{\frac{1}{2}-\delta}) \qquad \textit{as}\ \ \varepsilon\to 0^+.$$

Part (i) of this result was already proved in [22].

4. The r^{th} moment of the first exit time for cross-like scatterers

In this section we estimate the average of the first exit time for cross-like scatterers, thus proving Theorem 1.2. The first step towards estimating the integral $\int_I l_\varepsilon^r(\omega)\,d\omega = \int_I l_\varepsilon^{r-2}(\omega) l_\varepsilon^2(\omega)\,d\omega$ consists in approximating l_ε^{r-2} by a step function.

We take $I=[\alpha,\beta]\subseteq[0,\frac{\pi}{4}]$, $t_1=\tan\alpha$, $t_2=\tan\beta$, $J=[t_1,t_2]\subseteq[0,1]$. For consecutive Farey fractions $\frac{a}{q}<\frac{a'}{q'}$ from $J\cap\mathcal{F}_Q$, where $Q=\left[\frac{1}{\varepsilon}\right]$, we denote

$$\omega_1 = \arctan\frac{a}{q}\,,\quad \omega_2 = \arctan\frac{a+\varepsilon}{q}\,,\quad \omega_2' = \arctan\frac{a'-\varepsilon}{q'}\,,\quad \omega_3 = \arctan\frac{a'}{q'}\,.$$

The function l_ε^{r-2} will be approximated by the constants $l_\varepsilon^{r-2}(\omega_1) = \|(q,a)\|^{r-2}$ on $[\omega_1,\omega_2]$ and by $l_\varepsilon^{r-2}(\omega_3) = \|(q',a')\|^{r-2}$ on $[\omega_2,\omega_3]$ when $q<q'$, and respectively

by $l_\varepsilon^{r-2}(\omega_1)$ on $[\omega_1,\omega_2']$ and by $l_\varepsilon^{r-2}(\omega_3)$ on $[\omega_2',\omega_3]$ when $q>q'$. To be precise, we set

$$A_{r,J,\varepsilon}= \sideset{^J}{}\sum_{a/q} \|(q,a)\|^{r-2}\int_{\omega_1}^{\omega_2} l_\varepsilon^2(\omega)\,d\omega + \sideset{^J}{}\sum_{a/q} \|(q',a')\|^{r-2}\int_{\omega_2}^{\omega_3} l_\varepsilon^2(\omega)\,d\omega,$$

$$B_{r,J,\varepsilon}= \sideset{}{^J}\sum_{a/q} \|(q,a)\|^{r-2}\int_{\omega_1}^{\omega_2'} l_\varepsilon^2(\omega)\,d\omega + \sideset{}{^J}\sum_{a/q} \|(q',a')\|^{r-2}\int_{\omega_2'}^{\omega_3} l_\varepsilon^2(\omega)\,d\omega,$$

$$S_{r,J,\varepsilon}=A_{r,J,\varepsilon}+B_{r,J,\varepsilon}.$$

Next we estimate the quantities

$$E^{(1)}_{r,J,\varepsilon}=\left|\sideset{^J}{}\sum_{a/q}\int_{\omega_1}^{\omega_3} l_\varepsilon^r(\omega)\,d\omega - A_{r,J,\varepsilon}\right| \qquad \Big(\le E^{(1)}_{r,[0,1],\varepsilon}\Big),$$

and respectively

$$E^{(2)}_{r,J,\varepsilon}=\left|\sideset{}{^J}\sum_{a'/q'}\int_{\omega_1}^{\omega_3} l_\varepsilon^r(\omega)\,d\omega - B_{r,J,\varepsilon}\right| \qquad \Big(\le E^{(2)}_{r,[0,1],\varepsilon}\Big).$$

An inspection of the case $q<q'$ in the proof of Lemma 3.1 leads to

$$\sup_{\omega\in[\omega_1,\omega_2]}\left|l_\varepsilon^{r-2}(\omega)-l_\varepsilon^{r-2}(\omega_1)\right| \le \|(q,a+\varepsilon)\|^{r-2}-\|(q-\varepsilon,a)\|^{r-2} \ll_r \varepsilon Q^{r-3},$$

and to

$$\begin{aligned}\sup_{\omega\in[\omega_2,\omega_3]}\left|l_\varepsilon^{r-2}(\omega)-l_\varepsilon^{r-2}(\omega_3)\right| &\le \|(q',a')\|^{r-2}-\left\|\left(q',\frac{(a+\varepsilon)q'}{q}\right)\right\|^{r-2}\\ &\ll_r \left(a'-\frac{(a+\varepsilon)q'}{q}\right)Q^{r-3} \ll \frac{Q^{r-3}}{q}\,.\end{aligned}$$

Therefore

$$\begin{aligned}(4.1)\qquad &\left|\int_{\omega_1}^{\omega_3} l_\varepsilon^r(\omega)\,d\omega - l_\varepsilon^{r-2}(\omega_1)\int_{\omega_1}^{\omega_2} l_\varepsilon^2(\omega)\,d\omega - l_\varepsilon^{r-2}(\omega_3)\int_{\omega_2}^{\omega_3} l_\varepsilon^2(\omega)\,d\omega\right|\\ &\qquad \ll_r q^2(\omega_2-\omega_1)\varepsilon Q^{r-3}+q'^2(\omega_3-\omega_2)\,\frac{Q^{r-3}}{q}\,.\end{aligned}$$

But $\omega_2-\omega_1\le \frac{a+\varepsilon}{q}-\frac{a}{q}=\frac{\varepsilon}{q}$ and $\omega_3-\omega_2\le\frac{a'}{q'}-\frac{a+\varepsilon}{q}=\frac{1-\varepsilon q'}{qq'}<\frac{1}{qq'}$, so the right-hand side in (4.1) is

$$\ll_r q\varepsilon^2 Q^{r-3}+\frac{Q^{r-2}}{q^2}\ll\frac{Q^{r-2}}{q^2}\,.$$

As a result we infer that

$$(4.2)\qquad E^{(1)}_{r,[0,1],\varepsilon}=O_r\left(Q^{r-2}\sum_{q=1}^{Q}\frac{\varphi(q)}{q^2}\right)=O_r(Q^{r-2}\ln Q).$$

In the case $q' < q$ we get (in both subcases $t_{S'} \leq t_W$ and $t_{S'} > t_W$)

$$\sup_{\omega\in[\omega_1,\omega_2']} \left| l_\varepsilon^{r-2}(\omega) - l_\varepsilon^{r-2}(\omega_1)\right| \leq \left\| \left(q, \frac{(a'-\varepsilon)q}{q'} \right) \right\|^{r-2} - \|(q,a-\varepsilon)\|^{r-2}$$
$$\ll_r \left(\frac{(a'-\varepsilon)q}{q'} - a + \varepsilon \right) Q^{r-3} \leq \frac{Q^{r-3}}{q'}$$

and

$$\sup_{\omega\in[\omega_2',\omega_3]} \left| l_\varepsilon^{r-2}(\omega) - l_\varepsilon^{r-2}(\omega_3)\right| \leq \|(q',a')\|^{r-2} - \|(q',a'-\varepsilon)\|^{r-2} \ll_r \varepsilon Q^{r-3}.$$

Employing also $\omega_2' - \omega_1 = \frac{a'-\varepsilon}{q'} - \frac{a}{q} = \frac{1-\varepsilon q}{qq'} \leq \frac{1}{qq'}$ and $\omega_3 - \omega_2' = \frac{a'}{q'} - \frac{a'-\varepsilon}{q'} = \frac{\varepsilon}{q'}$, we get in the case $q' < q$ the estimate

$$\left| \int_{\omega_1}^{\omega_3} l_\varepsilon^r(\omega)\, d\omega - l_\varepsilon^{r-2}(\omega_1) \int_{\omega_1}^{\omega_2'} l_\varepsilon^2(\omega)\, d\omega - l_\varepsilon^{r-2}(\omega_3) \int_{\omega_2'}^{\omega_3} l_\varepsilon^2(\omega)\, d\omega \right|$$
$$\ll_r \frac{Q^{r-1}}{qq'^2} + \frac{Q^{r-3}}{q'} \ll \frac{Q^{r-2}}{q'}\,.$$

Hence

$$E^{(2)}_{r,[0,1],\varepsilon} = O_r\left(\sum_{(q,q')\in\Delta_Q} \frac{Q^{r-2}}{q'^2} \right) = O_r\left(Q^{r-2} \sum_{q'=1}^{Q} \frac{\varphi(q')}{q'^2} \right) = O_r(Q^{r-2}\ln Q). \tag{4.3}$$

Since the contribution of a single term $\int_{\omega_1}^{\omega_3} l_\varepsilon^r(\omega)\, d\omega$ is $\ll \frac{\max\{q,q'\}^r}{qq'} \leq Q^{r-1}$, we infer from (4.2) and (4.3) that

$$\int_I l_\varepsilon^r(\omega)\, d\omega = S_{r,J,\varepsilon} + O_r(\varepsilon^{1-r}). \tag{4.4}$$

Next, we adjust the second integral in the expression of $A_{r,J,\varepsilon}$, writing

$$\begin{aligned}
a'^2 + q'^2 &= q'^2 \left(\left(\frac{a'}{q'}\right)^2 + 1 \right) = q'^2 \left(\left(\frac{a}{q} + \frac{1}{qq'} \right)^2 + 1 \right) \\
&= \left(\frac{q'}{q}\right)^2 \left(\left(a + \frac{1}{q'} \right)^2 + q^2 \right) \\
&= \left(\frac{q'}{q}\right)^2 \left(a^2 + q^2 + O\left(\frac{a}{q'}\right) \right) \\
&= \left(\frac{q'}{q}\right)^2 (a^2+q^2) \left(1 + O\left(\frac{a}{q'(a^2+q^2)} \right) \right) \\
&= \left(\frac{q'}{q}\right)^2 (a^2+q^2) \left(1 + O\left(\frac{1}{qq'} \right) \right) \\
&= \left(\frac{q'}{q}\right)^2 (a^2+q^2) \left(1 + O\left(\frac{1}{Q} \right) \right).
\end{aligned}$$

This gives in turn

$$\|(q',a')\|^{r-2} = \left(\frac{q'}{q}\right)^{r-2} \|(q,a)\|^{r-2}\left(1+O_r\Big(\frac{1}{Q}\Big)\right). \tag{4.5}$$

In a similar way

$$\|(q,a)\|^{r-2} = \left(\frac{q}{q'}\right)^{r-2} \|(q',a')\|^{r-2}\left(1+O_r\Big(\frac{1}{Q}\Big)\right). \tag{4.6}$$

For further use, it is also worth to note

$$\frac{1+\frac{a'}{q'}}{1+(\frac{a'}{q'})^2} = \frac{\big(1+\frac{a}{q}\big)\big(1+O(\frac{1}{Q})\big)}{\big(1+(\frac{a}{q})^2\big)\big(1+O(\frac{1}{Q})\big)} = \frac{1+\frac{a}{q}}{1+(\frac{a}{q})^2}\left(1+O\Big(\frac{1}{Q}\Big)\right). \tag{4.7}$$

Making use of (see the proof of Lemma 3.1)

$$\int_{\omega_1}^{\omega_2} l_\varepsilon^2(\omega)\,d\omega = 2\,\mathrm{area}(\triangle OAN) + O(\varepsilon^2) = \varepsilon q + O(\varepsilon^2),$$

$$\int_{\omega_2}^{\omega_3} l_\varepsilon^2(\omega)\,d\omega = 2\,\mathrm{area}(\triangle ON_0'A') = \frac{q'(1-\varepsilon q')}{q}\,,$$

and (4.5), we see that $A_{r,J,\varepsilon}$ can be expressed as

$$\sum_{a/q}^{J} \|(q,a)\|^{r-2}\left(\frac{q'}{q}\right)^{r-2}\frac{q'(1-\varepsilon q')}{q}\left(1+O_r\Big(\frac{1}{Q}\Big)\right) + \varepsilon \sum_{a/q}^{J}\|(q,a)\|^{r-2}q + O_r(1). \tag{4.8}$$

If $\bar{x}$ denotes the inverse of the integer $x \pmod q$ in $[1,q]$, then $a'q-aq'=1$ gives $a=q-\bar{q'}$. Since $\frac{1-\varepsilon q'}{q}<\varepsilon$, the error in the first sum in (4.8) is $\ll_r Q^2Q^{r-1}\varepsilon Q^{-1} = Q^{r-1}$, and so $A_{r,J,\varepsilon}$ is equal up to an error term of order $O_r(Q^{r-1})$ to

$$\begin{aligned} &\sum_{a/q}^{J}\|(q,a)\|^{r-2}\left(\varepsilon q + \frac{q'^{r-1}}{q^{r-1}} - \frac{\varepsilon q'^r}{q^{r-1}}\right) \\ &= \sum_{q=1}^{Q} \sum_{\substack{\max\{q,Q-q\}<x\le Q\\ \bar{x}\in J_q^{(1)}}} \big((q-\bar{x})^2+q^2\big)^{\frac{r-2}{2}}\left(\varepsilon q + \frac{x^{r-1}}{q^{r-1}} - \frac{\varepsilon x^r}{q^{r-1}}\right), \end{aligned} \tag{4.9}$$

with $J_q^{(1)}$ as defined in Section 2.

When $q>q'$, we employ (see the proof of Lemma 3.1)

$$\int_{\omega_1}^{\omega_2'} l_\varepsilon^2(\omega)\,d\omega = 2\,\mathrm{area}(\triangle OAS_0') + O(\varepsilon^2) = \frac{(1-\varepsilon q)q}{q'} + O(\varepsilon^2),$$

$$\int_{\omega_2'}^{\omega_3} l_\varepsilon^2(\omega)\,d\omega = 2\,\mathrm{area}(\triangle OA'S') = \varepsilon q',$$

together with $Q^{r-2}\varepsilon^2\#\mathcal{F}_Q \leq Q^{r-2}$, relation (4.6), and the fact that $a'q - aq' = 1$ implies $a' = \bar{q} \pmod{q'}$, to infer that $B_{r,J,\varepsilon}$ is expressible as

$$\begin{aligned}
&\sum_{a'/q'}^{J} \|(q,a)\|^{r-2}\frac{(1-\varepsilon q)q}{q'} + \sum_{a'/q'}^{J}\left(\frac{q'}{q}\right)^{r-2}\|(q,a)\|^{r-2}\varepsilon q' + O_r(Q^{r-1})\\
&= \sum_{a'/q'}^{J} \|(q',a')\|^{r-2}\left(\varepsilon q' + \frac{q^{r-1}}{q'^{r-1}} - \frac{\varepsilon q^r}{q'^{r-1}}\right) + O_r(Q^{r-1})\\
&= \sum_{q'=1}^{Q} \sum_{\substack{\max\{q',Q-q'\}<x\leq Q\\ \bar{x}\in J_{q'}^{(2)}}} \left(\bar{x}^2 + q'^2\right)^{\frac{r-2}{2}}\left(\varepsilon q' + \frac{x^{r-1}}{q'^{r-1}} - \frac{\varepsilon x^r}{q'^{r-1}}\right) + O_r(Q^{r-1}),
\end{aligned}$$

where $\bar{x}$ denotes the inverse of an integer $x \pmod{q'}$ in $[1,q']$. Changing notation, $B_{r,J,\varepsilon}$ can be rewritten as

$$\sum_{q=1}^{Q} \sum_{\substack{\max\{q,Q-q\}<x\leq Q\\ \bar{x}\in J_{q}^{(2)}}} \left(\bar{x}^2 + q^2\right)^{\frac{r-2}{2}}\left(\varepsilon q + \frac{x^{r-1}}{q^{r-1}} - \frac{\varepsilon x^r}{q^{r-1}}\right) + O_r(Q^{r-1}). \tag{4.10}$$

By (4.4), (4.9) and (4.10), we infer that

$$\int_I l_\varepsilon^r(\omega)\,d\omega = T_{r,J,\varepsilon} + O_r(Q^{r-1}), \tag{4.11}$$

where $T_{r,J}(\varepsilon) = S_1(Q) + S_2(Q)$, with

$$S_k(Q) = \sum_{q=1}^{Q} \sum_{\substack{\max\{Q-q,q\}<x\leq Q\\ \bar{x}\in J_{q}^{(k)}}} f_k(x,\bar{x},q), \qquad k=1,2, \tag{4.12}$$

and

$$\begin{aligned}
f_2(x,y,z) &= \left(y^2+z^2\right)^{\frac{r-2}{2}}\left(\varepsilon z + \frac{x^{r-1}}{z^{r-1}} - \frac{\varepsilon x^r}{z^{r-1}}\right),\\
f_1(x,y,z) &= f_2(x,z-y,z).
\end{aligned}$$

For each $q \in [1,Q]$, the functions $f_k(\cdot,\cdot,q)$, defined on $[1,Q]\times[1,q]$, manifestly satisfy the estimates

$$\|f_k(\cdot,\cdot,q)\|_\infty \ll_r \frac{Q^{r-1}}{q} \tag{4.13}$$

and

$$\|Df_k(\cdot,\cdot,q)\|_\infty \ll_r \frac{Q^{r-2}}{q} \leq \frac{Q^{r-1}}{q^2}\,. \tag{4.14}$$

Thus we may consider in Lemma 2.2 for each $q \in [1,Q]$ the function $f_k(\cdot,\cdot,q)$, the intervals $\mathcal{I} = (\max\{Q-q,q\},Q]$ and $\mathcal{J} = J_q^{(k)}$ with $|\mathcal{I}| \leq q$ and $|\mathcal{J}| = q|I|$, and take $T = [Q^\alpha]$, to infer that the inner sum in (4.12) can be expressed as

$$\frac{\varphi(q)}{q^2} \int\limits_{\max\{Q-q,q\}}^{Q} dx \int\limits_{J_q^{(k)}} dy\ f_k(x,y,q)$$
$$+ O_{\delta,r}(q^{-\frac{1}{2}+\delta} Q^{r-1+2\alpha} + q^{-\frac{1}{2}+\delta} Q^{r-1+\alpha} + |I| Q^{r-1-\alpha}).$$

Summing up over $q \in [1,Q]$, we arrive at

$$S_k(Q) = \sum_{q=1}^{Q} \frac{\varphi(q)}{q}\, g(q) + O_{\delta,r}(Q^{r-\frac{1}{2}+2\alpha+\delta} + |I| Q^{r-\alpha}), \tag{4.15}$$

where

$$\begin{aligned} g(z) &= \frac{1}{z} \int\limits_{\max\{Q-z,z\}}^{Q} dx \int\limits_{(1-t_2)z}^{(1-t_1)z} dy\ f_1(x,y,z) \\ &= \frac{1}{z} \int\limits_{\max\{Q-z,z\}}^{Q} dx \int\limits_{t_1 z}^{t_2 z} dy\ f_2(x,y,z), \qquad z \in [1,Q]. \end{aligned}$$

The formulas

$$\begin{aligned} \frac{d}{dz}\left(\frac{1}{z}\int\limits_{z}^{Q} dx \int\limits_{az}^{bz} dy\, h(x,y,z)\right) &= -\frac{1}{z^2}\int\limits_{z}^{Q} dx \int\limits_{az}^{bz} dy\, h(x,y,z) \\ &\quad + \frac{1}{z}\int\limits_{z}^{Q} dx \int\limits_{az}^{bz} dy\, \frac{\partial h}{\partial z}(x,y,z) - \frac{1}{z}\int\limits_{az}^{bz} h(z,y,z)\, dy \\ &\quad + \frac{b}{z}\int\limits_{z}^{Q} h(x,bz,z)\, dx - \frac{a}{z}\int\limits_{z}^{Q} h(x,az,z)\, dx, \\ \frac{d}{dz}\left(\frac{1}{z}\int\limits_{Q-z}^{Q} dx \int\limits_{az}^{bz} dy\, h(x,y,z)\right) &= -\frac{1}{z^2}\int\limits_{Q-z}^{Q} dx \int\limits_{az}^{bz} dy\, h(x,y,z) \\ &\quad + \frac{1}{z}\int\limits_{Q-z}^{Q} dx \int\limits_{az}^{bz} dy\, \frac{\partial h}{\partial z}(x,y,z) + \frac{1}{z}\int\limits_{az}^{bz} h(Q-z,y,z)\, dy \\ &\quad + \frac{b}{z}\int\limits_{Q-z}^{Q} h(x,bz,z)\, dx - \frac{a}{z}\int\limits_{Q-z}^{Q} h(x,az,z)\, dx, \end{aligned}$$

and the estimates (4.13) and (4.14) show that $|g'(z)| \ll_r \frac{Q^{r-1}}{z}$. As a result we get $\int_1^Q |g'(z)|\, dz \ll_r Q^{r-1} \ln Q$. It is also clear that $\|g\|_\infty \ll_r Q^{r-1}$, so we are in the position of being able to apply Lemma 2.3 in [3] to g, collecting

$$\sum_{q=1}^{Q} \frac{\varphi(q)}{q}\, g(q) = \frac{1}{\zeta(2)} \int_1^Q g(z)\, dz + O\left(\left(\|g\|_\infty + \int_1^Q |g'(z)|\, dz \right) \ln Q \right)$$
$$= \frac{1}{\zeta(2)} \int_1^Q g(z)\, dz + O_r(Q^{r-1} \ln^2 Q).$$

Comparing the previous relation with (4.15), we infer that both $S_1(Q)$ and $S_2(Q)$ can now be expressed as

$$\frac{1}{\zeta(2)} \int_1^Q g(z)\, dz + O_{\delta,r}(Q^{r-\frac{1}{2}+2\alpha+\delta} + |I| Q^{r-\alpha}). \tag{4.16}$$

Taking into account (4.11) and (4.12), we gather

$$\int_I l_\varepsilon^r(\omega)\, d\omega = \frac{2}{\zeta(2)} \int_1^Q g(z)\, dz + O_{\delta,r}(Q^{r-\frac{1}{2}+2\alpha+\delta} + |I| Q^{r-\alpha}). \tag{4.17}$$

Integrating with respect to y in the formula that gives g and changing then z into Qz, we may express the main term in (4.17) as

$$\frac{K_{r,I}}{\zeta(2)} \int_1^Q dz \int_{\max\{Q-z,z\}}^{Q} dx \left(\varepsilon + \frac{x^{r-1}}{z^r} - \frac{\varepsilon x^r}{z^r} \right) z^{r-1} \tag{4.18}$$
$$= \frac{K_{r,I} Q^{r+1}}{\zeta(2)} \int_{1/Q}^{1} dz \int_{\max\{1-z,z\}}^{1} dx \left(\varepsilon z^{r-1} + \frac{x^{r-1}}{Qz} - \frac{\varepsilon x^r}{z} \right),$$

where

$$K_{r,I} = 2 \int_{t_1}^{t_2} (1+t^2)^{\frac{r-2}{2}}\, dt = 2 \int_I \frac{dx}{\cos^r x}\,.$$

Up to an error term of order $O_r(Q^{-2})$, the double integral in the right-hand side of (4.18) is given by

$$\varepsilon \int_0^{1/2} z(z^{r-1} + (1-z)^{r-1})\, dz + \frac{1}{rQ} \int_0^{1/2} \frac{1-(1-z)^r}{z(1-z)}\, dz - \frac{\varepsilon}{r+1} \int_0^{1/2} \frac{1-(1-z)^{r+1}}{z(1-z)}\, dz.$$

Since $Q^{r+1} = \frac{1}{\varepsilon^{r+1}} + O_r(\frac{1}{\varepsilon^r})$, we now infer from (4.17) and (4.18) the equality

$$\int_I l_\varepsilon^r(\omega)\, d\omega = \frac{c_r}{\varepsilon^r} \int_I \frac{dx}{\cos^r x} + O_{\delta,r}(\varepsilon^{-r+\frac{1}{2}-2\alpha-\delta} + |I| \varepsilon^{-r+\alpha}), \tag{4.19}$$

with c_r as in Theorem 1.1. The proof of Theorem 1.2 is now complete.

5. The moments of the number of reflections

We take as before $Q=[\frac{1}{\varepsilon}]$ and keep up with the notation from the beginning of Section 4. An inspection of the proof of Lemma 3.1 shows that if $\frac{a}{q}<\frac{a'}{q'}$ are consecutive in $\mathcal{F}_Q$, then

$$R_\varepsilon(\omega)=\begin{cases}\begin{cases}q+a & \text{if } \tan\omega\in[\frac{a}{q},\frac{a}{q-\varepsilon})\\ q+a+1 & \text{if } \tan\omega\in[\frac{a}{q-\varepsilon},\frac{a+\varepsilon}{q})\\ q'+a' & \text{if } \tan\omega\in[\frac{a+\varepsilon}{q},\frac{a'}{q'}]\end{cases} & \text{if } q<q';\\ \begin{cases}q+a & \text{if } \tan\omega\in[\frac{a}{q},\frac{a'-\varepsilon}{q'})\\ q'+a' & \text{if } \tan\omega\in[\frac{a'-\varepsilon}{q'},\frac{a'}{q'}]\end{cases} & \text{if } q>q' \text{ and } t_{S'}\le t_W;\\ \begin{cases}q+a & \text{if } \tan\omega\in[\frac{a}{q},\frac{a}{q-\varepsilon})\\ q+a+1 & \text{if } \tan\omega\in[\frac{a}{q-\varepsilon},\frac{a'-\varepsilon}{q'})\\ q'+a' & \text{if } \tan\omega\in[\frac{a'-\varepsilon}{q'},\frac{a'}{q'}]\end{cases} & \text{if } q>q' \text{ and } t_{S'}>t_W.\end{cases}$$

A first immediate remark is that we may replace $q+a+1$ by $q+a$ in the above formulas, since the contribution of the corresponding arcs is small, as we see from $|\arctan x-\arctan y|\le|x-y|$, and from

$$\sum_{a/q\in\mathcal{F}_Q}\left(\frac{a+\varepsilon}{q}-\frac{a}{q-\varepsilon}\right)=\sum_{a/q\in\mathcal{F}_Q}\frac{\varepsilon(q-a-\varepsilon)}{q(q-\varepsilon)}\ll\varepsilon\sum_{a/q\in\mathcal{F}_Q}\frac{1}{q}\le 1$$

and

$$\sum_{a/q\in\mathcal{F}_Q}\left(\frac{a'-\varepsilon}{q'}-\frac{a}{q-\varepsilon}\right)=\sum_{a/q\in\mathcal{F}_Q}\frac{1-\varepsilon(q+a')+\varepsilon^2}{q'(q-\varepsilon)}\ll\sum_{a/q\in\mathcal{F}_Q}\frac{1}{qq'}=1.$$

As a result we may write

$$\int_I R_\varepsilon^r(\omega)\,d\omega=T^{(1)}_{r,J,\varepsilon}+T^{(2)}_{r,J,\varepsilon}+O_r(Q^{r-1}),$$

where we set

$$T^{(1)}_{r,J,\varepsilon}={}^J\sum_{a/q}(q+a)^r\left(\arctan\frac{a+\varepsilon}{q}-\arctan\frac{a}{q}\right)+{}^J\sum_{a/q}(q'+a')^r\left(\arctan\frac{a'}{q'}-\arctan\frac{a+\varepsilon}{q}\right)$$

and

$$T^{(2)}_{r,J,\varepsilon}=\sum_{a/q}{}^J(q+a)^r\left(\arctan\frac{a'-\varepsilon}{q'}-\arctan\frac{a}{q}\right)+\sum_{a/q}{}^J(q'+a')^r\left(\arctan\frac{a'}{q'}-\arctan\frac{a'-\varepsilon}{q'}\right).$$

Employing

$$\arctan(x+h)-\arctan x=\frac{h}{1+x^2}+O(h^2)=\frac{h}{1+(x+h)^2}+O(h^2)$$

we now arrive at

$$\int_I R_\varepsilon^r(\omega)\, d\omega = S_{r,J,\varepsilon} + T_{r,J,\varepsilon} + O_r(Q^{r-1}), \tag{5.1}$$

where

$$S_{r,J,\varepsilon} = {\sum_{a/q}}^{J} \left(\frac{\frac{\varepsilon}{q}}{1+(\frac{a}{q})^2}\,(q+a)^r + \frac{\frac{1-\varepsilon q'}{qq'}}{1+(\frac{a'}{q'})^2}\,(q'+a')^r \right) \tag{5.2}$$

and

$$T_{r,J,\varepsilon} = \sum_{a/q}^{J} \left(\frac{\frac{1-\varepsilon q}{qq'}}{1+(\frac{a}{q})^2}\,(q+a)^r + \frac{\frac{\varepsilon}{q'}}{1+(\frac{a'}{q'})^2}\,(q'+a')^r \right). \tag{5.3}$$

To further simplify the expressions in (5.2) and (5.3) we employ

$$1+\frac{a'}{q'} = 1+\frac{a}{q}+\frac{1}{qq'} = 1+\frac{a}{q}+O\left(\frac{1}{Q}\right) = \left(1+\frac{a}{q}\right)\left(1+O\Big(\frac{1}{Q}\Big)\right),$$
$$\left(1+\frac{a'}{q'}\right)^r = \left(1+\frac{a}{q}\right)^r\left(1+O_r\Big(\frac{1}{Q}\Big)\right),$$

and

$$\frac{1}{Q}\sum_{a/q\in\mathcal{F}_Q}\frac{1}{qq'}\,q'^r \le Q^{r-2}\sum_{a/q\in\mathcal{F}_Q}\frac{1}{q} \le Q^{r-1},$$

to infer that

$$S_{r,J,\varepsilon} = {\sum_{a/q}}^{J} \frac{(1+\frac{a}{q})^r}{1+(\frac{a}{q})^2}\left(\varepsilon q^{r-1} + \frac{1-\varepsilon q'}{q}\,q'^{r-1}\right) + O_r(Q^{r-1}), \tag{5.4}$$

and also that

$$T_{r,J,\varepsilon} = \sum_{a/q}^{J} \frac{(1+\frac{a}{q})^r}{1+(\frac{a}{q})^2}\left(\varepsilon q^{r-1} + \frac{1-\varepsilon q'}{q}\,q'^{r-1}\right) + O_r(Q^{r-1}).$$

The main term in (5.4) can now be conveniently expressed as

$$A_{r,J,\varepsilon} = \sum_{q=1}^{Q}\ \sum_{\substack{\max\{Q-q,q\}<x\le Q\\ \bar{x}\in J_q^{(1)}}} f(x,\bar{x},q), \tag{5.5}$$

where $\bar{x}$ is the multiplicative inverse of x in $\{1,\dots,q-1\}$, and this time we set

$$\mathcal{I} = (\max\{Q-q,q\},Q], \qquad \mathcal{J} = J_q^{(1)},$$
$$f(x,y) = f(x,y,q) = \frac{(1+\frac{q-y}{q})^r}{q^2+(\frac{q-y}{q})^2}\left(\varepsilon q^{r-1} + \frac{1-\varepsilon x}{q}\,x^{r-1}\right)$$
$$= \frac{q(2q-y)^r}{q^2+(q-y)^2}\left(\varepsilon + \frac{1-\varepsilon x}{q}\Big(\frac{x}{q}\Big)^{r-1}\right).$$

Since $0 \le 1-\varepsilon x < 1$, it is easy to check that

$$\begin{aligned}
\|f\|_{\infty,\mathcal{I}\times J_q^{(1)}} &\ll \frac{(2q-y)^r}{q^2+(q-y)^2} \ll_r q^{r-2},\\
\left\|\frac{\partial f}{\partial x}\right\|_{\infty,\mathcal{I}\times J_q^{(1)}} &\ll \frac{q(2q-y)^r}{q^2+(q-y)^2}\cdot\frac{1}{q^2} \ll_r q^{r-3},\\
\left\|\frac{\partial f}{\partial y}\right\|_{\infty,\mathcal{I}\times J_q^{(1)}} &\ll \left\|\frac{\partial}{\partial y}\left(\frac{(2q-y)^r}{q^2+(q-y)^2}\right)\right\|_{\infty} \ll_r q^{r-3}.
\end{aligned}$$

Taking these estimates into account and applying Lemma 2.2 for each $q \in [1,Q]$ to $f=f(\cdot,\cdot,q)$, $\mathcal{I}=(\max\{Q-q,q\},Q]$, $\mathcal{J}=J_q^{(1)}$ and $T=[Q^\alpha]$, we infer that the inner sum in (5.5) can be expressed as

$$\begin{aligned}
&\frac{\varphi(q)}{q^2}\int\limits_{\max\{Q-q,q\}}^{Q}\left(\varepsilon+\frac{1-\varepsilon x}{q}\Big(\frac{x}{q}\Big)^{r-1}\right)dx\int\limits_{J_q^{(1)}}\frac{q(2q-y)^r}{q^2+(q-y)^2}\,dy\\
&\qquad+O_{r,\delta}(Q^{2\alpha}q^{\frac12+\delta}q^{r-2}+Q^{\alpha}q^{\frac32+\delta}q^{r-3}+|I|Q^{-\alpha}q^2q^{r-3})\\
&=\frac{C_{r,I}\varphi(q)}{q}\,q^{r-1}g_r(q)+O_{r,\delta}(Q^{r-\frac32+2\alpha+\delta}+|I|Q^{r-1-\alpha}),
\end{aligned}$$

where

$$\begin{aligned}
C_{r,I}&=\frac{1}{q^{r-1}}\int\limits_{(1-t_2)q}^{(1-t_1)q}\frac{(2q-y)^r}{q^2+(q-y)^2}\,dy=\frac{1}{q^{r-1}}\int\limits_{t_1q}^{t_2q}\frac{(q+y)^r}{q^2+y^2}\,dy\\
&=\int\limits_{t_1}^{t_2}\frac{(1+t)^r}{1+t^2}\,dt=\int\limits_{I}(1+\tan x)^r\,dx
\end{aligned}$$

and

$$g_r(q)=\int\limits_{\max\{Q-q,q\}}^{Q}\left(\varepsilon+\frac{1-\varepsilon x}{q}\Big(\frac{x}{q}\Big)^{r-1}\right)dx.$$

We now arrive at

$$\sum_{\substack{\max\{Q-q,q\}<x\le Q\\ \bar{x}\in J_q^{(1)}}} f(x,\bar{x},q)=\frac{C_{r,I}}{q}\,h_r(q)+O_{r,\delta}(Q^{r-\frac32+2\alpha+\delta}+|I|Q^{r-\alpha-1}),\tag{5.6}$$

where

$$h_r(q)=\int\limits_{\max\{Q-q,q\}}^{Q}\left(\varepsilon q^{r-1}+\frac{1-\varepsilon x}{q}\,x^{r-1}\right)dx.$$

Since $\|h\|_\infty \ll Q^{r-1}$ and $\int_1^Q |h'(q)|\,dq \ll Q^{r-1}$, Lemma 2.3 in [3] together with (5.5) and (5.6) show that

$$\begin{aligned} S_{r,J,\varepsilon} &= C_{r,I} \sum_{q=1}^{Q} \frac{\varphi(q)}{q}\, h_r(q) + O_{r,\delta}(Q^{r-\frac{1}{2}+2\alpha+\delta} + |I| Q^{r-\alpha}) \\ &= \frac{C_{r,I}}{\zeta(2)} \int_1^Q h_r(q)\,dq + O_{r,\delta}(Q^{r-\frac{1}{2}+2\alpha+\delta} + |I| Q^{r-\alpha}). \end{aligned}$$

Making use of $\varepsilon Q = 1 + O(\varepsilon)$ we arrive by a straightforward computation to

$$\begin{aligned} \int_1^Q h_r(q)\,dq = Q^r \int_0^1 \bigg(& x^{r-1}\big(1-\max\{1-x,x\}\big) + \frac{1-\max\{1-x,x\}^r}{rx} \\ & - \frac{1-\max\{1-x,x\}^{r+1}}{(r+1)x} \bigg)\,dx + O_r(Q^{r-1}). \end{aligned}$$

The integral above is seen to coincide with

$$\int_0^{1/2} \left(x\big(x^{r-1} + (1-x)^{r-1}\big) + \frac{1-(1-x)^r}{rx(1-x)} - \frac{1-(1-x)^{r+1}}{(r+1)x(1-x)} \right) dx = \frac{\pi^2 c_r}{12}\ ,$$

hence

$$\int_1^Q h_r(q)\,dq = \frac{\pi^2 c_r Q^r}{12} + O_r(Q^{r-1}) = \frac{\pi^2 c_r \varepsilon^{-r}}{12} + O_r(\varepsilon^{-r+1})$$

and as a result

$$\begin{aligned} S_{r,J,\varepsilon} &= \frac{6}{\pi^2} \cdot \frac{\pi^2 c_r \varepsilon^{-r}}{12} \int_I (1+\tan x)^r\,dx + O_{r,\delta}(\varepsilon^{-r+\frac{1}{2}-2\alpha-\delta} + |I| \varepsilon^{-r+\alpha}) \\ &= \frac{c_r \varepsilon^{-r}}{2} \int_I (1+\tan x)^r\,dx + O_{r,\delta}(\varepsilon^{-r+\frac{1}{2}-2\alpha-\delta} + |I| \varepsilon^{-r+\alpha}). \end{aligned}$$

By reversing the roles of q and q' it is seen in a similar way that

$$T_{r,J,\varepsilon} = \frac{c_r \varepsilon^{-r}}{2} \int_I (1+\tan x)^r\,dx + O_r(\varepsilon^{-r+\frac{1}{2}-2\alpha-\delta} + |I| \varepsilon^{-r+\alpha}).$$

This concludes the estimates of $S_{r,J,\varepsilon}$ and $T_{r,J,\varepsilon}$. Theorem 1.5 now follows from (5.1).

6. The case of circular scatterers

Note first that the statements of Theorems 1.2 and 1.5 hold true if we replace the scatterers $C_\varepsilon + \mathbb{Z}^{2*}$ by $V_\varepsilon + \mathbb{Z}^{2*}$.

In this section we consider the circular scatterers $D_\varepsilon + \mathbb{Z}^{2*}$, where

$$D_\varepsilon = \{(x,y) \in \mathbb{R}^2\,;\, x^2 + y^2 = \varepsilon^2\}.$$

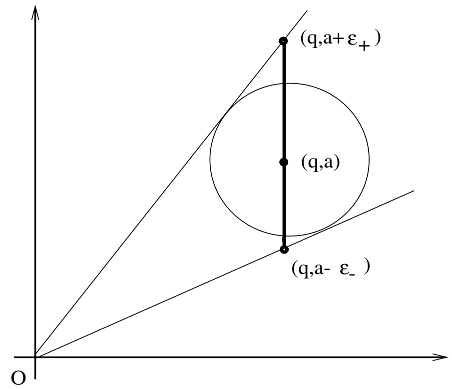

FIGURE 5. A circular scatterer

For each integer lattice point (q, a), let $(q, a \pm \varepsilon_\pm)$ denote the intersections of the line $x = q$ with the tangents from O to the circle

$$D_{\varepsilon,q,a} = (q, a) + D_\varepsilon = \{(x, y) \in \mathbb{R}^2\,;\, (x-q)^2 + (y-a)^2 = \varepsilon^2\},$$

where $\varepsilon_\pm = \varepsilon_\pm(q, a)$ are computed from the equality

$$\varepsilon = \frac{\left|a - \frac{a\pm\varepsilon_\pm}{q} q\right|}{\sqrt{1 + \left(\frac{a\pm\varepsilon_\pm}{q}\right)^2}} = \frac{\varepsilon_\pm q}{\sqrt{q^2 + (a\pm\varepsilon_\pm)^2}}\,,$$

which gives in turn

$$\varepsilon_\pm^2 q^2 = \varepsilon^2 q^2 + \varepsilon^2 (a \pm \varepsilon_\pm)^2,$$

or

$$(q^2 - \varepsilon^2)\varepsilon_\pm^2 \mp 2a\varepsilon^2\varepsilon_\pm - \varepsilon^2(q^2 + a^2) = 0.$$

The latter provides

$$\varepsilon_\pm = \pm\frac{a\varepsilon^2}{q^2 - \varepsilon^2} + \frac{\varepsilon}{q^2 - \varepsilon^2}\sqrt{q^4 + a^2q^2 - a^2\varepsilon^2}. \tag{6.1}$$

Employing also

$$\sqrt{q^4 + a^2q^2} - \sqrt{q^4 + a^2q^2 - a^2\varepsilon^2} \ll \varepsilon^2,$$
$$\frac{\varepsilon}{q^2 - \varepsilon^2} = \frac{\varepsilon}{q^2}\left(1 + O\left(\frac{\varepsilon^2}{q^2}\right)\right)$$

and

$$\frac{a\varepsilon^2}{q^2 - \varepsilon^2} = O\left(\frac{\varepsilon^2}{q}\right),$$

we arrive at

$$\varepsilon_\pm(q, a) = \varepsilon\sqrt{1 + \frac{a^2}{q^2}} + O\left(\frac{\varepsilon^2}{q}\right) = \frac{\varepsilon}{\cos\arctan\frac{a}{q}} + O\left(\frac{\varepsilon^2}{q}\right). \tag{6.2}$$

Proof of Theorem 1.1. We wish to compare $\int_I \widetilde{\tau}_\varepsilon^r(\omega)\,d\omega$ with $\int_I \widetilde{\widetilde{\tau}}_\varepsilon^r(\omega)\,d\omega$ where $\widetilde{\widetilde{\tau}}_\varepsilon(\omega)$, the smallest $\tau > 0$ for which

$$(\tau\cos\omega, \tau\sin\omega) \in \bigcup_{(q,a)\in\mathbb{Z}^{2*}} \{q\} \times [a - \varepsilon_-(q,a), a + \varepsilon_+(q,a)],$$

denotes the first exit time in the case where the scatterers are the vertical segments $\{q\} \times [a - \varepsilon_-(q,a), a + \varepsilon_+(q,a)]$. From Figure 5 it is apparent that

$$\sup_\omega |\widetilde{\tau}_\varepsilon(\omega) - \widetilde{\widetilde{\tau}}_\varepsilon(\omega)| \leq 2\varepsilon,$$

and so, since $\sup_\omega \widetilde{\widetilde{\tau}}_\varepsilon(\omega) \leq \sup_\omega l_\varepsilon(\omega) \leq \frac{\sqrt{2}}{\varepsilon}$, we get

$$\sup_\omega |\widetilde{\tau}_\varepsilon^r(\omega) - \widetilde{\widetilde{\tau}}_\varepsilon^r(\omega)| \ll_r \varepsilon \left(\frac{\sqrt{2}}{\varepsilon}\right)^{r-1} \ll_r \varepsilon^{2-r},$$

which gives

$$\int_I \widetilde{\tau}_\varepsilon^r(\omega)\,d\omega = \int_I \widetilde{\widetilde{\tau}}_\varepsilon^r(\omega)\,d\omega + O_r(\varepsilon^{2-r}). \tag{6.3}$$

To estimate $\int_I \widetilde{\widetilde{\tau}}_\varepsilon(\omega)\,d\omega$, we divide the interval I into $N = [\varepsilon^{-\theta}]$ intervals of equal size $I_j = [\omega_j, \omega_{j+1}]$ with $|I_j| = \frac{|I|}{N} \asymp \varepsilon^\theta$ for some $0 < \theta < \frac{1}{2}$. Then one has for all j that

$$\left| \frac{\cos\omega_j}{\varepsilon - \varepsilon^{3/2}} - \frac{\cos\omega_{j+1}}{\varepsilon + \varepsilon^{3/2}} \right| \ll \varepsilon^{\theta-1},$$

thus the integers $Q_j^+ = \left[\frac{\cos\omega_j}{\varepsilon - \varepsilon^{3/2}}\right] + 1$ and $Q_j^- = \left[\frac{\cos\omega_{j+1}}{\varepsilon + \varepsilon^{3/2}}\right]$ satisfy

$$0 < Q_j^+ - Q_j^- \ll \varepsilon^{\theta-1} \tag{6.4}$$

and

$$\frac{1}{Q_j^+} \leq \frac{\varepsilon - \varepsilon^{3/2}}{\cos\omega_j} \leq \frac{\varepsilon + \varepsilon^{3/2}}{\cos\omega_{j+1}} \leq \frac{1}{Q_j^-}\,. \tag{6.5}$$

Furthermore, it follows from (6.2) that there exists $\varepsilon_0 = \varepsilon_0(\theta) > 0$ such that for all $\varepsilon < \varepsilon_0$, all j, and all $\frac{a}{q} \in [\tan\omega_j, \tan\omega_{j+1}]$, one has

$$\frac{\varepsilon - \varepsilon^{3/2}}{\cos\omega_j} \leq \varepsilon_\pm(q,a) \leq \frac{\varepsilon + \varepsilon^{3/2}}{\cos\omega_{j+1}}\,.$$

This implies in conjunction with (6.5), for all $\frac{a}{q} \in [\tan\omega_j, \tan\omega_{j+1}]$, the inequalities

$$\frac{1}{Q_j^+} \leq \varepsilon_\pm(q,a) \leq \frac{1}{Q_j^-}\,. \tag{6.6}$$

Since $Q_j^\pm = \frac{\cos\omega_j}{\varepsilon} + O(\varepsilon^{\theta-1})$, one has

$$(Q_j^\pm)^r = \frac{\cos^r\omega_j}{\varepsilon^r} + O_r(\varepsilon^{-r+\theta}). \tag{6.7}$$

The first exit time increases when all the sizes of scatterers decrease. Thus we infer from (6.6) the inequalities

$$\int_{I_j} l^r_{\frac{1}{Q_j^-}}(\omega)\,d\omega \le \int_{I_j} \widetilde{\widetilde{\tau}}^r_\varepsilon(\omega)\,d\omega \le \int_{I_j} l^r_{\frac{1}{Q_j^+}}(\omega)\,d\omega. \tag{6.8}$$

But by Theorem 1.2 and by (6.7) we may write

$$\int_{I_j} l^r_{\frac{1}{Q_j^\pm}}(\omega)\,d\omega = c_r(Q_j^\pm)^r \int_{I_j} \frac{dx}{\cos^r x} + O_{r,\delta}(\varepsilon^{-r+\frac{1}{2}-2\alpha-\delta} + \varepsilon^{-r+\theta+\alpha}) \tag{6.9}$$

with the better error term $\varepsilon^{-\frac{3}{2}-\delta}$ for $r=2$. Also using $\int_{I_j} \frac{dx}{\cos^r x} \ll_r |I_j| \ll \varepsilon^\theta$ we infer that the first integral in (6.9) is expressible as

$$\frac{c_r \cos^r \omega_j}{\varepsilon^r} \int_{\omega_j}^{\omega_{j+1}} \frac{dx}{\cos^r x} + O_{r,\delta}(\varepsilon^{-r+2\theta} + \varepsilon^{-r+\frac{1}{2}-2\alpha-\delta} + \varepsilon^{-r+\theta+\alpha}), \tag{6.10}$$

with the better error term $\varepsilon^{-2+\theta} + \varepsilon^{-\frac{3}{2}-\delta}$ for $r=2$.

Summing up over j we infer from (6.8), (6.10) and (6.3) that

$$\begin{aligned}\int_I \widetilde{\tau}^r_\varepsilon(\omega)\,d\omega = \frac{c_r}{\varepsilon^r}\sum_{j=1}^N \cos^r\omega_j \int_{\omega_j}^{\omega_{j+1}} \frac{dx}{\cos^r x} \\ + O_{r,\delta}(\varepsilon^{-r+\frac{1}{2}-2\alpha-\theta-\delta} + \varepsilon^{-r+\alpha} + \varepsilon^{-r+\theta})\end{aligned} \tag{6.11}$$

with the better error term $\varepsilon^{-\frac{3}{2}-\theta-\delta} + \varepsilon^{-2+\theta}$ for $r=2$.

Finally, we apply the mean value theorem and chose some $\xi_j \in [\omega_j, \omega_{j+1}]$ to evaluate the sum

$$\sum_{j=1}^N \cos^r \omega_j \int_{\omega_j}^{\omega_{j+1}} \frac{dx}{\cos^r x}$$

as

$$\begin{aligned}\sum_{j=1}^N \frac{(\omega_{j+1}-\omega_j)\cos^r\omega_j}{\cos^r \xi_j} &= \sum_{j=1}^N (\omega_{j+1}-\omega_j)\Big(1 + O_r(\omega_{j+1}-\omega_j)\Big) \\ &= \sum_{j=1}^N (\omega_{j+1}-\omega_j)\Big(1 + O_r(\varepsilon^\theta)\Big) \\ &= |I| + O_r(\varepsilon^\theta).\end{aligned}$$

This implies Theorem 1.1 in conjunction with (6.11) by taking $\theta = \alpha = \frac{1}{8}$ for $r \neq 2$ and $\theta = \frac{1}{4}$ for $r = 2$. $\square$

Proof of Theorem 1.4. We proceed along the same line to estimate the moments of $\widetilde{R}$. Here we denote by $\widetilde{\widetilde{R}}(\omega)$ the number of reflections in the side cushions in the case of vertical scatterers (of variable size) $\{q\} \times [a - \varepsilon_-(q,a), a + \varepsilon_+(q,a)]$,

$(q, a) \in \mathbb{Z}^{2*}$. It is seen as in the proof of Theorem 1.1 that $\int_I \widetilde{R}_\varepsilon^r(\omega)\, d\omega$ differs from $\int_I \widetilde{\widetilde{R}}_\varepsilon^r(\omega)\, d\omega$ by an error term of order $O_r(\varepsilon^{2-r})$. One can also show that

$$\int\limits_{I_j} R^r_{\frac{1}{Q_j^-}}(\omega)\, d\omega \leq \int\limits_{I_j} \widetilde{\widetilde{R}}_\varepsilon^r(\omega)\, d\omega \leq \int\limits_{I_j} R^r_{\frac{1}{Q_j^+}}(\omega)\, d\omega. \tag{6.12}$$

Applying now Theorem 1.5 to the vertical scatterers $V_{1/Q_j^\pm}$ on the intervals $I_j = [\omega_j, \omega_{j+1}]$ of equal size $|I_j| = \frac{|I|}{N} \asymp \varepsilon^{\frac{1}{8}}$ with $\theta = \alpha = \frac{1}{8}$, and also using (6.7), we find that

$$\int\limits_{I_j} R^r_{\frac{1}{Q_j^\pm}}(\omega)\, d\omega = \frac{c_r \cos^r \omega_j}{\varepsilon^r} \int\limits_{I_j} (1 + \tan x)^r\, dx + O_{r,\delta}(\varepsilon^{-r+\frac{1}{4}-\delta}),$$

and thus

$$\int\limits_I \widetilde{R}_\varepsilon^r(\omega)\, d\omega = \frac{c_r}{\varepsilon^r} \sum_{j=1}^N \cos^r \omega_j \int\limits_{\omega_j}^{\omega_{j+1}} (1 + \tan x)^r\, dx + O_{r,\delta}(\varepsilon^{-r+\frac{1}{8}-\delta}). \tag{6.13}$$

By the mean value theorem we find $\xi_j, \eta_j \in I_j$ such that

$$\sum_{j=1}^N \cos^r \omega_j \int\limits_{\omega_j}^{\omega_{j+1}} (1 + \tan x)^r\, dx = \sum_{j=1}^N (\omega_{j+1} - \omega_j) \cos^r \omega_j (1 + \tan \xi_j)^r, \tag{6.14}$$

and respectively

$$\int\limits_I (\sin x + \cos x)^r\, dx = \sum_{j=1}^N \int\limits_{\omega_j}^{\omega_{j+1}} (\sin x + \cos x)^r\, dx$$
$$= \sum_{j=1}^N (\omega_{j+1} - \omega_j)(\sin \eta_j + \cos \eta_j)^r = \sum_{j=1}^N (\omega_{j+1} - \omega_j) \cos^r \eta_j (1 + \tan \eta_j)^r.$$

From

$$\cos^r \omega_j = \cos^r \eta_j + O_r(\omega_{j+1} - \omega_j) = \cos^r \eta_j + O_r(\varepsilon^{\frac{1}{8}})$$

and

$$(1 + \tan \xi_j)^r = (1 + \tan \eta_j)^r + O_r(|\tan \xi_j - \tan \eta_j|) = (1 + \tan \eta_j)^r + O_r(\varepsilon^{\frac{1}{8}})$$

we infer that the sum in (6.14) is equal to

$$\sum_{j=1}^N (\omega_{j+1} - \omega_j)\Big(\cos^r \eta_j + O_r(\varepsilon^{\frac{1}{8}})\Big)\Big((1 + \tan \eta_j)^r + O_r(\varepsilon^{\frac{1}{8}})\Big)$$
$$= \int\limits_I (\sin x + \cos x)^r\, dx + O_r(\varepsilon^{\frac{1}{8}}).$$

This can be combined with (6.13) to collect

$$\int\limits_I \widetilde{R}_\varepsilon^r(\omega)\, d\omega = \frac{c_r}{\varepsilon^r} \int\limits_I (\sin x + \cos x)^r\, dx + O_{r,\delta}(\varepsilon^{-r+\frac{1}{8}-\delta}),$$

which concludes the proof of Theorem 1.4. $\square$

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