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On generalized Jacobi matrices and orthogonal polynomials

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Abstract. We consider systems of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ which satisfy a recurrence relation that can be written in a matrix form: $Jp(\lambda) = \lambda^N p(\lambda), p = (p_0(\lambda), p_1(\lambda), \dots)^T, \lambda \in C, N \in N,$ J is a $(2N+1)$-diagonal, semi-infinite, Hermitian complex numerical matrix. For such systems we obtained orthonormality relations on radial rays. To prove these relations we used the standard method of scattering theory. We showed that these relations are characteristic. From the relations it is easily shown that systems of orthonormal polynomials on the real line, systems of Sobolev orthogonal polynomials with discrete measure at zero, systems or orthonormal polynomials on radial rays with a scalar measure are such systems of polynomials. Also we consider a connection with matrix orthonormal polynomials on the real line.

Contents

1. Introduction

Let us consider a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n is of the n-th degree) such that

$$(1) \qquad \sum_{i=1}^{N}(\overline{\alpha_{k-i,i}}p_{k-i}(\lambda) + \alpha_{k,i}p_{k+i}(\lambda)) + \alpha_{k,0}p_k(\lambda) = \lambda^N p_k(\lambda),\ k = 0, 1, 2, \dots,$$

where $\alpha_{m,n} \in \mathbb{C},\ m = 0, 1, 2, \dots;\ n = 0, 1, 2, \dots, N;\ \alpha_{m,N} \neq 0,\ \alpha_{m,0} \in \mathbb{R};\ N \in \mathbb{N};$ $\lambda \in \mathbb{C}$ and where the $\alpha_{m,n}$ and p_k which appear here with negative indices are equal to zero. We can equivalently write

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$$
(2) \quad \begin{pmatrix}
\alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} & \dots & \alpha_{0,N} & 0 & 0 & \dots \\
\overline{\alpha_{0,1}} & \alpha_{1,0} & \alpha_{1,1} & \dots & \alpha_{1,N-1} & \alpha_{1,N} & 0 & \dots \\
. & . & . & . & . & . & . & . \\
\overline{\alpha_{0,N}} & \overline{\alpha_{1,N-1}} & \overline{\alpha_{2,N-2}} & . & . & . & . & . \\
0 & \overline{\alpha_{1,N}} & \overline{\alpha_{2,N-1}} & . & . & . & . & . \\
. & . & . & . & . & . & . & .
\end{pmatrix}
\begin{pmatrix} p_0(\lambda) \\ p_1(\lambda) \\ . \\ p_N(\lambda) \\ p_{N+1}(\lambda) \\ . \end{pmatrix}
= \lambda^N \begin{pmatrix} p_0(\lambda) \\ p_1(\lambda) \\ . \\ p_N(\lambda) \\ p_{N+1}(\lambda) \\ . \end{pmatrix}.
$$

Denote the matrix on the left of (2) by J and $p(\lambda) = (p_0(\lambda), p_1(\lambda), \dots)^T$. Then

$$
(3) \qquad Jp(\lambda) = \lambda^N p(\lambda).
$$

Systems of polynomials which satisfy (1) with real $\alpha_{m,n}$ were first studied by Duran in 1993 in [5], following a suggestion of Marcellán. Duran showed that these polynomials were orthonormal with respect to the functional $B_\mu(p,q)$ [5]:

$$
(4) \quad B_\mu(p,q) = \sum_{m,m'=1}^{N} \int_R R_{N,m-1}(p)\, \overline{R_{N,m'-1}(q)}\, d\mu_{m,m'}, \ p,q \in \mathbb{P},
$$

$$
R_{N,m}(p)(t) = \sum_n \frac{p^{(nN+m)}(0)}{(nN+m)!} t^n,
$$

where μ is a measure, and $\mathbb{P}$ is the space of all polynomials, and we have used the notations from [6]. This can be written as

$$
(5) \quad B_\mu(p,q) = \int_R (R_{N,0}(p)(x), R_{N,1}(p)(x), \dots, R_{N,N-1}(p)(x))\, d\mu(x) \overline{\begin{pmatrix} R_{N,0}(q)(x) \\ R_{N,1}(q)(x) \\ \vdots \\ R_{N,N-1}(q)(x) \end{pmatrix}},
$$

where $\mu = (\mu_{m,m'})_{m,m'=1}^N$. Also in [5] Duran studied the Sobolev type orthogonal polynomials on $\mathbb{R}$ and their connection with polynomials in (1). In [6], Duran improved his result and showed that the measure μ in (4) can be taken to be positive ([6, Theorem 3.1, p. 93]). For this purpose he used operator theory tools. Duran and Van Assche [8] studied a connection between the polynomials from (1) with matrix orthogonal polynomials on the real line. They called the $(2N+1)$-banded matrix J in (3) the N-Jacobi matrix. The Theorem in [8] asserts that the polynomials

$$
(6) \quad P_n(x) = \begin{pmatrix}
R_{N,0}(p_{nN})(x) & R_{N,1}(p_{nN})(x) & \dots & R_{N,N-1}(p_{nN})(x) \\
R_{N,0}(p_{nN+1})(x) & R_{N,1}(p_{nN+1})(x) & \dots & R_{N,N-1}(p_{nN+1})(x) \\
\dots & \dots & \dots & \dots \\
R_{N,0}(p_{nN+N-1})(x) & R_{N,1}(p_{nN+N-1})(x) & \dots & R_{N,N-1}(p_{nN+N-1})(x)
\end{pmatrix},
$$

are orthonormal polynomials on the real line. And conversely, from a given set of matrix orthogonal polynomials on $\mathbb{R}$ (suitably normed) one can define scalar polynomials by

$$p_{nN+m}(x) = \sum_{j=0}^{N-1} x^j P_{n,m,j}(x^N),\ n \in \mathbb{N}, 0 \le m \le N-1. \tag{7}$$

Then these polynomials satisfy a $(2N+1)$-term relation of the form (1). To prove this, Duran and Van Assche showed how (1) implies the matrix three-term recurrence relation:

$$D_{n+1}P_{n+1}(x) + E_n P_n(x) + D_n^* P_{n-1}(x) = xP_n(x), \tag{8}$$

where D_n are $(N \times N)$ matrices and $\det D_n \neq 0$. And conversely, they showed how (1) follows from (8). It remained to use the Favard theorem for matrix orthogonal polynomials. Unfortunately, Duran and Van Assche [8, p. 263] erroneously referred in this question to [2]. But in [2, p. 328] it was written that the Jacobi matrix L with matrix entries $\{V_j, E_j\}_{j=0}^{\infty} : V_j^* = V_j, \det E_j \neq 0$ always define in $l_2(\mathbb{C}^N)$ a self-adjoint operator. But even for numerical Jacobi matrices this is not always valid. Moreover, the Favard theorem for matrix-valued polynomials was established in [3, Chapter 6, The considerations after Theorem 6.8.1] (for operator-valued polynomials the Favard theorem was obtained in [4, Chapter 7, §2, Theorem 2.4]). So, it is correct to refer to [3] on this question.

In [8] Duran and Van Assche considered a more general relation when in (1) there was $h(\lambda)p_k(\lambda)$ on the right, where $h(\lambda)$ is a polynomial of degree N, and obtained analogous results. Also they studied a connection with the Sobolev type orthogonal polynomials. We refer also to the papers [7], [12], [13] for related questions.

We began to study polynomials which satisfy (1) with $N = 2$ in 1998 [16] (in fact, independently from Marcellán, Duran and Van Assche) following a suggestion of Zolotarev. In [16] different examples of five-diagonal Jacobi matrices were given and the measures of orthogonality were presented explicitly (see also [19]). For example, when

$$J = \begin{pmatrix} -\beta^2+\frac14 & 0 & \frac14 & 0 & 0 & \dots \\ 0 & -\beta^2+\frac12 & 0 & \frac14 & 0 & \dots \\ \frac14 & 0 & -\beta^2+\frac12 & 0 & \frac14 & \dots \\ 0 & \frac14 & 0 & -\beta^2+\frac12 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where β is a real parameter, we obtained [16, p. 260]:

$$\begin{aligned} p_{2k}(\lambda) &= U_{2k}(\sqrt{\lambda^2+\beta^2}), \\ p_{2k+1}(\lambda) &= \frac{\lambda - \beta i}{\sqrt{\lambda^2+\beta^2}} U_{2k+1}(\sqrt{\lambda^2+\beta^2}), \end{aligned} \qquad k = 0, 1, 2, \dots, \tag{9}$$

where

$$U_n(\lambda) = \frac{\sin((n+1)\arccos\lambda)}{\sqrt{1-\lambda^2}}$$

are the Chebyshev polynomials of the second kind, and [16, p. 262]:

$$(10)\qquad \int_{-\sqrt{1-\beta^2}}^{\sqrt{1-\beta^2}} (p_n(\lambda), p_n(-\lambda)) \begin{pmatrix} \lambda^2+\beta^2 & -\beta(\beta-\lambda i) \\ -\beta(\beta+\lambda i) & \lambda^2+\beta^2 \end{pmatrix} \times \frac{\sqrt{1-(\lambda^2+\beta^2)}\,d\lambda}{|\lambda|\sqrt{\lambda^2+\beta^2}} \overline{\begin{pmatrix} p_m(\lambda) \\ p_m(-\lambda) \end{pmatrix}} + \int_{-|\beta|i}^{|\beta|i} (p_n(\lambda), p_n(-\lambda)) \begin{pmatrix} \beta(\beta-\lambda i) & -(\lambda^2+\beta^2) \\ -(\lambda^2+\beta^2) & \beta(\beta+\lambda i) \end{pmatrix} \times \frac{\sqrt{1-(\lambda^2+\beta^2)}\,d\lambda}{i|\lambda|\sqrt{\lambda^2+\beta^2}} \overline{\begin{pmatrix} p_m(\lambda) \\ p_m(-\lambda) \end{pmatrix}} = \delta_{nm}, n,m = 0,1,2,\ldots.$$

A corresponding symmetric moment problem was considered. Also the following question was studied: Given a five-diagonal matrix J and a vector of polynomials p such that $Jp(\lambda) = \lambda^2 p(\lambda)$, when does it follow that $J_3 p(\lambda) = \lambda p(\lambda)$ with a tridiagonal matrix J_3? The general question of orthogonality of polynomials connected with five-diagonal matrices was studied in [17, Theorem 2] and it was corrected in [21]. It turns out that if polynomials $\{p_n(\lambda)\}_{n=0}^\infty$ satisfy (1) with $N=2$ then they are orthonormal on the real and the imaginary axes in the complex plane:

$$(11)\qquad \int_{\mathbb{R}\cup T} (p_n(\lambda), p_n(-\lambda)) J_\lambda\, d\sigma(\lambda) J_\lambda^* \overline{\begin{pmatrix} p_m(\lambda) \\ p_m(-\lambda) \end{pmatrix}} = \delta_{nm},\ n,m = 0,1,2,\ldots,$$

where $J_\lambda = \begin{pmatrix} 1 & \frac{1}{\lambda} \\ 1 & \frac{1}{-\lambda} \end{pmatrix}$, and $\sigma(\lambda)$ is a non-decreasing (2×2) matrix function on $\mathbb{R}\cup T, T = (-i\infty, i\infty)$.

Also the basic set of solutions of (1) ($N=2$) was built in [17].

The main aim of our present investigation is to obtain the following orthonormality relations for polynomials $\{p_n(\lambda)\}_{n=0}^\infty$:

$$(12)\qquad \int_{P_N} (p_n(\lambda), p_n(\lambda\varepsilon), p_n(\lambda\varepsilon^2), \ldots, p_n(\lambda\varepsilon^{N-1})) J_\lambda\, d\sigma(\lambda) \times J_\lambda^* \overline{\begin{pmatrix} p_m(\lambda) \\ p_m(\lambda\varepsilon) \\ \cdot \\ \cdot \\ p_m(\lambda\varepsilon^{N-1}) \end{pmatrix}} = \delta_{nm},$$

$n,m = 0,1,2,\ldots$. Here ε is a primitive N-th root of unity; $J_\lambda = (a_{i,j})_{i,j=1}^N$, with $a_{i,j} = \frac{1}{\varepsilon^{ij}\lambda^j}$; $P_N = \{\lambda \in \mathbb{C} : \lambda^N \in \mathbb{R}\}$; and $\sigma(\lambda)$ is a non-decreasing matrix-valued function on P_N (i.e., on each of the $2N$ rays in P_N) such that $\sigma(0) = 0$. When $\lambda = 0$ one must take the limit values as $\lambda \to 0$ of $(p_n(\lambda), p_n(\lambda\varepsilon), \ldots, p_n(\lambda\varepsilon^{N-1}))J_\lambda$

and $J_\lambda^* \overline{\begin{pmatrix} p_m(\lambda) \\ p_m(\lambda\varepsilon) \\ . \\ . \\ p_m(\lambda\varepsilon^{N-1}) \end{pmatrix}}$ under the integral.

$$(13)\qquad \int_{P_N} (p_n(\lambda), p_n(\lambda\varepsilon), p_n(\lambda\varepsilon^2), \dots, p_n(\lambda\varepsilon^{N-1}))\, dW(\lambda) \overline{\begin{pmatrix} p_m(\lambda) \\ p_m(\lambda\varepsilon) \\ . \\ . \\ p_m(\lambda\varepsilon^{N-1}) \end{pmatrix}}$$

$$+ (p_n(0), p_n'(0), p_n''(0), \dots, p_n^{(N-1)}(0)) M \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ . \\ . \\ p_m^{(N-1)}(0) \end{pmatrix}}$$

$$= \delta_{nm},\ n, m = 0, 1, 2, \dots,$$

where $W(\lambda)$ is a non-decreasing matrix-valued function on $P_N \backslash \{0\}$. With the integral at $\lambda = 0$ we understand the improper integral. $M \geq 0$ is a $(N \times N)$ complex numerical matrix.

In order to prove relation (12) we shall use a method which is analogous to the standard method of the direct scattering problem for the discrete Sturm-Liouville operator (or the Jacobi matrix) (see for example [3, Chapters 4,5]) and also we use some estimates in spirit of [1, Chapter 2, Proof of Theorem 2.1.1] (see also [11]). To obtain (13) from (12) we shall use properties of the Radon-Nikodym derivative of a measure σ (see [10], [15]).

Then we shall show the converse statement: (1) follows from (12) or (13).

In Section 3 we shall consider a connection with matrix orthogonal polynomials that is analogous to that discussed in [8], [7].

Finally, in Section 4 we compare our results with results of other authors and discuss further perspectives on this topic.

2. Orthonormality on radial rays

The following theorem is valid (see [20]):

Theorem 1. *Let a system of polynomials $\{p_n(\lambda)\}_{n=0}^\infty$ (p_n is of the n-th degree) satisfy* (1)*. Then an orthonormality relation* (12) *on radial rays in the complex plane holds.*

Proof. Let a system of polynomials $\{p_n(\lambda)\}_{n=0}^\infty$ (p_n is of the n-th degree) satisfy (1). Denote by J_M a matrix obtained by an intersection of the first M columns and the first M rows of J, $M \in \{N+1, N+2, \dots\}$. Consider the following vectors:

$$(14)\qquad \vec{r}_m^{\,k}(\lambda) = \begin{pmatrix} R_{m,N}(p_0)(\lambda) \\ R_{m,N}(p_1)(\lambda) \\ \vdots \\ R_{m,N}(p_{k-1})(\lambda) \end{pmatrix}, \qquad \vec{r}_m(\lambda) = \begin{pmatrix} R_{m,N}(p_0)(\lambda) \\ R_{m,N}(p_1)(\lambda) \\ \vdots \\ R_{m,N}(p_{N-1})(\lambda) \\ \vdots \end{pmatrix},$$

$m = 0, 1, 2, \ldots, N-1$; $k \in \mathbb{N}$ and $R_{m,N}(p)(\lambda)$ which appear here were introduced in [5, p. 90] in the following way:

$$R_{m,N}(p)(\lambda) = \sum_{j=0}^{[l/N]} a_{Nj+m}\lambda^{Nj}, \qquad \text{for } p = \sum_{i=0}^{l} a_i\lambda^i. \tag{15}$$

Notice that $R_{m,N}(p)(\lambda) \neq R_{N,m}(\lambda)$, where $R_{N,m}$ are from (4). In fact,

$$R_{m,N}(p)(\lambda) = R_{N,m}(p)(\lambda^N). \tag{16}$$

The polynomials $R_{m,N}$ will be useful for the construction of the measure of orthogonality. They satisfy the following relation (see [5]):

$$R_{m,N}(p)(\lambda) = \frac{1}{N\lambda^m}\sum_{k=0}^{N-1} \varepsilon^{-mk} p(\lambda\varepsilon^k), \tag{17}$$

where ε is a N-th primitive root of unity, $\lambda \neq 0$.

This relation can be checked directly by substitution of $p = \sum_{i=0}^{l} a_i\lambda^i$ in the right-hand side of (17) and by calculating the sum.

Consider a matrix $W = (\vec{r}_0^{\,N}(\lambda), \vec{r}_1^{\,N}(\lambda), \ldots, \vec{r}_{N-1}^{\,N}(\lambda))$. If $p_n(\lambda) = \mu_{n,n}\lambda^n + \mu_{n,n-1}\lambda^{n-1} + \cdots + \mu_{n,0}$, then using (15) we obtain:

$$W = \begin{pmatrix} \mu_{0,0} & 0 & 0 & \ldots & 0 \\ \mu_{1,0} & \mu_{1,1} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{N-1,0} & \mu_{N-1,1} & \mu_{N-1,2} & \cdots & \mu_{N-1,N-1} \end{pmatrix}. \tag{18}$$

So, W is a nonsingular numerical matrix.

Let $c_1, c_2, \ldots, c_M$ be the set of eigenvalues of J_M (an eigenvalue of the multiplicity p appears p times) and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_M$ be the corresponding orthonormal eigenvectors. Let $\lambda_i = \sqrt[N]{c_i}, i = 1, 2, \ldots, M$ (we take an arbitrary branch of the root). Then

$$J_M\vec{v}_i = \lambda_i^N\vec{v}_i, \qquad i = 1, 2, \ldots, M. \tag{19}$$

Lemma 1. *The eigenvector $\vec{v}_i$ of the cut matrix J_M $(M \geq N+1)$ corresponding to the eigenvalue λ_i, $i = 1, 2, \ldots, M$, can be written as a linear combination of N vectors $\vec{r}_n^{\,M}(\lambda_i)$, $n = 0, 1, 2, \ldots, N-1$:*

$$\vec{v}_i = \sum_{k=1}^{N} \beta_{k,i}\vec{r}_{k-1}^{\,M}(\lambda_i), \qquad i = 1, 2, \ldots, M. \tag{20}$$

Proof. Since the matrix W from (18) is nonsingular for any λ, a system of linear algebraic equations,

$$\sum_{k=1}^{N} \beta_{k,i}\vec{r}_{k-1}^{\,N}(\lambda_i) = \vec{v}_i^{\,N}, \qquad i = 1, 2, \ldots, M, \tag{21}$$

where $\beta_{k,i} \in \mathbb{C}$ and $\vec{v}_i^{\,N}$ is the vector of the first N components of $\vec{v}_i$, has a unique solution $\{\beta_{k,i}\}$.

So, for these $\{\beta_{k,i}\}$ the first N components of the vectors on the left and on the right of (20) will coincide. But the rest of the components satisfy the same recurrence relation as in (19). Really, the components of $\vec{r}^{M}_{k-1}(\lambda_i)$ are linear combinations of polynomials $p(\lambda\varepsilon^k)$, as follows from (17), which satisfy (1). Hence the components satisfy (19) (for the case $\lambda = 0$ we can take the limit values of polynomials $R_{m,N}$). So, relation (20) is valid. □

Lemma 2. *For polynomials* $\{p_n(\lambda)\}_{n=0}^{M-1}$ *the following orthonormality relation holds:*

$$\begin{aligned}(22)\qquad &\int_{P_N} (R_{0,N}(p_i)(\lambda), R_{1,N}(p_i)(\lambda), \ldots, R_{N-1,N}(p_i)(\lambda))\, dB_M(\lambda) \\ &\qquad\times \overline{\begin{pmatrix} R_{0,N}(p_j)(\lambda) \\ R_{1,N}(p_j)(\lambda) \\ . \\ R_{N-1,N}(p_j)(\lambda) \end{pmatrix}} \\ &= \delta_{ij}, i, j = 0, 1, \ldots, M-1.\end{aligned}$$

Here $B_M(\lambda)$ *is a non-decreasing* $(N \times N)$ *matrix function on* P_N*.*

Proof. Since J_M is a self-adjoint matrix, vectors $\{\vec{v}_i\}$ form an orthonormal basis in $\mathbb{C}^M$. For arbitrary $\vec{x}, \vec{y} \in \mathbb{C}^M$ we can write:

$$\vec{x} = \sum_{i=1}^{M} (\vec{x}, \vec{v}_i)\vec{v}_i, \qquad \vec{y} = \sum_{i=1}^{M} (\vec{y}, \vec{v}_i)\vec{v}_i,$$

where $(.,.)$ denotes the inner product in $\mathbb{C}^M$. Using Lemma 1 for the inner product we have:

$$\begin{aligned}(\vec{x}, \vec{y}) &= \sum_{i=1}^{M} (\vec{x}, \vec{v}_i)\, \overline{(\vec{y}, \vec{v}_i)} \\ &= \sum_{i=1}^{M} \sum_{k=1}^{N} \overline{\beta_{k,i}}\, (\vec{x}, \vec{r}^{M}_{k-1}(\lambda_i)) \overline{\sum_{l=1}^{N} \overline{\beta_{l,i}}\, (\vec{y}, \vec{r}^{M}_{l-1}(\lambda_i))} \\ &= \sum_{i=1}^{M} \sum_{k,l=1}^{N} \overline{\beta_{k,i}}\, \beta_{l,i} (\vec{x}, \vec{r}^{M}_{k-1}(\lambda_i))\, \overline{(\vec{y}, \vec{r}^{M}_{l-1}(\lambda_i)}.\end{aligned}$$

Let us define mappings $f^k \colon \mathbb{C}^N \to \mathbb{P}$ in the following way:

$$f^k(\vec{x})(\lambda) = (\vec{x}, \vec{r}^{M}_{k-1}(\lambda)) = \sum_{l=1}^{M} x_l R_{k-1,N}(p_{l-1})(\lambda),$$

if $\vec{x} = (x_1, x_2, \ldots, x_M)^T \in \mathbb{C}^M$. Using this we can write:

$$
\begin{aligned}
(23)\quad (\vec{x},\vec{y}\,) &= \sum_{i=1}^{M}\sum_{k,l=1}^{N}\overline{\beta_{k,i}}\,\beta_{l,i}f^{k}(\vec{x}\,)(\lambda_i)\,\overline{f^{l}(\vec{y}\,)(\lambda_i)}\\
&= \sum_{i=1}^{M}(f^{1}(\vec{x}\,)(\lambda_i), f^{2}(\vec{x}\,)(\lambda_i),\dots,f^{N}(\vec{x}\,)(\lambda_i))\\
&\quad\times\begin{pmatrix}\overline{\beta_{1,i}}\,\beta_{1,i} & \overline{\beta_{1,i}}\,\beta_{2,i} & \dots & \overline{\beta_{1,i}}\,\beta_{N,i}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \overline{\beta_{N,i}}\,\beta_{1,i} & \overline{\beta_{N,i}}\,\beta_{2,i} & \dots & \overline{\beta_{N,i}}\,\beta_{N,i}\end{pmatrix}\overline{\begin{pmatrix}f^{1}(\vec{y}\,)(\lambda_i)\\ f^{2}(\vec{y}\,)(\lambda_i)\\ \cdot\\ f^{N}(\vec{y}\,)(\lambda_i)\end{pmatrix}}\\
&= \int_{P_N}(f^{1}(\vec{x}\,)(\lambda), f^{2}(\vec{x}\,)(\lambda),\dots,f^{N}(\vec{x}\,)(\lambda))\,dB_M(\lambda)\overline{\begin{pmatrix}f^{1}(\vec{y}\,)(\lambda)\\ f^{2}(\vec{y}\,)(\lambda)\\ \cdot\\ f^{N}(\vec{y}\,)(\lambda)\end{pmatrix}},
\end{aligned}
$$

where $B_M(\lambda)$ is piecewise constant on each ray of P_N, $B_M(0)=0$. It has jumps at $\lambda_i, i=1,2,\dots,M$ equal to $(b^{i}_{k,l})^{N}_{k,l=1}, b^{i}_{k,l}=\sum_{j:\lambda_j=\lambda_i}\overline{\beta_{k,j}}\,\beta_{l,j}$. At $\lambda=0$ we assume the jump is on a ray $[0,+\infty)$. With the integral we understand the sum of integrals over each radial ray in P_N.

If we take vectors of the standard basis in $\mathbb{C}^M$ $e_i, i=1,2,\dots,M$ as vectors $\vec{x},\vec{y}$ in (23) then we have:

$$
\begin{aligned}
(\vec{e}_i,\vec{e}_j) = \int_{P_N}&(R_{0,N}(p_{i-1})(\lambda), R_{1,N}(p_{i-1})(\lambda),\dots,R_{N-1,N}(p_{i-1})(\lambda))\,dB_M(\lambda)\\
&\times\overline{\begin{pmatrix}R_{0,N}(p_{j-1})(\lambda)\\ R_{1,N}(p_{j-1})(\lambda)\\ \cdot\\ R_{N-1,N}(p_{j-1})(\lambda)\end{pmatrix}}.
\end{aligned}
$$

Changing indices $i-1$ to i and $j-1$ to j we get (22). □

Lemma 3. *There exists a subsequence of positive integers $\{M_k\}_{k=0}^{\infty}$ and functions $B_{i,j}(\lambda)$, $i,j=0,1,2,\dots,N$, such that for any continuous function $f(\lambda)$ on a compact subset K of P_N the following is true:*

$$
(24)\qquad \lim_{k\to\infty}\int_K f(\lambda)\,dB_{M_k;i,j}(\lambda) = \int_K f(\lambda)\,dB_{i,j}(\lambda),\qquad i,j=0,1,\dots,N-1.
$$

Proof. Analogously as for the system of equations (21) we can say that the following system of linear algebraic equations:

$$
(25)\qquad \sum_{i=0}^{N-1}\xi^{k}_{i}(R_{0,N}(p_i)(\lambda), R_{1,N}(p_i)(\lambda),\dots,R_{N-1,N}(p_i)(\lambda)) = \vec{e}^{\,T}_{k},\qquad k=0,1,\dots,N-1,
$$

has a unique solution $\{\xi^{k}_{i}\}$.

Multiplying (22) by $\xi_i^k\,\overline{\xi_j^l}$, taking sum in (22) over i from 0 to $N-1$ and over l from 0 to $N-1$, and then using (25) we get:

$$(26)\quad \int_{P_N} \vec{e}_k^{\,T}\, dB_M(\lambda)\,\overline{\vec{e}_l^{\,T}} = \sum_{i=0}^{N-1} \xi_i^k \sum_{l=0}^{N-1} \overline{\xi_j^l}\,\delta_{ij} = \sum_{i=0}^{N-1} \xi_i^k\,\overline{\xi_i^l}, \quad k,l = 0,1,\ldots,N-1.$$

Note that

$$B_{M;i,i}(\lambda) = \int_0^\lambda \vec{e}_i^{\,T}\, dB_M(\lambda)\,\overline{\vec{e}_i^{\,T}} \le \int_{P_N} \vec{e}_i^{\,T}\, dB_M(\lambda)\,\overline{\vec{e}_i^{\,T}} = \sum_{j=0}^{N-1} |\xi_j^i|^2,$$

where $B_{M;i,j}(\lambda)$ are the entries of $B_M(\lambda)$.

Applying Helly's theorems to $B_{M;i,i}(\lambda)$, $M = N+1, N+2, \ldots$, we can find a function $B_{i,i}(\lambda) = \lim_{k\to\infty} B_{M_k;i,i}(\lambda)$, where M_k is a subsequence of $\{N+1, N+2, \ldots\}$. Note that

$$(27)\quad \operatorname{Re}\ B_{M;i,j}(\lambda) = \frac{1}{2}\left(\int_0^\lambda (\vec{e}_i + \vec{e}_j)^T\, dB_M(\lambda)\,\overline{(\vec{e}_i + \vec{e}_j)^T} - \int_0^\lambda \vec{e}_i^{\,T}\, dB_M(\lambda)\,\overline{\vec{e}_i^{\,T}} - \int_0^\lambda \vec{e}_j^{\,T}\, dB_M(\lambda)\,\overline{\vec{e}_j^{\,T}}\right), \quad \text{and}$$

$$(28)\quad \operatorname{Im}\ B_{M;i,j}(\lambda) = \frac{1}{2}\left(\int_0^\lambda (\vec{e}_i + i\vec{e}_j)^T\, dB_M(\lambda)\,\overline{(\vec{e}_i + i\vec{e}_j)^T} - \int_0^\lambda \vec{e}_i^{\,T}\, dB_M(\lambda)\,\overline{\vec{e}_i^{\,T}} - \int_0^\lambda \vec{e}_j^{\,T}\, B_M(\lambda)\,\overline{\vec{e}_j^{\,T}}\right).$$

The matrix functions B_M on the right of (14) and (15) are non-decreasing functions and they are bounded. So we can also use Helly's theorems and obtain the limit functions $B_{i,j}^{\mathrm{Re}}(\lambda)$ and $B_{i,j}^{\mathrm{Im}}(\lambda)$. Then we put $B_{i,j}(\lambda) = B_{i,j}^{\mathrm{Re}}(\lambda) + iB_{i,j}^{\mathrm{Im}}(\lambda)$.

Let K be a compact subset of P_N. For any continuous function $f(\lambda)$ on K the following is true:

$$\lim_{k\to\infty} \int_K f(\lambda)\, dB_{M_k;i,j} = \int_K f(\lambda)\, dB_{i,j}, \qquad i,j = 0,1,\ldots,N-1,$$

as follows from Helly's theorems (we subtracted subsequences sequently for each entry of $B_M(\lambda)$ and indexes M_k here correspond to the last subsequence). So, relation (24) holds. □

Put by definition $\vec{R}(u)(\lambda) = (R_{0,N}(u)(\lambda), R_{1,N}(u)(\lambda), ..., R_{N-1,N}(u)(\lambda))$, where $u(\lambda) \in \mathbb{P}$. With $\vec{R}^*(u)(\lambda)$ we shall denote the complex conjugated vector.

Put $B(\lambda) = (B_{i,j}(\lambda))_{i,j=0}^{N-1}$. It is a non-decreasing, Hermitian matrix function on P_N because $B_M(\lambda)$ are such functions. Let us check that for $B(\lambda)$ the orthonormality relation (12) is fulfilled. Using Lemma 2 we can write:

(29)

$$\begin{aligned}\delta_{nm} &= \int_{P_N} \vec{R}(p_n)(\lambda)\, dB_{M_k}(\lambda)\vec{R}^*(p_m)(\lambda)\\ &= \int_{K} \vec{R}(p_n)(\lambda)\, dB_{M_k}(\lambda)\vec{R}^*(p_m)(\lambda)\\ &\quad + \int_{P_N\setminus K} \vec{R}(p_n)(\lambda)\, dB_{M_k}(\lambda)\vec{R}^*(p_m)(\lambda), \qquad n,m < M_k,\ k=0,1,2,\dots.\end{aligned}$$

Define a functional:

$$\sigma_M^K(u(\lambda),v(\lambda)) = \int_{P_N\setminus K} \vec{R}(u)(\lambda)dB_M(\lambda)\vec{R}^*(v)(\lambda),$$
$$u,v\in P,\ M=N+1,N+2,\dots.$$

It is correctly defined because $B_M(\lambda)$ is piecewise constant.

It is not hard to see that the functional σ_M^K is bilinear, $\sigma_M^K(u,u)\geq 0$, and $\overline{\sigma_M^K(u,v)} = \sigma_M^K(v,u)$. Using the same arguments as for the scalar product in a Hilbert space we can show that

$$|\sigma_M^K(u,v)|^2 \leq \sigma_M^K(u,u)\sigma_M^K(v,v), \qquad u,v\in\mathbb{P}. \tag{30}$$

In particular, from this it follows that

$$|\sigma_M^K(p_n,p_m)|^2 \leq \sigma_M^K(p_n,p_n)\sigma_M^K(p_m,p_m). \tag{31}$$

Let us take $K=[-A,B]\cup[-Ci,Di]$, $A,B,C,D>0$.

Lemma 4. *For the set $K=[-A,B]\cup[-Ci,Di]$, $A,B,C,D>0$, the following estimates are fulfilled:*

$$\sigma_M^K(p_n,p_n) \leq \frac{\widetilde{A}_n}{\widetilde{K}^{2N}}, \qquad \text{for } n=0,1,2,\dots,M-N-1, \tag{32}$$

where $\widetilde{K}=\min(A,B,C,D)$ and $\widetilde{A}_n\geq 0$ does not depend of M.

Proof. Using the definition of σ_M^K we write:

$$\begin{aligned}\sigma_M^K(p_n,p_n) &= \int_{P_N\setminus K} \vec{R}(p_n)(\lambda)dB_M(\lambda)\vec{R}^*(p_n)(\lambda)\\ &= \int_{P_N\setminus K} \frac{1}{|\lambda^N|^2}\vec{R}(p_n\lambda^N)(\lambda)dB_M(\lambda)\vec{R}^*(p_n\lambda^N)(\lambda)\\ &\leq \frac{1}{\widetilde{K}^{2N}}\int_{P_N\setminus K} \vec{R}(p_n\lambda^N)(\lambda)dB_M(\lambda)\vec{R}^*(p_n\lambda^N)(\lambda),\end{aligned}$$

where we used the relation:

$$R_{m,N}(p_n(\lambda)\lambda^N) = \lambda^N R_{m,N}(p_n(\lambda)), \qquad m=0,1,\dots,N-1,$$

which follows from (17), and $\widetilde{K}=\min(A,B,C,D)$.

Then

$$\begin{aligned}
&\sigma_M^K(p_n,p_n)\\
&\le \frac{1}{\widetilde{K}^{2N}}\int_{P_N}\vec{R}(p_n\lambda^N)(\lambda)\,dB_M(\lambda)\vec{R}^*(p_n\lambda^N)(\lambda)\\
&= \frac{1}{\widetilde{K}^{2N}}\int_{P_N}\left(\sum_{i=1}^{N}\left(\overline{\alpha_{n-i,i}}\,\vec{R}(p_{n-i})(\lambda)+\alpha_{n,i}\vec{R}(p_{n+i})(\lambda)\right)+\alpha_{n,0}\vec{R}(p_n)(\lambda)\right)dB_M\\
&\quad\cdot(\lambda)\left(\sum_{j=1}^{N}\left(\alpha_{n-j,j}\,\vec{R}^*(p_{n-j})(\lambda)+\overline{\alpha_{n,j}}\vec{R}^*(p_{n+j})(\lambda)\right)+\alpha_{n,0}\vec{R}^*(p_n)(\lambda)\right),
\end{aligned}$$

where we used recurrence relation (1) and the linearity of $R_{m,N}$.

Then using (22) we can write

$$\sigma_M^K(p_n,p_n)\le\frac{1}{\widetilde{K}^{2N}}\left(\sum_{i=1}^{N}(|\alpha_{n-i,i}|^2+|\alpha_{n,i}|^2)+|\alpha_{n,0}|^2\right),$$

where $n<M-N$.

Put $\widetilde{A}_n=\sum_{i=1}^{N}(|\alpha_{n-i,i}|^2+|\alpha_{n,i}|^2)+|\alpha_{n,0}|^2, n=0,1,2,\dots$. Note that $\widetilde{A}_n$ does not depend of M. Then from the last inequality we obtain (32). □

Let us fix some positive integers n,m. Then we can write (29) for all k such that: $M_k-N>n, M_k-N>m$.

Let us pass to the limit in (29) as $k\to\infty$ (for selected numbers k). The first term on the right of (29) has the limit value $\int_K\vec{R}(p_n)(\lambda)\,dB(\lambda)\,\vec{R}^*(p_m)(\lambda)$ as $k\to\infty$ as follows from Lemma 3. So, the second term also has the limit value as $k\to\infty$. We can write:

$$\left|\delta_{nm}-\int_K\vec{R}(p_n)(\lambda)\,dB(\lambda)\,\vec{R}^*(p_m)(\lambda)\right|=\left|\lim_{k\to\infty}\sigma_{M_k}^K(p_n,p_m)\right|.$$

But

$$\lim_{k\to\infty}|\sigma_{M_k}^K(p_n,p_m)|\le\lim_{k\to\infty}\sqrt{\sigma_{M_k}^K(p_n,p_n)}\sqrt{\sigma_{M_k}^K(p_m,p_m)},$$

as follows from (31).

From Lemma 4 it follows that

$$\sigma_{M_k}^K(p_n,p_n)\le\frac{\widetilde{A}_n}{\widetilde{K}^{2N}},\qquad \sigma_{M_k}^K(p_m,p_m)\le\frac{\widetilde{A}_m}{\widetilde{K}^{2N}}.$$

Then

$$\lim_{k\to\infty}\sqrt{\sigma_{M_k}^K(p_n,p_n)}\sqrt{\sigma_{M_k}^K(p_m,p_m)}\le\lim_{k\to\infty}\sqrt{\frac{\widetilde{A}_n}{\widetilde{K}^{2N}}}\sqrt{\frac{\widetilde{A}_m}{\widetilde{K}^{2N}}}=\frac{\sqrt{\widetilde{A}_n\widetilde{A}_m}}{\widetilde{K}^{2N}};$$

and

$$\left|\delta_{nm}-\int_K\vec{R}(p_n)(\lambda)\,dB(\lambda)\,\vec{R}^*(p_m)(\lambda)\right|\le\frac{\sqrt{\widetilde{A}_n\widetilde{A}_m}}{\widetilde{K}^{2N}}.\tag{33}$$

From this it follows that

$$\lim_{A,B,C,D\to\infty}\left|\delta_{nm}-\int_K\vec{R}(p_n)(\lambda)\,dB(\lambda)\vec{R}^*(p_m)(\lambda)\right|=0.\tag{34}$$

But (34) means that an integral

$$\int_{P_N} \vec{R}(p_n)(\lambda)\, dB(\lambda)\, \vec{R}^*(p_m)(\lambda)$$

exists and

$$\int_{P_N} \vec{R}(p_n)(\lambda)\, dB(\lambda)\, \vec{R}^*(p_m)(\lambda) = \delta_{nm}. \tag{35}$$

Using the representation for $R_{m,N}$ (17) we can write

$$\int_{P_N} (p_n(\lambda), p_n(\lambda\varepsilon), \dots, p_n(\lambda\varepsilon^{N-1}))\frac{1}{N^2} J_\lambda\, dB(\lambda) J_\lambda^* \overline{\begin{pmatrix} p_m(\lambda) \\ p_m(\lambda\varepsilon) \\ \cdot \\ p_m(\lambda\varepsilon^{N-1}) \end{pmatrix}} = \delta_{nm}, \tag{36}$$

where J_λ is from (12).

Let us put $\sigma(\lambda) = \frac{1}{N^2} B(\lambda)$. From (36) it follows that (12) is fulfilled for $\sigma(\lambda)$.

The proof is complete. □

Theorem 2. *Let a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n is of the n-th degree) satisfy* (1). *Then the relation* (13) *is valid.*

Proof. Suppose a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n is of the n-th degree) satisfies (1). Then from Theorem 1 we conclude (12). The entries of matrix $\sigma(\lambda)$ in (13) are denoted by $\sigma_{i,j}(\lambda)$, $i, j = 0, 1, \dots, N-1$. They are absolutely continuous with respect to the trace measure $\tau(\lambda) = \sum_{i=0}^{N-1} \sigma_{i,i}(\lambda)$. This can be proved by the same arguments used for matrix functions on the real line [10]. Put by definition $\vec{S}(u)(\lambda) = (u(\lambda), u(\lambda\varepsilon), u(\lambda\varepsilon^2), \dots, u(\lambda\varepsilon^{N-1}))$, where $u(\lambda) \in \mathbb{P}$. With $\vec{S}^*(u)(\lambda)$ we shall denote the complex conjugated vector. Then we can write

$$\int_{P_N} \vec{S}(p_n)(\lambda) J_\lambda D(\lambda) J_\lambda^* \vec{S}^*(p_m)(\lambda) d\tau(\lambda) = \delta_{nm}, \tag{37}$$

$D(\lambda) = (d_{i,j}(\lambda))_{i,j=0}^{N-1}$ is a nonnegative matrix function and $|d_{i,j}| \le 1$.

Let us introduce a function

$$W(\lambda) = \int_{a_i}^{\lambda} J_\lambda D(\lambda) J_\lambda^*\, d\tau(\lambda),\ \lambda \in (0, a_i\infty), \qquad i = 1, 2, \dots, 2N, \tag{38}$$

where a_i are the all roots of 1 and (-1) of order N.

Note that $P_N \backslash \{0\} = \bigcup_{i=1}^{2N} (0, a_i\infty)$.

Denote the integral on the left of (37) by I_1.

$$I_1 = \lim_{A\to\infty (A>1)} \int_{P_N \cap S(A)} \vec{S}(p_n)(\lambda) J_\lambda D(\lambda) J_\lambda^* \vec{S}^*(p_m)(\lambda)\, d\tau(\lambda),$$

where $S(A) = \{\lambda \in C : |\lambda| \le A\}$.

Then

$$I_1 = \lim_{A\to\infty (A>1)} \int_{(P_N \cap S(A))\backslash S(\delta)} \vec{S}(p_n)(\lambda) J_\lambda D(\lambda) J_\lambda^* \vec{S}^*(p_m)(\lambda)\, d\tau(\lambda) + \int_{P_N \cap S(\delta)} \vec{S}(p_n)(\lambda) J_\lambda D(\lambda) J_\lambda^* \vec{S}^*(p_m)(\lambda)\, d\tau(\lambda),$$

where $0 < \delta < 1$.

Note that

$$\int_{(P_N\cap S(A))\setminus S(\delta)} \vec{S}(p_n)(\lambda) J_\lambda D(\lambda) J_\lambda^* \vec{S}^*(p_m)(\lambda)\, d\tau(\lambda)$$
$$= \int_{(P_N\cap S(A))\setminus S(\delta)} \vec{S}(p_n)(\lambda)\, dW(\lambda)\, \vec{S}^*(p_m)(\lambda), A > 1, \quad 0 < \delta < 1.$$

It is not hard to see considering the integral sums of the integrals. Then we can write:

$$I_1 = \lim_{A\to\infty} \int_{(P_N\cap S(A))\setminus S(\delta)} \vec{S}(p_n)(\lambda)\, dW(\lambda)\, \vec{S}^*(p_m)(\lambda) + \int_{P_N\cap S(\delta)} \vec{S}(p_n)(\lambda) J_\lambda\, d\sigma(\lambda) J_\lambda^* \vec{S}^*(p_m)(\lambda). \tag{39}$$

Denote the second term on the right of (39) by I_2. Then

$$I_2 = \int_{P_N\cap S(\delta)} \vec{R}(p_n) N^2 d\sigma(\lambda)\, \vec{R}^*(p_m),$$

and

$$\left| \int_{P_N\cap S(\delta)} \vec{R}(p_n) N^2\, d\sigma(\lambda)\, \vec{R}^*(p_m) - \vec{R}(p_n)(0) N^2 \Delta\sigma(0)\, \vec{R}^*(p_m)(0) \right|$$
$$= \left| \int_{P_N\cap S(\delta)} \vec{R}(p_n) N^2\, d\hat{\sigma}(\lambda)\, \vec{R}^*(p_m)(\lambda) \right|,$$

where $\Delta\sigma(0) = \sigma(+0)$, $\hat{\sigma}(\lambda) = \sigma(\lambda) - \sigma(+0)$.

The entries of $\hat{\sigma}(\lambda)$ $\hat{\sigma}_{i,j}(\lambda), i,j = 0,1,\ldots,N-1$, are absolutely continuous with respect to $\hat{\tau}(\lambda) = \sum_{i=0}^{N-1} \hat{\sigma}_{i,i}(\lambda)$. Then

$$\left| \int_{P_N\cap S(\delta)} \vec{R}(p_n) N^2\, d\hat{\sigma}(\lambda)\, \vec{R}^*(p_m) \right| = \left| \int_{P_N\cap S(\delta)} \vec{R}(p_n) N^2 \hat{D}(\lambda)\, \vec{R}^*(p_m)\ d\hat{\tau}(\lambda) \right|$$
$$\leq \int_{P_N\cap S(\delta)} \left| \vec{R}(p_n) N^2 \hat{D}(\lambda)\, \vec{R}^*(p_m) \right|\ d\hat{\tau}(\lambda),$$

where $\hat{D}(\lambda) = (\hat{d}_{i,j}(\lambda))_{i,j=0}^{N-1}$, $|\hat{d}_{i,j}(\lambda)| \leq 1$.

Since the expression $\vec{R}(p_n) N^2 \hat{D}(\lambda)\, \vec{R}^*(p_m)$ is bounded in $P_N \cap S(\delta)$ and a.e. continuous then

$$\lim_{\delta\to 0} \int_{P_N\cap S(\delta)} \left| \vec{R}(p_n) N^2 \hat{D}(\lambda)\, \vec{R}^*(p_m) \right|\ d\hat{\tau}(\lambda) = 0.$$

Thus we have

$$\left| I_2 - \vec{R}(p_n)(0) N^2 \Delta\sigma(0)\, \vec{R}^*(p_m)(0) \right| \to 0, \delta \to 0. \tag{40}$$

Proceeding to the limit as $\delta \to 0$ in equality (39) and using (40) we get

(41)

$$I_1 = \lim_{A\to\infty} \int_{P_N\cap S(A)} \vec{S}(p_n)(\lambda)\, dW(\lambda)\, \vec{S}^*(p_m)(\lambda) + \vec{R}(p_n)(0) N^2 \Delta\sigma(0)\, \vec{R}^*(p_m)(0).$$

Using property (16) and the definition of $R_{N,m}$ from (4) we can write:

$$\vec{R}(p_n)(0)N^2\Delta\sigma(0)\,\vec{R}^*(p_m)(0) \tag{42}$$

$$= \left(p_n(0), p_n'(0), \frac{p_n''(0)}{2!}, \dots, \frac{p_n^{(N-1)}(0)}{(N-1)!}\right) N^2\Delta\sigma(0) \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ \cdot \\ \frac{p_m^{(N-1)}(0)}{(N-1)!} \end{pmatrix}}$$

$$= (p_n(0), p_n'(0), p_n''(0), \dots, p_n^{(N-1)}(0))\,\mathrm{diag}\left(1, 1, \frac{1}{2}, \dots, \frac{1}{(N-1)!}\right) N^2\Delta\sigma(0)$$

$$\times\,\mathrm{diag}\left(1, 1, \frac{1}{2}, \dots, \frac{1}{(N-1)!}\right) \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ \cdot \\ p_m^{(N-1)}(0) \end{pmatrix}}.$$

Denote $M = \mathrm{diag}\left(1, 1, \frac{1}{2}, \dots, \frac{1}{(N-1)!}\right) N^2\Delta\sigma(0)\,\mathrm{diag}\left(1, 1, \frac{1}{2}, \dots, \frac{1}{(N-1)!}\right)$. Then from (41) and (42) we get

$$I = \int_{P_N} \vec{S}(p_n)(\lambda)\, dW(\lambda)\, \vec{S}^*(p_m)(\lambda)$$

$$+ (p_n(0), p_n'(0), p_n''(0), \dots, p_n^{(N-1)}(0)) M \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ \cdot \\ p_m^{(N-1)}(0) \end{pmatrix}}.$$

If we remember that $I = \delta_{nm}$ then we conclude (13). The theorem is proved. $\square$

The converse of these theorems is also true.

Theorem 3. *If a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n is of the n-th degree) satisfies* (12) *or* (13) *then it satisfies* (1) *with a set of numbers $\alpha_{m,n}$.*

Proof. Let a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n is of the n-th degree) satisfies (5) or (25). Define a functional

$$\sigma(u, v) = \int_{P_N} \vec{S}(u)(\lambda) J_\lambda\, d\sigma(\lambda) J_\lambda^*\, \vec{S}^*(v)(\lambda), \tag{43}$$

in the case (12), or

$$\sigma(u, v) = \int_{P_N} \vec{S}(u)(\lambda)\, dW(\lambda)\, \vec{S}^*(v)(\lambda) \tag{44}$$

$$+ \left(u(0), u'(0), \dots, u^{(N-1)}(0)\right) M \overline{\begin{pmatrix} v(0) \\ v'(0) \\ \cdot \\ \cdot \\ v^{(N-1)}(0) \end{pmatrix}},$$

in the case (13), $u, v \in \mathbb{P}$.

The functional $\sigma(u,v)$ is bilinear. Also

$$\overline{\sigma(u,v)} = \sigma(v,u). \tag{45}$$

From (12) (or (13)) we get:

$$\sigma(p_i,p_j) = \delta_{ij}, \qquad i,j = 0,1,2,\dots. \tag{46}$$

From (43), (44) it is clear that:

$$\sigma(\lambda^N p_i(\lambda), p_j(\lambda)) = \sigma(p_i(\lambda), \lambda^N p_j(\lambda)), \qquad i,j = 0,1,2,\dots. \tag{47}$$

The polynomial $p_i(\lambda)$ has degree i and so we can write:

$$\lambda^N p_k(\lambda) = \sum_{i=0}^{k+N} \xi_{k,i} p_i(\lambda), \qquad \text{for } \xi_{k,i} \in \mathbb{C}. \tag{48}$$

Using (45) and (46), from (48) we get:

$$\xi_{k,i} = \sigma(\lambda^N p_k(\lambda), p_i(\lambda)), \qquad i = 0,1,\dots,k+N. \tag{49}$$

Then, using (47) we write:

$$\xi_{k,i} = \sigma(\lambda^N p_k(\lambda), p_i(\lambda)) = \sigma(p_k(\lambda), \lambda^N p_i(\lambda)) = 0, \qquad i+N<k.$$

So,

$$\lambda^N p_k(\lambda) = \sum_{j=1}^{N} (\xi_{k,k-j} p_{k-j}(\lambda) + \xi_{k,k+j} p_{k+j}(\lambda)) + \xi_{k,k} p_k(\lambda), \tag{50}$$

where $p_{-i}(\lambda) = 0$.

Put $\alpha_{k,0} = \xi_{k,k}$, $\alpha_{k,j} = \xi_{k,k+j}$, $j = 1,2,\dots,N$.

From (49), (47), (44) we conclude that

$$\alpha_{k,0} = \sigma(\lambda^N p_k(\lambda), p_k(\lambda)) = \sigma(p_k(\lambda), \lambda^N p_k(\lambda)) = \overline{\sigma(\lambda^N p_k(\lambda), p_k(\lambda))} = \overline{\alpha_{k,0}}.$$

Also we get

$$\begin{aligned} \xi_{k,k-j} &= \sigma(\lambda^N p_k(\lambda), p_{k-j}(\lambda)) = \sigma(p_k(\lambda), \lambda^N p_{k-j}(\lambda)) \\ &= \overline{\sigma(\lambda^N p_{k-j}(\lambda), p_k(\lambda))} = \overline{\xi_{k-j,k}} = \overline{\alpha_{k-j,j}}. \end{aligned}$$

Using this, from (50) we obtain (1). The theorem is proved. □

The class of systems of polynomials which satisfy (13) includes the following classes of systems of polynomials:

1) Orthonormal polynomials on the real line. Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n is of the n-th degree) be a system of orthonormal polynomials on the real line:

$$\int_{\mathbb{R}} p_n(\lambda) p_m(\lambda)\, d\tau(\lambda) = \delta_{nm}, \qquad n,m = 0,1,2,\dots,$$

where $\tau(\lambda)$ is a non-decreasing function on $\mathbb{R}$. Then

$$\int_{\mathbb{R}} p_n(\lambda) p_m(\lambda)\, d\tau(\lambda) = \int_{[0,+\infty)\cup[0,-\infty)} p_n(\lambda)\, d\mu(\lambda)\, \overline{p_m(\lambda)} + p_n(0) p_m(0) \widetilde{M},$$

where $\mu(\lambda) = \begin{cases} \tau(\lambda), & \lambda > 0 \\ -\tau(\lambda), & \lambda < 0 \end{cases}$, $\widetilde{M} = \tau(+0) - \tau(-0)$, and the integral on the right is understood as improper at $\lambda = 0$. Then

$$\int_{\mathbb{R}} p_n(\lambda) p_m(\lambda)\, d\tau(\lambda)$$
$$= \int_{P_N} \vec{S}(p_n)(\lambda)\, d \begin{pmatrix} \mu(\lambda) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \vec{S}^*(p_m)(\lambda)$$
$$+ (p_n(0), p_n'(0), \dots, p_n^{(N-1)}(0)) \begin{pmatrix} \widetilde{M} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ \vdots \\ p_m^{(N-1)}(0) \end{pmatrix}}.$$

So, for $\{p_n(\lambda)\}_{n=0}^{\infty}$ a relation of type (13) is fulfilled. Here $\mu(\lambda) = 0$, $\lambda \notin R$.

2) Sobolev orthonormal polynomials with a discrete measure concentrated at zero. Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n is of the n-th degree) be a system of polynomials such that:

$$\int_R p_n(\lambda) p_m(\lambda)\, d\tau(\lambda) + (p_n(0), p_n'(0), \dots, p_n^{(N-1)}(0)) M \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ \vdots \\ p_m^{(N-1)}(0) \end{pmatrix}}$$
$$= \delta_{nm}, \qquad n, m = 0, 1, 2, \dots,$$

where $\tau(\lambda)$ is a non-decreasing function on R, $M \geq 0$ is a complex numerical matrix. Using the notations from 1) for $\mu(\lambda)$ and $\widetilde{M}$, we can write

$$\int_{P_N} \vec{S}(p_n)(\lambda)\, d \begin{pmatrix} \mu(\lambda) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \vec{S}^*(p_m)(\lambda)$$
$$+ (p_n(0), p_n'(0), \dots, p_n^{(N-1)}(0)) \left(M + \begin{pmatrix} \widetilde{M} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right) \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ \vdots \\ p_m^{(N-1)}(0) \end{pmatrix}}.$$

Then for $\{p_n(\lambda)\}_{n=0}^{\infty}$ a relation of type (13) holds true.

3) Orthonormal polynomials on the radial rays with respect to a scalar measure. Orthogonal polynomials on the radial rays with respect to a scalar measure are studied by G. V. Milovanović in [14] and in [13]. Let us consider a system

of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n is of the n-th degree) such that:

$$\int_{P_N} p_n(\lambda)\, dw(\lambda)\, \overline{p_m(\lambda)} = \delta_{nm}, \qquad n,m = 0,1,2,\ldots,$$

where $w(\lambda)$ is a nondecreasing function on P_N, $w(0) = 0$ (the integral is understood as a sum of integrals on each ray in P_N).

If $w(\lambda)$ is absolutely continuous it is not hard to see that we obtain polynomials studied by Milovanović. We can write

$$\int_{P_N} \vec{S}(p_n)(\lambda)\, d \begin{pmatrix} w(\lambda) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \vec{S}^*(p_m)(\lambda)$$

$$+ (p_n(0), p_n'(0), \ldots, p_n^{(N-1)}(0)) \begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ \vdots \\ p_m^{(N-1)}(0) \end{pmatrix}} = \delta_{nm},$$

where the integral at $\lambda = 0$ is understood as improper, and A is a sum of the jumps of $w(\lambda)$ at $\lambda = 0$ on each ray in P_N. So, for $\{p_n(\lambda)\}_{n=0}^{\infty}$ a relation of type (13) is valid.

3. On a connection with orthonormal matrix polynomials

Now we shall establish a connection between the polynomials from (12) and orthogonal matrix polynomials. From the proof of Theorem 1 it is clear that there exists a function $\sigma(\lambda)$ for which (12) is valid and which has support on $R^+ \cup M^+$, where $R^+ = [0, +\infty), M^+ = [0, w\infty)$, w is a fixed root of (-1). Really, the function $B_M(\lambda)$ from (23) had jumps in $\lambda_i, i = 1,2,\ldots,M$ and λ_i by definition are $\lambda_i = \sqrt[N]{c_i}; c_i \in \mathbb{R}$. So, in the proof we could take a fixed branch of the root for λ_i. Let us write the orthonormality relation:

$$\int_{R^+ \cup M^+} \vec{S}(p_n)(\lambda) J_\lambda\, d\sigma(\lambda) J_\lambda^*\, \vec{S}^*(p_m)(\lambda) = \delta_{nm}, \qquad n,m = 0,1,2,\ldots. \tag{51}$$

Then

$$\int_{R^+ \cup M^+} \vec{R}(p_n)(\lambda)\, d\sigma(\lambda)\, \vec{R}^*(p_m)(\lambda). \tag{52}$$

Put by definition

$$R_k'(\lambda) = \begin{pmatrix} R_{0,N}(p_{Nk})(\lambda) & R_{1,N}(p_{Nk})(\lambda) & \ldots & R_{N-1,N}(p_{Nk})(\lambda) \\ R_{0,N}(p_{Nk+1})(\lambda) & R_{1,N}(p_{Nk+1})(\lambda) & \ldots & R_{N-1,N}(p_{Nk+1})(\lambda) \\ . & . & . & . \\ R_{0,N}(p_{Nk+N-1})(\lambda) & R_{1,N}(p_{Nk+N-1})(\lambda) & \ldots & R_{N-1,N}(p_{Nk+N-1})(\lambda) \end{pmatrix},$$

and with $R_k'^*(\lambda)$ we shall denote the complex conjugated matrix. Then

$$\int_{R^+ \cup M^+} R_k'(\lambda) N^2\, d\sigma(\lambda) R_l'^*(\lambda) = I\delta_{kl}, \qquad k,l = 0,1,2,\ldots. \tag{53}$$

Make the substitution $t = \lambda^N, \lambda = \sqrt[N]{t}$. Then

$$\int_{R^+ \cup [0,-\infty)} R'_k(\sqrt[N]{t}) N^2 \, d\sigma(\sqrt[N]{t}) R'^*_l(\sqrt[N]{t}) = \int_R R'_k(\sqrt[N]{t}) \, d\hat{\sigma}(t) R'^*_l(\sqrt[N]{t}) = I\delta_{kl},$$

$$k, l = 0, 1, 2, \ldots, \tag{54}$$

where $\hat{\sigma}(\lambda) = \begin{cases} N^2\sigma(\sqrt[N]{t}), & t \in [0, +\infty) \\ -N^2\sigma(\sqrt[N]{t}), & t \in (-\infty, 0] \end{cases}$.

Put

$$P_k(t) = R'_k(\sqrt[N]{t}), k = 0, 1, 2, \ldots. \tag{55}$$

Then

$$\int_R P_k(t) \, d\hat{\sigma}(t) \, \overline{P_l(t)} = I\delta_{kl}, \qquad k, l = 0, 1, 2, \ldots. \tag{56}$$

From (55) it follows that $P_k(t)$ is a matrix polynomial of degree k and its leading coefficient is a lower triangular nonsingular matrix. So, $\{P_k(t)\}_0^\infty$ is a set of orthonormal matrix polynomials on the real line.

Conversely, let $\{P_k(t)\}_0^\infty$ be a set of orthonormal matrix polynomials on the real line and suppose (56) holds, with a non-decreasing function $\hat{\sigma}(t)$. Also we assume that the leading coefficient of $P_k(t)$ is a lower triangular matrix. Then we put

$$p_{Nk+i}(t) = P_{k;i,0}(t^N) + tP_{k;i,1}(t^N) + t^2 P_{k;i,2}(t^N) + \cdots + t^{N-1}P_{k;i,N-1}(t^N), \quad i = 0, 1, \ldots, N-1, \tag{57}$$

where $P_{k;i,j}$ are the entries of $P_k(t)$.

It is not hard to verify that $p_{Nk+i}(t)$ is a polynomial of degree $Nk+i$. Using the definition of $p_l(t)$ from (57) we can write $P_k(t)$ in form (55). From property (56) for $\{P_k(t)\}_0^\infty$ it follows (54). Then we deduce (53), (52) and (51).

4. Conclusion

From the previous sections we have understood that if a system of polynomials $\{p_n(\lambda)\}_{n=0}^\infty$ ($\deg p_n = n$) satisfies recurrence (1), or, equivalently, in a matrix form, (2), (3), then polynomials are orthonormal in a sense (4)((5)), (12) or (13). Notice first of all that (4) was obtained for real $\{\alpha_{n,m}\}$ and an extension of the operator theory arguments in the proof of (4) in [6] are not trivial. Second, relations (12), (13) have a different form from (4)((5)). Maybe they will be more convenient in algebraic expressions because they are transparent (for example, $\sigma(\lambda^N p, q) = \sigma(p, \lambda^N q)$ is obvious from (12), (13)). Also, since these forms do not use operators $R_{n,m}$, the integrals in (12), (13) can be considered not only for infinitely differentiable functions at zero. In particular, one can consider corresponding L_2 spaces of measurable functions on P_N such that:

$$\int_{P_N} (f(\lambda), f(\lambda\varepsilon), f(\lambda\varepsilon^2), \ldots, f(\lambda\varepsilon^{N-1})) J_\lambda \, d\sigma(\lambda) J^*_\lambda \overline{\begin{pmatrix} f(\lambda) \\ f(\lambda\varepsilon) \\ \cdot \\ \cdot \\ f(\lambda\varepsilon^{N-1}) \end{pmatrix}} < \infty;$$

or,

$$\int_{P_N} (f(\lambda), f(\lambda\varepsilon), f(\lambda\varepsilon^2), \ldots, f(\lambda\varepsilon^{N-1}))\, dW(\lambda) \overline{\begin{pmatrix} f(\lambda) \\ f(\lambda\varepsilon) \\ \cdot \\ \cdot \\ f(\lambda\varepsilon^{N-1}) \end{pmatrix}}$$

$$+ (f(0), f'(0), f''(0), \ldots, f^{(N-1)}(0))M \overline{\begin{pmatrix} f(0) \\ f'(0) \\ \cdot \\ \cdot \\ f^{(N-1)}(0) \end{pmatrix}} < \infty,$$

where one needs only the existence of $f^{(k)}(0), k = 0, 1, 2, \ldots, N-1$.

Third, the method of proof of (12), i.e., the step by step construction of the spectral measure, was used to construct explicit measures like (10) in [16], [19]. It can be used in further investigations. Also, the method admits a numerical calculation of the spectral measure. We notice that the $(2N+1)$-banded matrix in (3) can be considered as a Jacobi matrix with matrix entries of size $(N \times N)$. Also the corresponding set of matrix polynomials $\{\widetilde{P}_n(\lambda)\}_0^\infty$ can be considered. The construction of spectral measure in terms of matrix polynomials $\widetilde{P}_n$ and a matrix orthonormality relation can be found in [3, Chapter 6, §8] (see also [2] for special Jacobi matrices).

Let us consider now relations (56), (55) and (6). It is not hard to see that polynomials from (55) and (6) coincide. It seems that the relation between orthonormal polynomials on rays P_N and matrix orthonormal polynomials on $\mathbb{R}$ will be usefull in the both directions. For example, orthonormality relations (10) can be translated on the language of the matrix orthonormality relations for matrix polynomials. Also there are different results for the Sobolev type orthonormal polynomials with a discrete measure which consequently imply results for the matrix orthonormal polynomials.

The interrelation between orthonormal polynomials on rays and matrix orthonormal polynomials may be compared with interrelation between orthonormal polynomials on the unit circle and on the real segment. For instance, this interrelation was usefull for calculating the asymptotics of orthogonal polynomials on the real segment [9]. But classes of systems of polynomials on the unit circle and on a real segment have different general properties for their roots. This also holds for the case of orthonormal polynomial on radial rays and matrix orthogonal polynomials on the real line.

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