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Band-pass moves and the Casson–Walker–Lescop invariant

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Abstract. Computable formulas involving only linking numbers and surgery coefficients are presented for computing how the Casson–Walker–Lescop invariant changes under band pass moves in a framed link presenting a 3-manifold. There is a discussion of the relevant equivalence classes and the potential for further formulas of this type.

Contents

1. Introduction

In 1985 Casson introduced an integer-valued invariant of homology spheres [AM]. Many geometric questions were addressed via the connections between this invariant and the Rohlin invariant. Since then, this invariant has been extended to arbitrary 3-manifolds [W, Ls], and shown to have connections to both the Seiberg–Witten invariants [N] and quantum 3-manifold invariants [HB]. This invariant may be computed by either detailed representation theory or by Lescop's lengthy combinatorial formula. In [J1] we presented a formula in terms of linking numbers and surgery coefficients for computing how the Casson–Walker invariant changes under crossing changes in framed links presenting 3-manifolds. This crossing change formula only applies to a crossing within one component; we would also like to have a formula for changes between components. In general, changes between components do not admit this approach, but here we describe a crossing change formula between components in a particular case.

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The concept of surgery modification is presented in [Lv]:

Definition 1. *Surgery modification* of a framed link $\mathbf{L}$ is the effect on $\mathbf{L}$ of integer surgery on a null-homologous unlink $\mathbf{B}$ in the complement of $\mathbf{L}$.

The formula in [J1] was derived by viewing the crossing change as a surgery modification of the original framed link. It is natural to consider what other surgery modifications would produce useful formulas. Using the technique of changing the link by surgery on an extra component, K_0, we will not discover a general crossing change formula between components because surgery modification preserves linking numbers. On the other hand, this limitation does not prohibit computation for the effect of crossings that preserve linking numbers. In particular, we can compute the effects of a band crossing change between components on the Casson–Walker invariant of surgery on a link. The formula, though technical, depends only on the computation of linking numbers. This result and related discussions are presented in this paper.

In Section 2 the formulas are presented with examples illustrating their use. In Section 3 they are placed in context through a discussion of the relevant equivalence classes of links. In Section 4 we present the proof of the band pass formula between components

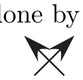
In Section 5 we reexamine the invariant $\beta(\mathbf{L}, \mathbf{s})$ defined in [J1]. Results in [J2] show that this invariant is polynomial in $\mathbf{s}$ for any $\mathbf{L}$ link homologous to (i.e., having the same linking numbers as) $\mathbf{L}^0$. The external band crossing formula provides a geometric skein approach to computing this invariant for $\mathbf{L}$ surgery equivalent to $\mathbf{L}^0$. We conclude this paper by considering methods to compute $\beta(\mathbf{L}, \mathbf{s})$ for $\mathbf{L}$ link homologous but not surgery equivalent to $\mathbf{L}^0$.

Acknowledgements. Some of the original work on this paper was completed while I was a graduate student at Indiana University. I would like to thank my advisor Charles Livingston who encouraged me to follow many of the directions pursued in this paper.

2. Formulas and examples

Let us set some notation and recall some formulas. Let $\mathbf{L} = (K_1, K_2, ..., K_n)$ be an ordered n-component link in S^3, let $\mathbf{s} = (s_1, s_2, ..., s_n)$ be a rational vector, let $\mathbf{L_s}$ be the 3-manifold obtained by $\mathbf{s}$ surgery on $\mathbf{L}$, and let $A(\mathbf{L}, \mathbf{s})$ be the associated linking matrix. Let $\lambda(\mathbf{L_s})$ be the Casson–Walker–Lescop invariant of $\mathbf{L_s}$, and if $\mathbf{L_s}$ is a rational homology sphere, or equivalently if $\det(A(\mathbf{L}, \mathbf{s})) \neq 0$, let $\lambda_w(\mathbf{L_s})$ be the Casson–Walker invariant of $\mathbf{L_s}$ [AM, W, Ls].

When we perform a crossing change we can consider passing through an intermediate stage of a link with one singular point from a self-intersection in the first component. If we have precisely one singularity there are two "lobes" determined. To see one lobe start at the singularity and traverse the knot until returning to the singularity. The other lobe is the remaining portion of the component. We will denote the two lobes of K_1 as K_1^a and K_1^b. The process of separating the lobes is called "smoothing", and it is done by locally replacing

 with .

Given this background, the crossing change formula of [J1] can be restated. For $\mathbf{L}^-$ and $\mathbf{L}^+$, two links in S^3 with the same framing which differ by a crossing change in the first component, where $\det(A(\mathbf{L}^\pm, \mathbf{s})) \neq 0$,

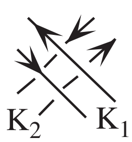
$$(*) \qquad \lambda_w(\mathbf{L}^-_\mathbf{s}) - \lambda_w(\mathbf{L}^+_\mathbf{s}) = 2\frac{\begin{vmatrix} l & k^a_{12} & \dots & k^a_{1n} \\ k^b_{12} & s_2 & \dots & n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k^b_{1n} & n_{2n} & \dots & s_n \end{vmatrix}}{\begin{vmatrix} s_1 & n_{12} & \dots & n_{1n} \\ n_{12} & s_2 & \dots & n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ n_{1n} & n_{2n} & \dots & s_n \end{vmatrix}}$$

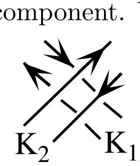
where s_i's are the surgery coefficients,

$$n_{ij} = \mathrm{lk}(K_i, K_j), \qquad k^a_{1j} = \mathrm{lk}(K^a_1, K_j),$$
$$k^b_{1j} = \mathrm{lk}(K^b_1, K_j), \qquad l = \mathrm{lk}(K^a_1, K^b_1).$$

For reference let us refer to this formula as the "internal crossing change formula".

Can we get similar results to this theorem? The internal crossing change formula was proven using the fact that we can produce such a crossing change using a surgery modification. In this paper we derive the following formula for how the Casson–Walker invariant changes under band-pass moves. A band-pass move is the local over-under change between two oppositely oriented strands in one component and two oppositely oriented strands in another component. Visually this is the change

from 

to 

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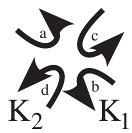
In this setting the following formula predicts the effects on the Casson–Walker–Lescop invariant on the surgered manifold presented by the given link.

$$\lambda_w(\text{band 2 over band 1}) - \lambda_w(\text{band 1 over band 2}) =$$
$$4\frac{\begin{vmatrix} n_{ac} & n_{ad} & k^a_{13} & \dots & k^a_{1n} \\ n_{bc} & n_{bd} & k^b_{13} & \dots & k^b_{1n} \\ k^c_{23} & k^d_{23} & s_3 & \dots & n_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k^c_{2n} & k^d_{2n} & n_{3n} & \dots & s_n \end{vmatrix}}{\begin{vmatrix} s_1 & n_{12} & \dots & n_{1n} \\ n_{12} & s_2 & \dots & n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ n_{1n} & n_{2n} & \dots & s_n \end{vmatrix}}$$

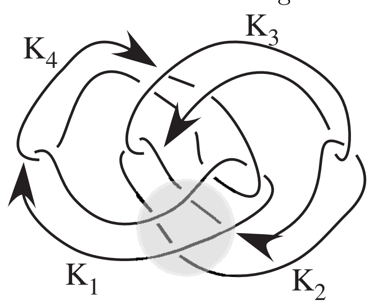
where both the first and second component are smoothed at the band pass. After this smoothing, the first component is separated into K^a_1 and K^b_1. Likewise the second component is separated into K^c_2 and K^d_2.

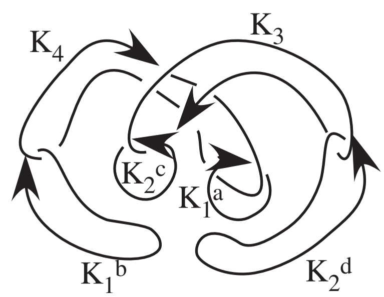
This smoothing can be seen here: 

.

In the matrix used for the formula $n_{ac} = \mathrm{lk}(K_1^a, K_2^c)$ and $k_{13}^a = \mathrm{lk}(K_1^a, K_3)$. The other symbols are defined analogously.

Let us examine a particular example to clarify the use of this formula. The most natural example to which this theorem applies is the Bing double of the Hopf link. Consider the following two diagrams of this link: the diagram with the highlighted band crossing before the change and the diagram of the new components created after the double smoothing.





Changing the band crossing of this link results in the unlink. Therefore we have

$$\lambda_w(0_1^4) - \lambda_w(\text{Bing double of Hopf link}) = 4\frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & s_3 & 0 \\ 0 & 0 & 0 & s_4 \end{vmatrix}}{\begin{vmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & s_4 \end{vmatrix}} = 0.$$

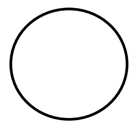
Therefore any surgery on the four component unlink producing a rational homology sphere will result in a manifold with the same Casson–Walker invariant as the surgery on the Bing double of the Hopf link. These results extend to Lescop's invariant by the proof in [J2], so the above statement can be made for any surgery. Using this new formula along with the internal crossing change formula allows us to compute $\lambda(\mathbf{L}'_\mathbf{s}) - \lambda(\mathbf{L}_\mathbf{s})$ for any $\mathbf{L}'$ surgery equivalent to $\mathbf{L}$. (See the following section for a definition of surgery equivalent links.)

3. A comparison of equivalence classes

In this section we consider the relations among several equivalence classes of links. Two links are said to be link homotopic if there is a homotopy between them allowing crossing changes *within* each component but no changes *between* compmonents. Milnor's μ invariants with distinct indices are link homotopy invariants [F]. Two links are said to be pass-equivalent if one can be obtained from the other by a combination of ambient isotopy and band-pass moves. Band-pass moves were introduced by Kauffman [K] and studied extensively for their connections with the

Arf invariant. The Arf invariant is a pass-equivalence invariant for knots [K]. Two links are said to be surgery equivalent if one can be obtained from the other by a sequence of surgery modifications [Lv]. Milnor's μ_{ijk} for distinct indices is a surgery equivalent invariant [Lv]. Two links are said to be link homologous if they have the same pairwise linking numbers.

Gallery of Examples: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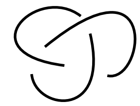
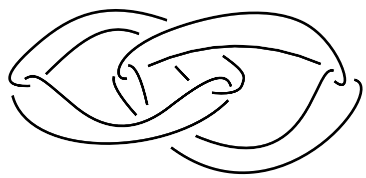
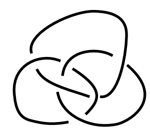

trefoil

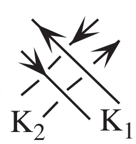

Bing-double of Hopf link

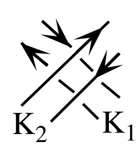

Borromean rings

Surgery equivalence is generated by band-pass moves and link homotopy [Lv]. Both of these moves are necessary. The Arf invariant detects knots that are not pass-equivalent to the unknot. In particular, the trefoil is not pass-equivalent to the unknot, but it is homotopic to the unknot and therefore surgery equivalent. The Bing-double of the Hopf link is not homotopic to the unknot because μ_{1234} is an invariant of link homotopy and $\mu_{1234} = \pm 1$ for the Bing-double of the Hopf link and $\mu_{1234} = 0$ for the unlink [C]. If two links are surgery equivalent then they have the same linking numbers. The Borromean rings and the three component unlink have the same linking numbers but they are not surgery equivalent because μ_{123} is an invariant of surgery equivalence and $\mu_{123} = \pm 1$ for the Borromean rings and $\mu_{123} = 0$ for the three component unlink.

4. The two-component band crossing formula

When we change a presenting link in S^3

from 

to 

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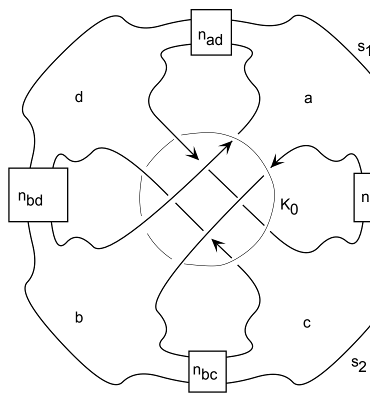
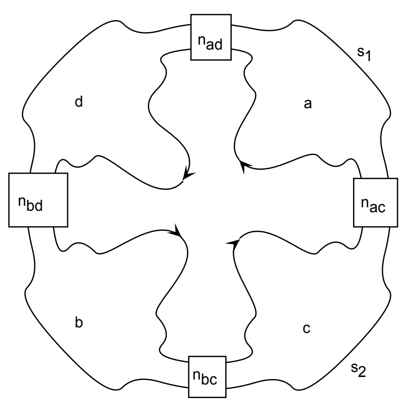
there is a computable formula which describes how the Casson–Walker invariant of surgery on the links changes.

Consider this general schematic diagram of a link before changing a band crossing:



In this diagram we see two components of the link. The first component, with the s_1 framing, encloses regions denoted a and b, and the second component, with s_2 framing, encloses regions c and d. The numbers in boxes indicate the linking numbers between lobes. Knottingness within components is omitted for clarity. The unchanging components are not included in the diagram, but have pairwise linking numbers and linking numbers with the four lobes drawn in this diagram. Also included in this diagram is the null-homologous knot K_0 which will be used in Walker's crossing change formula to effect the band change. This knot is not part of the link which is undergoing a band change.

To compute the band crossing formula, pinch each of the first two components into two lobes each:



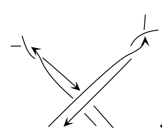
here labeled a and b in the first component and c and d in the second component. Then let n_{ac} indicate the linking number between lobes a and c and respectively for the other lobes. Note that a could link b and c could link d, but this linking does not appear in the final formula, so it seems unnecessary to include it in the diagram. Although they are not indicated in the diagram, for $i > 2$ let k^a_{1i} indicate the linking number between lobe a of the first component and all of the ith component.

Theorem 1. *When changing a framed link in S^3 as indicated, λ_w changes by*

$$\lambda_w(\text{band 2 over band 1}) - \lambda_w(\text{band 1 over band 2}) = 4\frac{\begin{vmatrix} n_{ac} & n_{ad} & k_{13}^a & \dots & k_{1n}^a \\ n_{bc} & n_{bd} & k_{13}^b & \dots & k_{1n}^b \\ k_{23}^c & k_{23}^d & s_3 & \dots & n_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{2n}^c & k_{2n}^d & n_{3n} & \dots & s_n \end{vmatrix}}{\begin{vmatrix} s_1 & n_{12} & \dots & n_{1n} \\ n_{12} & s_2 & \dots & n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ n_{1n} & n_{2n} & \dots & s_n \end{vmatrix}}.$$

For later notational purposes, let us denote the matrix whose determinant appears in the numerator of this formula as M_C. Note that this formula is independent of the orientation of any of the components. Changing the orientation of a component that is not involved in the band change results in changing the sign of one row and one column of both M_C and the framing matrix, and hence does not change the determinant. Changing the orientation of the first component changes the sign of two rows of M_C and does not change the determinant, and changing the orientation of the second component changes the sign of two columns of M_C and does not change the determinant.

Proof. We will effect this band change through a two step process. First we will compute the difference between $+1$ and $\frac{1}{0}$ surgery on K_0 as in the proof of the internal crossing change in [J1] in order to perform a right-hand twist by the Kirby–Rolfsen moves. While this procedure will exchange which band passes over, it will also introduce twists in each band like this: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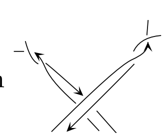
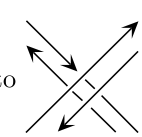
Conveniently, as these twists are internal to the components we may undo them using the internal crossing change formula. If we compute the effect of surgery around a band and then the effects of fixing the twists in the bands, we may put them all together to find how the Casson–Walker invariant changes when changing the entire band crossing.

Using the internal crossing change formula, it is apparent that changing the twisted bands

from  to 

results in a change of

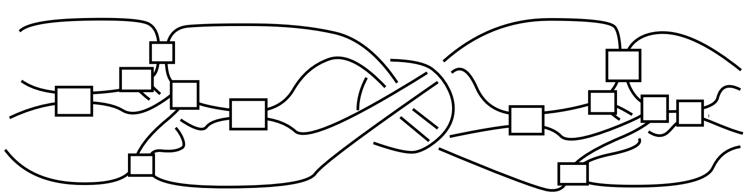
$$-2\frac{\begin{vmatrix} n_{ab} & n_{ac}+n_{ad} & k^a_{13} & \dots & k^a_{1n} \\ n_{bc}+n_{bd} & s_2 & n_{23} & \dots & n_{2n} \\ k^b_{13} & n_{23} & s_3 & \dots & n_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k^b_{1n} & n_{2n} & n_{3n} & \dots & s_n \end{vmatrix}}{\begin{vmatrix} s_1 & n_{12} & \dots & n_{1n} \\ n_{12} & s_2 & \dots & n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ n_{1n} & n_{2n} & \dots & s_n \end{vmatrix}}$$

from the change in the first component and

$$-2\frac{\begin{vmatrix} s_1 & n_{ad}+n_{bd} & n_{13} & \dots & n_{1n} \\ n_{ac}+n_{bc} & n_{cd} & k^c_{23} & \dots & k^c_{2n} \\ n_{13} & k^d_{23} & s_3 & \dots & n_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_{1n} & k^d_{2n} & n_{3n} & \dots & s_n \end{vmatrix}}{\begin{vmatrix} s_1 & n_{12} & \dots & n_{1n} \\ n_{12} & s_2 & \dots & n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ n_{1n} & n_{2n} & \dots & s_n \end{vmatrix}}$$

from the change in the second component (note: both of these expressions are negative because we are changing from a negative to a positive crossing). Let us denote the two matrices appearing in the numerators above to be M_4 and M_5, respectively.

This leaves the Dehn surgery. Just as we did in the internal crossing change formula, we will take the infinite cyclic cover of $S^3 \backslash K_0$ by opening up along a Seifert surface. This time we get this schematic of the cover:



Using this cover, we get the following presentation matrix for $H_1(S^3 \backslash \widetilde{\mathrm{im}(K_0)})$:

$$\begin{bmatrix} s_1+n_{ab}(t+t^{-1}-2) & n_{bc}t+n_{ad}t^{-1}+n_{ac}+n_{bd} & k^a_{13}+k^b_{13}t & \dots & k^a_{1n}+k^b_{1n}t \\ n_{bc}t^{-1}+n_{ad}t+n_{ac}+n_{bd} & s_2+n_{cd}(t+t^{-1}-2) & k^c_{23}+k^d_{23}t & \dots & k^c_{2n}+k^d_{2n}t \\ k^a_{13}+k^b_{13}t^{-1} & k^c_{23}+k^d_{23}t^{-1} & s_3 & \dots & n_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k^a_{1n}+k^b_{1n}t^{-1} & k^c_{2n}+k^d_{2n}t^{-1} & n_{3n} & \dots & s_n \end{bmatrix}.$$

The determinant of this matrix will be some Laurent polynomial of the form

$$v_2t^2 + v_1t + v_0 + v_{-1}t^{-1} + v_{-2}t^{-2}.$$

The polynomial can be seen to be symmetric (the matrix is nearly symmetric, with t exchanged for t^{-1} in the transpose), and so it is actually of the form

$$v_2t^2 + v_1t + v_0 + v_1t^{-1} + v_2t^{-2}.$$

At $t = 1$, the determinant is the determinant of the framing matrix A (note $n_{ac} + n_{bc} + n_{ad} + n_{bd} = n_{12}$).

Therefore

$$\begin{aligned}\Delta_{K_0}(t) &= \frac{1}{\det(A)}(v_2t^2 + v_1t + v_0 + v_1t^{-1} + v_2t^{-2}),\\ \frac{d^2}{dt^2}\Delta_{K_0}(t) &= \frac{1}{\det(A)}(2v_2 + 2v_1t^{-3} + 6v_2t^{-4})\end{aligned}$$

and hence $\frac{d^2}{dt^2}\Delta_{K_0}(1) = \frac{1}{\det(A)}(8v_2 + 2v_1)$, so we need the v_2 and v_1 coefficients from the determinant of the presentation matrix.

We can compute v_2 by taking all t coefficients in the top two rows, and constant coefficients in the remaining rows. So,

$$v_2 = \begin{vmatrix} n_{ab} & n_{bc} & k_{13}^b & \dots & k_{1n}^b \\ n_{ad} & n_{cd} & k_{23}^d & \dots & k_{1n}^d \\ k_{13}^a & k_{23}^c & s_3 & \dots & n_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{1n}^a & k_{2n}^c & n_{3n} & \dots & s_n \end{vmatrix} =: M_1.$$

Then v_1 can be produced by taking the t coefficient in the first row and constant coefficients in the remaining rows, then the t coefficient in the second row and the remainder constants coefficients again, and finally by taking t coefficients in the first two rows and successively taking the t^{-1} coefficient in each of the remaining rows.

$$\begin{aligned} v_1 = &\begin{vmatrix} n_{ab} & n_{bc} & k_{13}^b & \dots & k_{1n}^b \\ n_{ac} + n_{bd} & s_2 - 2n_{cd} & k_{23}^c & \dots & k_{2n}^c \\ k_{13}^a & k_{23}^c & s_3 & \dots & n_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{1n}^a & k_{2n}^c & n_{3n} & \dots & s_n \end{vmatrix} \\ &+ \begin{vmatrix} s_1 - 2n_{ab} & n_{ac} + n_{bd} & k_{13}^a & \dots & k_{1n}^a \\ n_{ad} & n_{cd} & k_{23}^d & \dots & n_{2n}^d \\ k_{13}^a & k_{23}^c & s_3 & \dots & n_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{1n}^a & k_{2n}^c & n_{3n} & \dots & s_n \end{vmatrix} + \end{aligned}$$

$$
+\begin{vmatrix} n_{ab} & n_{bc} & k_{13}^b & \dots & k_{1n}^b \\ n_{ad} & n_{cd} & k_{23}^d & \dots & k_{1n}^d \\ k_{13}^b & k_{23}^d & 0 & \dots & 0 \\ k_{14}^a & k_{24}^c & n_{34} & \dots & n_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{1n}^a & k_{2n}^c & n_{3n} & \dots & s_n \end{vmatrix} + \begin{vmatrix} n_{ab} & n_{bc} & k_{13}^b & \dots & k_{1n}^b \\ n_{ad} & n_{cd} & k_{23}^d & \dots & k_{1n}^d \\ k_{13}^a & k_{23}^c & s_3 & \dots & n_{3n} \\ k_{14}^b & k_{24}^d & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix} + \cdots .
$$

Note: For an n-component link, this sum has n terms (the band-change itself only makes sense for at least two components). Let us denote this sum of determinants of matrices, one-by-one, as

$$v_1 = \det(M_2) + \det(M_3) + \sum_{i=3}^{n} \det(M_{3i}).$$

Putting together what has been said so far, we now have that

$$
\begin{aligned}
\lambda_w(\text{band 1 over band 2}) - \lambda_w(\text{band 2 over band 1}) = \\
\frac{1}{\det(A)} \left(8\det(M_1) + 2\det(M_2) + 2\det(M_3)\right) \\
+ \frac{1}{\det(A)} \left(\left[\sum_{i=3}^{n} 2\det(M_{3i}) \right] - 2\det(M_4) - 2\det(M_5) \right).
\end{aligned}
$$

Hence we have *a* formula, but it is not a very helpful formula, as the sum of so many determinants. Fortunately, it may be simplified.

Claim 1.

$$
\begin{aligned}
\frac{4\det(M_C)}{\det(A)} &= \frac{1}{\det(A)} \left(8\det(M_1) + 2\det(M_2) + 2\det(M_3)\right) \\
&\quad + \frac{1}{\det(A)} \left(\left[\sum_{i=3}^{n} 2\det(M_{3i}) \right] - 2\det(M_4) - 2\det(M_5) \right).
\end{aligned}
$$

Given the previous statement, the theorem follows from this claim. After multiplying both sides by $\det(A)$ and dividing by 2, we can see that to prove the claim it suffices to show

$$
\begin{aligned}
2\det(M_C) &= 4\det(M_1) + \det(M_2) + \det(M_3) \\
&\quad + \sum_{i=3}^{n} \det(M_{3i}) - \det(M_4) - \det(M_5).
\end{aligned}
$$

To demonstrate this we will expand both sides out into polynomials in several variables and show that the terms simplify to be the same on each side. Let us denote the left side polynomial as P_L and the right side polynomial as P_R. That is, let

$$P_L = 2\det(M_C)$$

$$P_R = 4\det(M_1) + \det(M_2) + \det(M_3) + \sum_{i=3}^{n} \det(M_{3i}) - \det(M_4) - \det(M_5).$$

Let F be the framing matrix for the last $n-2$ components. Recall that this matrix is symmetric. Let the notation k_1^a be shorthand for the vector $\begin{bmatrix} k_{13}^a & k_{14}^a & \dots & k_{1n}^a \end{bmatrix}$, and likewise for similar symbols. Then

$$M_1 = \begin{bmatrix} n_{ab} & n_{bc} & k_1^b \\ n_{ad} & n_{cd} & k_2^d \\ k_1^a & k_2^c & F \end{bmatrix}$$

$$M_2 = \begin{bmatrix} n_{ab} & n_{bc} & k_1^b \\ n_{ac}+n_{bd} & s_2 - 2n_{cd} & k_2^c \\ k_1^a & k_2^c & F \end{bmatrix}$$

$$M_3 = \begin{bmatrix} s_1 - 2n_{ab} & n_{ac}+n_{bd} & k_1^a \\ n_{ad} & n_{cd} & k_2^d \\ k_1^a & k_2^c & F \end{bmatrix}.$$

Then $M_{3i} = M_1$ except row i is replaced by $\begin{bmatrix} k_{1i}^b & k_{2i}^d & 0 & 0 & \cdots & 0 \end{bmatrix}$.

$$M_4 = \begin{bmatrix} n_{ab} & n_{ac}+n_{ad} & k_1^a \\ n_{bc}+n_{bd} & s_2 & k_2^c + k_2^d \\ k_1^b & k_2^c + k_2^d & F \end{bmatrix}$$

$$M_5 = \begin{bmatrix} s_1 & n_{ad}+n_{bd} & k_1^a + k_1^b \\ n_{ac}+n_{bc} & n_{cd} & k_2^c \\ k_1^a + k_1^b & k_2^d & F \end{bmatrix}$$

and

$$M_C = \begin{bmatrix} n_{ac} & n_{ad} & k_1^a \\ n_{bc} & n_{bd} & k_1^b \\ k_2^c & k_2^d & F \end{bmatrix}.$$

Our approach will be to verify directly that $P_L = P_R$ for the terms containing each of the variables that appear in the upper left 2×2 submatrices. Then, as it holds for each of these variables, we will set them *all* equal to zero, then verify that $P_L = P_R$ for the remaining variables. We will check the following variables in the given order: s_1, s_2, n_{ac}, n_{bd}, n_{bc}, n_{ad}, n_{ab}, n_{cd}.

Subclaim 1. *The terms in P_L containing s_1 equal the terms in P_R containing s_1.*

Proof. Since s_1 does not appear in M_C, we must show the term cancels on the right. We get the following coefficients on s_1 from M_3 and M_5:

$$\begin{vmatrix} n_{cd} & k_2^d \\ k_2^c & F \end{vmatrix} - \begin{vmatrix} n_{cd} & k_2^c \\ k_2^d & F \end{vmatrix} = 0.$$

□

Subclaim 2. *The terms in P_L containing s_2 equal the terms in P_R containing s_2.*

Proof. In a similar fashion as s_1 we get coefficients on s_2 from M_2 and M_4:

$$\begin{vmatrix} n_{ab} & k_1^b \\ k_1^a & F \end{vmatrix} - \begin{vmatrix} n_{ab} & k_1^a \\ k_1^b & F \end{vmatrix} = 0.$$

□

Subclaim 3. *The terms in P_L containing n_{ac} equal the terms in P_R containing n_{ac}.*

Proof. As n_{ac} does appear in M_C, we must verify that the right side terms give the term from M_C. We get the following coefficients from M_2, M_3, M_4 and M_5 (taking the sign of each as determined by the formula and the position of n_{ac} in each matrix):

$$-\begin{vmatrix} n_{bc} & k_1^b \\ k_2^c & F \end{vmatrix} - \begin{vmatrix} n_{ad} & k_2^d \\ k_1^a & F \end{vmatrix} + \begin{vmatrix} n_{bc}+n_{bd} & k_2^c+k_2^d \\ k_1^b & F \end{vmatrix} + \begin{vmatrix} n_{ad}+n_{bd} & k_1^a+k_1^b \\ k_2^d & F \end{vmatrix}.$$

Taking transposes of M_3 and M_4 yields

$$-\begin{vmatrix} n_{bc} & k_1^b \\ k_2^c & F \end{vmatrix} - \begin{vmatrix} n_{ad} & k_1^a \\ k_2^d & F \end{vmatrix} + \begin{vmatrix} n_{bc}+n_{bd} & k_1^b \\ k_2^c+k_2^d & F \end{vmatrix} + \begin{vmatrix} n_{ad}+n_{bd} & k_1^a+k_1^b \\ k_2^d & F \end{vmatrix}.$$

And then using multilinearity on M_4 and M_5 we get

$$-\begin{vmatrix} n_{bc} & k_1^b \\ k_2^c & F \end{vmatrix} - \begin{vmatrix} n_{ad} & k_1^a \\ k_2^d & F \end{vmatrix} + \begin{vmatrix} n_{bc} & k_1^b \\ k_2^c & F \end{vmatrix} + \begin{vmatrix} n_{bd} & k_1^b \\ k_2^d & F \end{vmatrix} + \begin{vmatrix} n_{ad} & k_1^a \\ k_2^d & F \end{vmatrix} + \begin{vmatrix} n_{bd} & k_1^b \\ k_2^d & F \end{vmatrix}.$$

Cancelling and collecting, we have

$$2\begin{vmatrix} n_{bd} & k_1^b \\ k_2^d & F \end{vmatrix}$$

which is as desired for $2\det(M_C)$. □

Subclaim 4. *The terms in P_L containing n_{bd} equal the terms in P_R containing n_{bd}.*

Proof. The argument is entirely analogous to that for n_{ac}. □

Subclaim 5. *The terms in P_L containing n_{bc} equal the terms in P_R containing n_{bc}.*

Proof. This is more tedious than the previous cases, as n_{bc} appears in each of the matrices in the formula but one. Care and caution are required here. Take terms from M_1, M_2, M_4, M_5 and then all of the M_{3i}'s to get

$$-4\begin{vmatrix} n_{ad} & k_2^d \\ k_1^a & F \end{vmatrix} - \begin{vmatrix} n_{ac}+n_{bd} & k_2^c \\ k_1^a & F \end{vmatrix} + \begin{vmatrix} n_{ac}+n_{ad} & k_1^a \\ k_2^c+k_2^d & F \end{vmatrix}$$
$$+\begin{vmatrix} n_{ad}+n_{bd} & k_1^a+k_1^b \\ k_2^d & F \end{vmatrix} - \begin{vmatrix} 0 & k_2^d \\ k_{13}^b & 0 \\ 0 & F \end{vmatrix} - \begin{vmatrix} 0 & k_2^d \\ 0 & F \\ k_{14}^b & 0 \\ 0 & F \end{vmatrix} - \cdots - \begin{vmatrix} 0 & k_2^d \\ 0 & F \\ k_{1n}^b & 0 \end{vmatrix}.$$

Note for the M_{3i} terms we are using the fact that if we have a row of all zeroes except for one entry, we may assume without changing the determinant that the column with the nonzero row element has zeroes except for that element. Looking above we see that the M_{3i} terms collapse and we can expand M_2, M_4 and M_5 by multilinearity, transposing when useful, to get

$$-4\begin{vmatrix} n_{ad} & k_1^a \\ k_2^d & F \end{vmatrix} - \begin{vmatrix} n_{ac} & k_1^a \\ k_2^c & F \end{vmatrix} - \begin{vmatrix} n_{bd} & 0 \\ 0 & F \end{vmatrix} + \begin{vmatrix} n_{ac} & k_1^a \\ k_2^c & F \end{vmatrix}$$
$$+\begin{vmatrix} n_{ad} & k_1^a \\ k_2^d & F \end{vmatrix} + \begin{vmatrix} n_{ad} & k_1^a \\ k_2^d & F \end{vmatrix} + \begin{vmatrix} n_{bd} & k_1^b \\ k_2^d & F \end{vmatrix} - \begin{vmatrix} 0 & k_2^d \\ k_1^b & F \end{vmatrix}.$$

Cancelling and combining yields

$$-2\begin{vmatrix} n_{ad} & k_1^a \\ k_2^d & F \end{vmatrix} - \begin{vmatrix} n_{bd} & 0 \\ 0 & F \end{vmatrix} + \begin{vmatrix} n_{bd} & k_1^b \\ k_2^d & F \end{vmatrix} - \begin{vmatrix} 0 & k_1^b \\ k_2^d & F \end{vmatrix}$$

and then the last three matrices cancel together to leave us with

$$-2\begin{vmatrix} n_{ad} & k_1^a \\ k_2^d & F \end{vmatrix}$$

as desired for M_C. □

Subclaim 6. *The terms in P_L containing n_{ad} equal the terms in P_R containing n_{ad}.*

Proof. An analogous argument to the previous subclaim proves the n_{ad} coefficients to match up as well. □

Subclaim 7. *The terms in P_L containing n_{ab} equal the terms in P_R containing n_{ab}.*

Proof. In fact, there are no terms involving n_{ab} in either P_L or P_R. This is somewhat surprising as the related variable in the internal crossing change formula, l, plays a prominent role. Here we will see how it cancels out of P_R. As in the preceding verification, n_{ab} appears in each matrix but one. In order to simplify notation, recall that the M_{3i} terms will collapse, so we will write them that way from the beginning. So, taking terms from M_1, M_2, M_3, M_4, and finally from the M_{3i}'s we have as our n_{ab} coefficient

$$4\begin{vmatrix} n_{cd} & k_2^d \\ k_2^c & F \end{vmatrix} + \begin{vmatrix} s_2 - 2n_{cd} & k_2^c \\ k_2^c & F \end{vmatrix} - 2\begin{vmatrix} n_{cd} & k_2^d \\ k_2^c & F \end{vmatrix} - \begin{vmatrix} s_2 & k_2^c + k_2^d \\ k_2^c + k_2^d & F \end{vmatrix} + \begin{vmatrix} 0 & k_2^d \\ k_2^d & F \end{vmatrix}.$$

Note: the -2 coefficient on the third matrix comes from the coefficient on n_{ab} in M_3. Now, combining and canceling, using multilinearity on M_2 and twice on M_4 produces

$$2\begin{vmatrix} n_{cd} & k_2^d \\ k_2^c & F \end{vmatrix} + \begin{vmatrix} s_2 & k_2^c \\ k_2^c & F \end{vmatrix} - 2\begin{vmatrix} n_{cd} & 0 \\ 0 & F \end{vmatrix} - \begin{vmatrix} s_2 & k_2^c \\ k_2^c & F \end{vmatrix}$$
$$- \begin{vmatrix} 0 & k_2^c \\ k_2^d & F \end{vmatrix} - \begin{vmatrix} 0 & k_2^d \\ k_2^c & F \end{vmatrix} - \begin{vmatrix} 0 & k_2^d \\ k_2^d & F \end{vmatrix} + \begin{vmatrix} 0 & k_2^d \\ k_2^d & F \end{vmatrix}.$$

And some more combining and canceling gives us finally

$$2\begin{vmatrix} n_{cd} & k_2^d \\ k_2^c & F \end{vmatrix} - 2\begin{vmatrix} n_{cd} & 0 \\ 0 & F \end{vmatrix} - 2\begin{vmatrix} 0 & k_2^d \\ k_2^c & F \end{vmatrix} = 0.$$

□

Subclaim 8. *The terms in P_L containing n_{cd} equal the terms in P_R containing n_{cd}.*

Proof. The reasoning for n_{cd} is analogous to that for n_{ab}. □

We set each of these upper left variables equal to zero in P_L and P_R, and go from there.

Subclaim 9. *The terms in P_L containing none of the following variables: s_1, s_2, n_{ac}, n_{bd}, n_{bc}, n_{ad}, n_{ab}, n_{cd} equal the terms in P_R omitting the same variables.*

To see this, let us work to simplify what remains of P_R. Note: row i of M_{3i} can be seen as $[k^b_{1i}+0 \quad 0+k^d_{2i} \quad 0 \quad \dots \quad 0]$ and thus we can envision $\det(M_{3i})$ as

$$\begin{vmatrix} 0 & 0 & M_1 \\ M_1 & 0 & M_1 \\ 0 & k^d_{2i} & 0 \\ M_1 & 0 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & M_1 \\ 0 & M_1 & M_1 \\ k^b_{1i} & 0 & 0 \\ 0 & M_1 & F \end{vmatrix}$$

where the M_1 entries indicate blocks from the M_1 matrix. If we then take the sum over all i, we can encapsulate all the M_{3i} terms as

$$\begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^a_1 & k^d_2 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^b_1 & k^c_2 & F \end{vmatrix}.$$

Setting the variables to zero, and making this substition for the M_{3i} terms we are left with

$$\begin{aligned} P_R = 4&\begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^a_1 & k^c_2 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^c_2 \\ k^a_1 & k^c_2 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^d_2 \\ k^a_1 & k^c_2 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^a_1 & k^d_2 & F \end{vmatrix} \\ &+ \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^b_1 & k^c_2 & F \end{vmatrix} - \begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^c_2+k^d_2 \\ k^b_1 & k^c_2+k^d_2 & F \end{vmatrix} - \begin{vmatrix} 0 & 0 & k^a_1+k^b_1 \\ 0 & 0 & k^c_2 \\ k^a_1+k^b_1 & k^d_2 & F \end{vmatrix}. \end{aligned}$$

Via multilinearity twice on both M_4 and M_5 we have

$$\begin{aligned} P_R = \\ 4&\begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^a_1 & k^c_2 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^c_2 \\ k^a_1 & k^c_2 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^d_2 \\ k^a_1 & k^c_2 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^a_1 & k^d_2 & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^b_1 & k^c_2 & F \end{vmatrix} \\ -&\begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^c_2 \\ k^b_1 & k^c_2 & F \end{vmatrix} - \begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^d_2 \\ k^b_1 & k^c_2 & F \end{vmatrix} - \begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^c_2 \\ k^b_1 & k^d_2 & F \end{vmatrix} - \begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^d_2 \\ k^b_1 & k^d_2 & F \end{vmatrix} \\ -&\begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^c_2 \\ k^a_1 & k^d_2 & F \end{vmatrix} - \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^c_2 \\ k^a_1 & k^d_2 & F \end{vmatrix} - \begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^c_2 \\ k^b_1 & k^d_2 & F \end{vmatrix} - \begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^c_2 \\ k^b_1 & k^d_2 & F \end{vmatrix}. \end{aligned}$$

Combining and canceling once again we see

$$P_R = 2\begin{vmatrix} 0 & 0 & k^b_1 \\ 0 & 0 & k^d_2 \\ k^a_1 & k^c_2 & F \end{vmatrix} - 2\begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^d_2 \\ k^b_1 & k^c_2 & F \end{vmatrix}.$$

Let us recall that after setting variables to zero

$$P_L = 2\begin{vmatrix} 0 & 0 & k^a_1 \\ 0 & 0 & k^b_1 \\ k^c_2 & k^d_2 & F \end{vmatrix}.$$

We are much closer than when we entered this algebraic morass. Now for the *coup de grace*: if we rewrite the asserted equality $P_L = P_R$, cancel the twos, move all the determinants to one side, transpose and exchange two rows in the matrix for

P_L, we see that the subclaim, claim and theorem will all be established if we can demonstrate the following:

Lemma 1. *For any symmetric $m \times m$ matrix F and any m-vectors k_1^a, k_1^b, k_2^c, and k_2^d,*

$$\begin{vmatrix} 0 & 0 & k_1^a \\ 0 & 0 & k_2^c \\ k_2^d & k_1^b & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k_1^a \\ 0 & 0 & k_2^d \\ k_1^b & k_2^c & F \end{vmatrix} + \begin{vmatrix} 0 & 0 & k_1^a \\ 0 & 0 & k_1^b \\ k_2^c & k_2^d & F \end{vmatrix} = 0.$$

Proof. Since F is symmetric, we can diagonalize F by conjugation with an orthogonal matrix P. Call D the diagonalization of F, and let λ_i be the eigenvalues of F. Conjugate each of the above matrices by $\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$; this preserves determinants and yields:

$$\begin{vmatrix} 0 & 0 & \widetilde{a} \\ 0 & 0 & \widetilde{c} \\ \widetilde{d}^t & \widetilde{b}^t & D \end{vmatrix} + \begin{vmatrix} 0 & 0 & \widetilde{a} \\ 0 & 0 & \widetilde{d} \\ \widetilde{b}^t & \widetilde{c}^t & D \end{vmatrix} + \begin{vmatrix} 0 & 0 & \widetilde{a} \\ 0 & 0 & \widetilde{b} \\ \widetilde{c}^t & \widetilde{d}^t & D \end{vmatrix}$$

where $\widetilde{a}$ is $(k_1^a)P^{-1}$, and likewise for the others.

We take the Laplace expansion of each determinant by the first two rows. This produces terms like

$$\left(\begin{vmatrix} \widetilde{a}_1 & \widetilde{a}_2 \\ \widetilde{c}_1 & \widetilde{c}_2 \end{vmatrix} \begin{vmatrix} \widetilde{d}_1 & \widetilde{b}_1 \\ \widetilde{d}_2 & \widetilde{b}_2 \end{vmatrix} + \begin{vmatrix} \widetilde{a}_1 & \widetilde{a}_2 \\ \widetilde{d}_1 & \widetilde{d}_2 \end{vmatrix} \begin{vmatrix} \widetilde{b}_1 & \widetilde{c}_1 \\ \widetilde{b}_2 & \widetilde{c}_2 \end{vmatrix} + \begin{vmatrix} \widetilde{a}_1 & \widetilde{a}_2 \\ \widetilde{b}_1 & \widetilde{b}_2 \end{vmatrix} \begin{vmatrix} \widetilde{c}_1 & \widetilde{d}_1 \\ \widetilde{c}_2 & \widetilde{d}_2 \end{vmatrix} \right) \lambda_3 \dots \lambda_{n-2}$$

and so on for the other combinations of λ_i's.

We can view this as a polynomial in the λ_i's and examine the coefficients. The coefficient of $\frac{\lambda_1 \dots \lambda_{n-2}}{\lambda_i \lambda_j}$ will then be

$$\pm \left(\begin{vmatrix} \widetilde{a}_i & \widetilde{a}_j \\ \widetilde{c}_i & \widetilde{c}_j \end{vmatrix} \begin{vmatrix} \widetilde{d}_i & \widetilde{b}_i \\ \widetilde{d}_j & \widetilde{b}_j \end{vmatrix} + \begin{vmatrix} \widetilde{a}_i & \widetilde{a}_j \\ \widetilde{d}_i & \widetilde{d}_j \end{vmatrix} \begin{vmatrix} \widetilde{b}_i & \widetilde{c}_i \\ \widetilde{b}_j & \widetilde{c}_j \end{vmatrix} + \begin{vmatrix} \widetilde{a}_i & \widetilde{a}_j \\ \widetilde{b}_i & \widetilde{b}_j \end{vmatrix} \begin{vmatrix} \widetilde{c}_i & \widetilde{d}_i \\ \widetilde{c}_j & \widetilde{d}_j \end{vmatrix} \right).$$

If we let $\hat{a} = \begin{bmatrix} \widetilde{a}_i & \widetilde{a}_j & 0 \end{bmatrix}$ and likewise for b, c and d, then

$$\hat{a} \times \hat{b} = \begin{bmatrix} 0 & 0 & \begin{vmatrix} \widetilde{a}_i & \widetilde{a}_j \\ \widetilde{b}_i & \widetilde{b}_j \end{vmatrix} \end{bmatrix}$$

and so on. What we have here can be interpreted as

$$(\hat{a} \times \hat{c}) \cdot (\hat{d} \times \hat{b}) + (\hat{a} \times \hat{d}) \cdot (\hat{b} \times \hat{c}) + (\hat{a} \times \hat{b}) \cdot (\hat{c} \times \hat{d}).$$

Note: $(r \times s) \cdot [t \times u] = r \cdot (s \times [t \times u])$ as the triple product is cyclic and the dot product is commutative. The above expression then equals

$$\begin{aligned} &\hat{a} \cdot (\hat{c} \times (\hat{d} \times \hat{b})) + \hat{a} \cdot (\hat{d} \times (\hat{b} \times \hat{c})) + \hat{a} \cdot (\hat{b} \times (\hat{c} \times \hat{d})) \\ &= \hat{a} \cdot [(\hat{c} \times (\hat{d} \times \hat{b})) + (\hat{d} \times (\hat{b} \times \hat{c})) + (\hat{b} \times (\hat{c} \times \hat{d}))]. \end{aligned}$$

The expression in the brackets is zero by the Jacobi identity for $\times$ on $\mathbb{R}^3$. So the entire expression equals zero, which concludes the proof of the lemma. □

Also we have proven the subclaim, the claim, and the theorem, i.e., the band crossing change formula. □

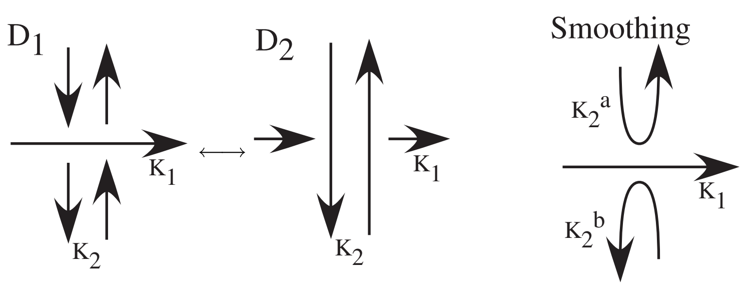
A consequence of [J2] is that this band crossing change formula holds for Lescop's invariant of abitrary 3-manifolds.

In [J1] it is demonstrated that $\beta(\times \times \times) = 0$, i.e., that β is type 2 within the link homotopy class of the base link. The analogous result here is that $\beta($ two band changes $) = 0$ because the self linking numbers (n_{ab} and n_{cd} canceled out of the formula for β).

5. Computing the invariant β

In [J1] β is defined as a polynomial on the link homotopy equivalence class of a base link $\mathbf{L}^0$ as $\beta(\mathbf{L}, \mathbf{s}) = \frac{det(A)}{2}(\lambda_w(\mathbf{L}, \mathbf{s}) - \lambda_w(\mathbf{L}^0, \mathbf{s}))$. The argument in [J2] demonstrates that in fact this invariant equivalently defined in terms of Lescop's invariant is polynomial for the link homology equivalence class of the base link. In this paper and [J1] we have presented geometrically computable formulas involving only linking numbers of components and smoothed pieces for links surgery equivalent to the base link.

This suggests the natural question of finding a formula for computing links that are link homologous but not surgery equivalent. Along with link homotopy the following move generates link homology classes [A]:



This particular move does not admit a surgery formula in terms of linking numbers as this would imply a similar formula for the z^{n+1} coefficient of the Conway polynomial (here denoted a_1) and no such is possible. This is impossible by the following argument due to J. Hoste. Suppose that

$$a_1(D_1) - a_1(D_2) = f(\text{linking numbers among}\{K_1, K_2^a, K_2^b, K_3, K_4, \dots\}).$$

Rotating each diagram about a vertical axis will not change a_1. Neither will changing the orientations of all the crossings. Therefore, $a_1(D_1) - a_1(D_2) = a_1(D_1') - a_1(D_2') = a_1(D_2) - a_1(D_1)$. Now locally the roles have changed, but the linking numbers remain the same, so we may compute this quantity as $-f$(linking numbers among $\{K_1, K_2^a, K_2^b, K_3, K_4, \dots\}$). So either this difference is identically zero or we have a contradiction. This difference is not identically zero because the Borromean rings have $a_1 = 1$ and can be changed via this move into the unlink which has $a_1 = 0$. Therefore it is impossible to compute this difference solely in terms of linking numbers of lobes determined at this crossing.

This leaves us with the question: is there a local difference between $\mathbf{L}^1$ and $\mathbf{L}^2$ such that this move along with link homotopy and pass moves would generate link homology equivalence such that $\lambda(\mathbf{L}^1_\mathbf{s}) - \lambda(\mathbf{L}^2_\mathbf{s})$ is given by a nice formula in terms of linking numbers of components and smoothed pieces? If so, then $\beta(\mathbf{L}, \mathbf{s})$

is geometrically computable if $\mathbf{L}$ is link homologous to $\mathbf{L}^0$, which is the most we can hope for, as β is not defined more broadly.

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