New York Journal of Mathematics

New York J. Math. **10** (2004) 117–131.

Derangements and asymptotics of the Laplace transforms of large powers of a polynomial

Liviu I. Nicolaescu

Abstract. We use a probabilistic approach to produce sharp asymptotic estimates as $n \to \infty$ for the Laplace transform of P^n, where P is a fixed complex polynomial. As a consequence we obtain a new elementary proof of a result of Askey-Gillis-Ismail-Offer-Rashed, [1, 3] in the combinatorial theory of derangements.

Contents

1. Statement of the main results

The generalized derangement problem in combinatorics can be formulated as follows. Suppose X is a finite set and $\sim$ is an equivalence relation on X. For each $x \in X$ we denote by $\hat{x}$ the equivalence class of x. $\hat{X}_\sim$ will denote the set of equivalence classes. The counting function of $\sim$ is the function

$$\nu = \nu_\sim : \hat{X} \longrightarrow \mathbb{Z}, \;\; \nu(\hat{x}) = |\hat{x}| = \text{ the cardinality of } \hat{x}.$$

A $\sim$-*derangement* of x is a permutation $\varphi : X \longrightarrow X$ such that

$$x \notin \hat{x}, \;\; \forall x \in X.$$

Received February 10, 2004.

Mathematics Subject Classification. 44A10, 05A05, 05A10, 05A16, 41A60, 33C45.

Key words and phrases. derangements, Laplace transforms, asymptotics, multinomial distributions.

This work was partially suported by the NSF grant DMS-0303601.

ISSN 1076-9803/04

We denote by $\mathcal{N}(X,\sim)$ the number of $\sim$-derangements. The ratio

$$p(X,\sim) = \frac{\mathcal{N}(X,\sim)}{|X|!}$$

is the probability that a randomly chosen permutation of X is a derangement.

In [2] S. Even and J. Gillis have described a beautiful relationship between these numbers and the Laguerre polynomials

$$L_n(x) = e^x \frac{d^n}{dx^n}\Bigl(e^{-x}x^n\Bigr) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{k!},\quad n = 0, 1, \dots .$$

For example

$$L_0(x) = 1,\quad L_1(x) = 1 - x,\quad L_2(x) = \frac{1}{2!}(x^2 - 4x + 2).$$

We set

$$L_\sim := \prod_{c\in\hat{X}} (-1)^{\nu(c)}\nu(c)!\,L_{\nu(c)}(x).$$

Observe that the leading coefficient of $L_\sim$ is 1. We have the following result.

Theorem 1.1 (Even-Gillis)**.**

$$\mathcal{N}(X,\sim) = \int_0^\infty e^{-x}L_\sim(x)dx. \tag{1.1}$$

For several very elegant short proofs we refer to [1, 4].

Given $(X,\sim)$ as above and n a positive integer we define $(X_n,\sim_n)$ to be the disjoint union of n-copies of X

$$X_n = \bigcup_{k=1}^{n} X\times\{k\}$$

equipped with the equivalence relation

$$(x,j)\sim_n (y,k) \Longleftrightarrow j = k,\quad x\sim y.$$

We deduce

$$p(X_n,\sim_n) = \frac{1}{(n|X|)!}\int_0^\infty e^{-x}\bigl(L_\sim(x)\bigr)^n dx. \tag{1.2}$$

For example, consider the “marriage relation”

$$(C,\sim),\quad C = \{\pm 1\},\quad -1\sim 1.$$

In this case $\hat{C}$ consists of a single element and the counting function is the number $\nu = 2$. Then $(C_n,\sim_n)$ can be interpreted as a group of n married couples. If we set

$$\delta_n := p(C_n,\sim_n)$$

then we can give the following amusing interpretation for δ_n.

Couples mixing problem. *At a party attended by n couples, the guests were asked to put their names in a hat and then to select at random one name from that pile. Then the probability that nobody will select his/her name or his/her spouse's name is equal to δ_n.*

Using (1.2) we deduce

$$\delta_n = \frac{1}{(2n)!}\int_0^\infty e^{-x}\left(x^2 - 4x + 2\right)^n dx. \tag{1.3}$$

We can ask about the asymptotic behavior of the probabilities $p(X_n, \sim_n)$ as $n \to \infty$. In [1, 3], Askey-Gillis-Ismail-Offer-Rashed describe the first terms of an asymptotic expansion in powers of n^{-1}. To formulate their result let us introduce the "momenta"

$$\nu_r = \sum_{c \in \hat{X}} \nu(c)^r.$$

Theorem 1.2 (Askey-Gillis-Ismail-Offer-Rashed)**.**

(1.4)
$$p(X_n, \sim_n) = \exp\left(-\frac{\nu_2}{\nu_1}\right)\left(1 - \frac{\nu_1(2\nu_3 - \nu_2) - \nu_2^2}{2\nu_1^3} n^{-1} + O(n^{-2})\right) \quad \textit{as } n \to \infty.$$

For example we deduce from the above that

$$\delta_n = e^{-2}\left(1 - \frac{1}{2}n^{-1} + O(n^{-2})\right), \quad n \to \infty. \tag{1.5}$$

The proof in [3] of the asymptotic expansion (1.4) is based on the saddle point technique applied to the integrals in the RHS of (1.2) and special properties of the Laguerre polynomials. The proof in [1] is elementary but yields a result less precise than (1.4).

In this paper we will investigate the large n asymptotics of Laplace transforms

$$\mathcal{F}_n(\mathcal{Q}, z) = \frac{z^{dn+1}}{(dn)!}\int_0^\infty e^{-zt}\mathcal{Q}(t)^n dt, \quad \mathfrak{Re}\, z > 0, \tag{1.6}$$

where $\mathcal{Q}(t)$ is a degree d complex polynomial with leading coefficient 1. If we denote by $\mathcal{L}[f(t), z]$ the Laplace transform of $f(t)$

$$\mathcal{L}[f(t), z] = \int_0^\infty e^{-zt} f(t) dt$$

then

$$\mathcal{F}_n(\mathcal{Q}, z) = \frac{\mathcal{L}[\mathcal{Q}(t)^n, z]}{\mathcal{L}[t^{dn}, z]}.$$

The estimate (1.4) will follow from our results by setting

$$z = 1, \quad \mathcal{Q} = L_\sim.$$

To formulate the main result we first write $\mathcal{Q}$ as a product

$$\mathcal{Q}(t) = \prod_{i=1}^d (t + r_i).$$

We set

$$\vec{r} = (r_1, \dots, r_d) \in \mathbb{C}^d, \quad \mu_s = \mu_s(\vec{r}) = \frac{1}{d}\sum_{i=1}^d r_i^s.$$

Theorem 1.3 (Existence theorem)**.** *For every* $\mathfrak{Re}\, z > 0$ *we have an asymptotic expansion as* $n \to \infty$

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{k=0}^{\infty} A_k(z) n^{-k}. \tag{1.7}$$

Above, the term $A_k(z)$ *is a holomorphic function on* $\mathbb{C}$ *whose coefficients are universal elements in the ring of polynomials* $\mathbb{C}(d)[\mu_1, \mu_2, \dots, \mu_k]$*, where* $\mathbb{C}(d)$ *denotes the field of rational functions in the variable* $d = \deg \mathcal{Q}$*.*

The proof of this theorem is given in the second section of this paper and it is probabilistic in flavor. In the third section we compute the terms A_k in some cases. For example we have

$$A_0(z) = e^{\mu_1 z}, \quad A_1(z) = \frac{1}{2d} e^{\mu_1 z}(\mu_1^2 - \mu_2) z^2, \tag{1.8}$$

and we can refine (1.5) to

$$\delta_n = e^{-2}\left(1 - \frac{1}{2}n^{-1} - \frac{23}{96}n^{-2} + O(n^{-3})\right), \quad n \to \infty. \tag{1.9}$$

These computations will lead to a proof of the following result.

Theorem 1.4 (Structure theorem)**.** *For any* k *and any degree* d *we have*

$$A_k(z) = e^{\mu_1 z} B_k(z),$$

where $B_k \in \mathbb{C}(d)[\mu_1, \dots, \mu_k][z]$ *is a universal polynomial in* z *with coefficients in* $\mathbb{C}(d)[\mu_1, \dots, \mu_k]$*.*

The formulæ (1.8) have an immediate curious consequence which was mentioned as an open question in [3].

Corollary 1.5. *Suppose* $P(t) = t^d + at^{d-1} + \cdots$ *is a degree* d *polynomial with real coefficients. Then*

$$\int_0^\infty e^{-t} P(t)^n dt > 0, \quad \forall n \gg 0.$$

Notations. A d*-dimensional (multi)index* will be a vector $\vec{\alpha} \in \mathbb{Z}^d_{\geq 0}$. For every vector $\vec{x} \in \mathbb{C}^d$ and any d-dimensional index $\vec{\alpha}$ we define

$$\vec{x}^{\,\vec{\alpha}} = x_1^{\alpha_1} \dots x_d^{\alpha_d}, \quad |\vec{\alpha}| = \alpha_1 + \cdots + \alpha_d, \quad S(\vec{x}) = x_1 + \cdots + x_d.$$

If $n = |\vec{\alpha}|$ then we define the multinomial coefficient

$$\binom{n}{\vec{\alpha}} := \frac{n!}{\prod_{i=1}^d \alpha_i!}.$$

These numbers appear in the *multinomial formula*

$$S(\vec{x})^n = \sum_{|\vec{\alpha}|=n} \binom{n}{\vec{\alpha}} \vec{x}^{\,\vec{\alpha}}.$$

Acknowledgments I want to thank Adam Boocher, a high school student attending the Math Club I was organizing, for asking me if I know how to solve the Couples Mixing Problem. The present paper grew out of my attempts to answer his question. I also want to thank the referee for making available to me the hard to get reference [1].

2. Proof of the existence theorem

The key to our approach is the following elementary result.

Lemma 2.1. *If $P(x) = p_m t^m + \cdots + p_1 t + p_0$ is a degree m with complex coefficients then for every $\mathfrak{Re}\, z > 1$ we have*

$$\frac{\mathcal{L}[P(t), z]}{\mathcal{L}[t^m, z]} = \frac{z^{m+1}}{m!}\int_0^\infty e^{-zt}P(t)dt = \sum_{a+b=m} \frac{p_a}{\binom{m}{a}}\frac{z^b}{b!}. \tag{2.1}$$

Proof.

$$\begin{aligned}\frac{z^{m+1}}{m!}\int_0^\infty e^{-zt}P(t)dt &= \frac{z^{m+1}}{m!}\sum_{a=0}^m p_a \int_0^\infty e^{-zt}t^a dt\\ &= \frac{z^{m+1}}{m!}\sum_{a=0}^m p_a \frac{a!}{z^{a+1}} = \sum_{a+b=m}\frac{p_a}{\binom{m}{a}}\frac{z^b}{b!}.\end{aligned}$$

□

Denote by $\mathcal{Q}(n,a)$ the coefficient of t^a in $\mathcal{Q}(t)^n$. From (2.1) we deduce

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn}\frac{\mathcal{Q}(n,a)}{\binom{dn}{a}}\frac{z^b}{b!}. \tag{2.2}$$

Using the equality

$$\mathcal{Q}^n = \prod_{i=1}^d \underbrace{\left(\sum_{j+k=n}^n \binom{n}{i} t^j r_i^k\right)}_{(t+r_i)^n}$$

we deduce that if $a+b = dn$ then

$$\mathcal{Q}(n,a) = \sum_{|\vec{\alpha}|=b}\left(\prod_{i=1}^d \binom{n}{\alpha_j}\right)\vec{r}^{\alpha}. \tag{2.3}$$

For $|\vec{\alpha}| = b$ we set

$$B(n,\vec{\alpha}) := \prod_{i=1}^d \binom{n}{\alpha_j},\ \ P_{n,b}(\vec{\alpha}) := \frac{B(n,\vec{\alpha})}{\binom{dn}{b}},\ \ \rho_b(\vec{\alpha}) = \vec{r}^{\vec{\alpha}},$$

so that

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn}\left(\sum_{|\vec{\alpha}|=b} P_{n,b}(\vec{\alpha})\rho_b(\vec{\alpha})\right)\cdot\frac{z^b}{b!}. \tag{2.4}$$

Observe that we have

$$P_{n,b}(\vec{\alpha}) = \frac{\prod_{i=1}^d (1-\frac{1}{n})\cdots(1-\frac{\alpha_i-1}{n})}{\prod_{k=1}^{b-1}(1-\frac{k}{dn})}\cdot\underbrace{\frac{1}{d^b}\binom{b}{\vec{\alpha}}}_{:=P_b(\vec{\alpha})}. \tag{2.5}$$

The coefficients $P_b(\vec{\alpha})$ define the multinomial probability distribution P_b on the set of multiindices

$$\Lambda_b = \Big\{ \vec{\alpha} \in \mathbb{Z}^b_{\geq 0};\ \ |\vec{\alpha}| = b \Big\}.$$

For every random variable ζ on Λ_b we denote by $E_b(\zeta)$ its expectation with respect to the probability distribution P_b. For each n we have a random variable $\zeta_{n,b}$ on Λ_b defined by

$$\zeta_{n,b}(\vec{\alpha}) = \frac{\prod_{i=1}^d (1-\frac{1}{n})\cdots(1-\frac{\alpha_i-1}{n})}{\prod_{k=1}^{b-1}(1-\frac{k}{dn})}\rho_b(\vec{\alpha}).$$

Form (2.4) and (2.5) we deduce

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn} E_b(\zeta_{n,b})\frac{z^b}{b!}. \tag{2.6}$$

To find the asymptotic expansion for $\mathcal{F}_n$ we will find asymptotic expansions in powers of n^{-1} for the expectations $E_b(\zeta_{n,b})$ and them add them up using (2.6).

For every nonnegative integer α we define a polynomial

$$W_\alpha(x) = \begin{cases} 1 & \text{if } \alpha = 0,1 \\ \prod_{j=1}^{\alpha-1}(1-jx) & \text{if } \alpha > 1. \end{cases}$$

For a d-dimensional multiindex $\vec{\alpha}$ we set

$$W_{\vec{\alpha}}(x) = \prod_{i=1}^d W_{\alpha_i}(x).$$

We can now rewrite (2.5) as

$$P_{n,b}(\vec{\alpha}) = P_b(\vec{\alpha})\frac{W_{\vec{\alpha}}(\frac{1}{n})}{W_b(\frac{1}{dn})}.$$

We set

$$R_b(\vec{\alpha}, x) = W_{\vec{\alpha}}(x),\ \ K_b(\vec{\alpha}, x) = \frac{1}{W_b(\frac{x}{d})}R_b(\vec{\alpha}, x)\rho_b(\alpha).$$

We regard the correspondences

$$\vec{\alpha} \mapsto R_b(\vec{\alpha}, x),\ K_b(\vec{\alpha}, x)$$

as random variables $R_b(x)$ and $K_b(x)$ on Λ_b valued in the field of rational functions. We deduce

$$\zeta_{n,b} = K_b(n^{-1}).$$

Observe

$$E_b(x) = E_b(K_b(x)) = \frac{1}{W_b(x)}E_b(R_b(x)).$$

From the fundamental theorem of symmetric polynomials we deduce that the expectations $E_b(R_b(x))$ are *universal* polynomials

$$E_b(R_b(x)) \in \mathbb{C}[\mu_1, \dots, \mu_b][x],\ \ \deg_x E_b(R_b(x)) \leq b-d,$$

whose coefficients have degree b in the variables μ_i, $\deg\mu_i = i$. We deduce that $E_b(x)$ has a Taylor series expansion

$$E_b(x) = \sum_{m\geq 0} E_b(m)x^m$$

such that $E_b(m) \in \mathbb{C}(d)[\mu_1, \dots, \mu_b]$. The rational function $x \to K_b(\vec{\alpha}, x)$ has a Taylor expansion at $x = 0$ convergent for $|x| < \frac{d}{b-1}$ so the above series converges for $|x| < \frac{d}{b-1}$. We would like to estimate the size of the coefficients $E_b(m)$. The tricky part is that the radius of convergence of $E_b(x)$ goes to zero as $b \to \infty$.

Lemma 2.2. *Set*

$$R = \max_{1\leq i\leq d} |r_i|.$$

There exists a constant C which depends only on R and d such that for every $b \geq 0$ and every $1 \leq \lambda_b < \frac{b}{b-1}$ we have the inequality

$$|E_b(m)| \leq \left(\frac{b}{\lambda_b d}\right)^m C^b \frac{b^{b-1}}{(b-2)!\left(1-\lambda_b\frac{b-1}{b}\right)}. \tag{2.7}$$

Proof. Note first that

$$|\rho_b(\vec{\alpha})| \leq R^b, \ \ \forall |\vec{\alpha}| = b.$$

For $b = 0, 1$ we deduce form the definition of the polynomials W_α that $E_b(x) = 1$. Fix m and $b > 1$. Using the Cauchy residue formula we deduce

$$E_b(m) = \frac{1}{2\pi\sqrt{-1}} \int_{|x|=\hbar} \frac{1}{x^{m+1}} E_b(x)dx, \quad \hbar = \lambda_b \cdot \frac{d}{b}.$$

Hence

$$|E_b(m)| \leq \frac{1}{\hbar^m} \sup_{|x|=\hbar} |E_b(x)| \leq \frac{b^m R^b}{(\lambda_b d)^m \min_{|x|=\hbar} |W_b(x/d)|} \cdot \max_{|x|=\hbar} E_b(R_b(x)).$$

Next observe that

$$W_b(x/d) = (b-1)! \prod_{k=1}^{b-1} \left(\frac{1}{k} - x/d\right), \quad \hbar/d < 1/k, \forall k \leq b-1,$$

from which we conclude

$$\begin{aligned}\min_{|x|=\hbar} |W_b(x)| = W_b(\hbar) = \prod_{k=1}^{b-1}\left(1 - \frac{k\lambda_b}{b}\right) &= \frac{1}{b^{b-1}} \prod_{k=1}^{b-1}(b - k\lambda_b) \\ &\geq \frac{(b-2)!(1-\lambda_b\frac{b-1}{b})}{b^{b-1}}.\end{aligned}$$

To estimate $E_b(R_b(x))$ from above observe that for every $1 \leq k \leq (b-1)$ and $|x| = \hbar$ we have

$$|1 - kx| \leq 1 + k|x| = 1 + \frac{k\lambda_b d}{b} < 1 + d.$$

This shows that for every $|\vec{\alpha}| = b$ and $|x| = \hbar$ we have

$$|R_b(\vec{\alpha}, x)| < (1 + d)^b.$$

The lemma follows by assembling all the facts established above. $\square$

Define the formal power series

$$A_m(z) := \sum_{b\geq 0} E_b(m)\frac{z^b}{b!} \in \mathbb{C}[[z]].$$

The estimate (2.7) shows that this series converges for all z.

For every formal power series $f = \sum_{k\geq 0} a_k T^k$ and every nonnegative integer ℓ we denote by $J_T^\ell(f)$ its ℓ-th jet

$$J_T^\ell(f) = \sum_{k=0}^{\ell} a_k T^k.$$

For $x = n^{-1}$ we have

$$\mathcal{F}_x(z) = \mathcal{F}_n(\mathcal{Q}, z) = \sum_{b\leq d/x} E_b(x)\frac{z^b}{b!} = \sum_{b\leq d/x}\left(\sum_{m\geq 0} E_b(m)x^m\right)\frac{z^b}{b!}$$

$$= \sum_{m\geq 0}\left(\sum_{b\leq d/x} E_b(m)\frac{z^b}{b!}\right)x^m = \sum_{m\geq 0} J_z^{d/x}(A_m(z))x^m.$$

Consider the formal power series in x with coefficients in the ring $\mathbb{C}\{z\}$ of convergent power series in z

$$\mathcal{F}_\infty(z) = \sum_{m\geq 0} A_m(z)x^m \in \mathbb{C}\{z\}[[x]].$$

We will prove that for every $\ell \geq 0$ and every $z \in \mathbb{C}$ we have

$$|\mathcal{F}_n(z) - J_x^\ell \mathcal{F}_\infty(z)| = O(n^{-\ell-1}), \text{ as } n \to \infty. \tag{2.8}$$

To prove this it is convenient to introduce the "rectangles"

$$D_{u,v} = \Big\{(b,m) \in (\mathbb{Z}_{\geq 0})^2;\ \ b \leq u,\ \ m \leq v\Big\}.$$

In this notation we have ($x = n^{-1}$)

$$\mathcal{F}_n(z) = \sum_{(b,m)\in D_{n,\infty}} E_b(m)x^m\frac{z^b}{b!},\quad J_x^\ell\mathcal{F}_\infty(z) = \sum_{(b,m)\in D_{\infty,\ell}} E_b(m)x^m\frac{z^b}{b!}.$$

Then

$$\mathcal{F}_n(z) - J_x^\ell\mathcal{F}_\infty(z) = \underbrace{\sum_{b\leq dn}\left(\sum_{m>\ell} E_b(m)x^m\right)\frac{z^b}{b!}}_{S_1(n)} + \underbrace{\sum_{m\leq \ell}\left(\sum_{b>dn} E_b(m)\frac{z^b}{b!}\right)x^m}_{S_2(n)}.$$

We estimate each sum separately. Using (2.7) with a $\lambda_b > 1$ to be specified later we deduce

$$\sum_{m>\ell} |E_b(m)x^m| \leq \frac{C^b b^{b-1}}{(b-2)!(1-\lambda_b\frac{b-1}{b})}\sum_{m>\ell}\left(\frac{bx}{\lambda_b d}\right)^m.$$

The inequality $b \leq dn$ can be translated into $\frac{bx}{d} \leq 1$ so that the above series is convergent for $b \leq dn$ whenever $\lambda_b > 1$ so that

$$\sum_{m>\ell} |E_b(m)x^m| \leq \frac{C^b b^{b-1}}{(b-2)!(1-\lambda_b\frac{b-1}{b})}\left(\frac{bx}{\lambda_b d}\right)^{\ell+1}\frac{1}{1-\frac{bx}{\lambda_b d}}.$$

When $b \leq dn$ we have

$$1-\frac{bx}{\lambda_b d} > 1-\frac{1}{\lambda_b}.$$

If we choose

$$\lambda_b = \left(\frac{b}{b-1}\right)^{1/2}$$

we deduce

$$1-\lambda_b\frac{b-1}{b} = 1-\left(\frac{b-1}{b}\right)^{1/2} \Longrightarrow \frac{1}{1-\lambda_b\frac{b-1}{b}} < b$$

and, since $\frac{bx}{\lambda_b d} \leq \frac{b}{d}x$,

$$\frac{1}{1-\frac{bx}{\lambda_b d}} < \frac{1}{1-\frac{1}{\lambda_b}} < 2b.$$

Using the inequalities

$$k! > \left(\frac{k}{e}\right)^k, \ \ \forall k > 0$$

we conclude that for $b \leq dn$ we have

$$\sum_{m>\ell} |E_b(m)x^m| \leq C_1^b b^{\ell+2}x^{\ell+1}.$$

Since the series $\sum_{b\geq 0} C_1^b b^{\ell+2}\frac{z^b}{b!}$ converges we conclude that

$$S_1(n) = O(x^{\ell+1}).$$

To estimate the second sum we choose $\lambda_b = 1$ in (2.7) and we deduce

$$E_b(m) \leq C_3^b.$$

Hence

$$\left|\sum_{b>dn} E_b(m)\frac{z^b}{b!}\right| \leq \frac{(C_3|z|)^b b^2}{b!} < (2C_3|z|)^2 \sum_{b>dn}\frac{(|C_3|z|)^{b-2}}{(b-2)!}.$$

Using Stirling's formula we deduce that for fixed z we have

$$\sum_{b>dn}\frac{(|C_3|z|)^{b-2}}{(b-2)!} < C_4(z)n^{-\ell-1}.$$

Hence

$$|S_2(n)| \leq C_4(z)(\ell+1)n^{-\ell-1}.$$

This completes the proof of (2.8) and of Theorem 1.3. $\square$

3. Additional structural results

3.1. The case $d = 1$. Hence $\mathcal{Q}(t) = (t + \mu_1)$ so that

$$\int_0^\infty e^{-zt}(t+\mu_1)^n dt = e^{\mu_1 z}\int_0^\infty e^{-zt}t^n dt = e^{\mu_1 z}\frac{n!}{z^{n+1}}.$$

Hence in this case

$$\mathcal{F}_n(z) = e^{\mu_1 z}$$

and we deduce

$$A_0(z) = e^{\mu_1 z},\ \ A_k(z) = 0,\ \ \forall k \geq 1.$$

3.2. The case $d = 2$. This is a bit more complicated. We assume first that $\mu_1 = 0$ so that

$$\mathcal{Q}(t) = t^2 - \sigma^2.$$

Then

$$\mathcal{Q}(n,a) = \begin{cases} (-1)^k\sigma^{2(n-k)}\binom{n}{k} & \text{if } a = 2k \\ 0 & \text{if } a \text{ is odd,}\end{cases}$$

and we deduce

$$\begin{aligned}\mathcal{F}_n(z) &= \sum_{b=0}^n \frac{(-1)^b\binom{n}{n-b}}{\binom{2n}{2n-2b}}\frac{(\sigma z)^{2b}}{(2b)!} = \sum_{b=0}^n \frac{n!(2n-2b)!}{(n-b)!(2n)!}\frac{(-1)^b(\sigma z)^{2b}}{b!}\\ &= \sum_{b=0}^n \frac{n(n-1)\cdots(n-b+1)}{2n(2n-1)\cdots(2n-2b+1)}\frac{(-1)^b(\sigma z)^{2b}}{b!}\\ &= \sum_{b=0}^n \frac{1}{2^{2b}}n^{-b}\frac{(1-1/n)\cdots(1-(b-1)/n)}{(1-1/(2n)\cdots(1-(2b-1)/(2n)}\frac{(-1)^b(\sigma z)^{2b}}{b!}\\ &= 1 - \frac{1}{2}n^{-1}\frac{1}{1-1/(2n)}\frac{(\sigma z^)2}{2!}\\ &\quad + \frac{1}{2^4}n^{-2}\frac{(1-1/n)}{(1-1/(2n))(1-2/(2n))(1-3/(2n))}\frac{(\sigma z)^4}{4!} + \cdots .\end{aligned}$$

To obtain $A_k(z)$ we need to collect the powers n^{-k}. The above description shows that the coefficients of the monomials z^{2b} contain only powers n^{-k}, $k \geq b$. We conclude that $A_k(z)$ is a polynomial and

$$\deg_z A_k(z) \leq 2k.$$

Let us compute the first few of these polynomials. We have

$$\mathcal{F}_n(z) = 1 - \frac{1}{2}n^{-1}\left(1 + \frac{1}{2}n^{-1} + \cdots\right)\frac{(\sigma z)^2}{2!} + \frac{1}{2^4}n^{-2}\Big(1 + \cdots\Big)\frac{(\sigma z)^4}{4!} + \cdots .$$

We deduce

$$A_0(z) = 1,\ \ A_1(z) = -\frac{1}{4}(\sigma z)^2,\ \ A_2(z) = -\frac{1}{8}(\sigma z)^2 + \frac{1}{2^4 4!}(\sigma z)^4.$$

If $\mu_1 \neq 0$ so that

$$\mathcal{Q}(t) = (t + r_1)(t + r_2), \quad r_1 + r_2 = 2\mu_1,$$

then we make the change in variables $t = s - \mu_1$ so that

$$\mathcal{Q}(t) = P(s) = s^2 - r^2, \quad \sigma^2 = (r_1 - \mu_1)^2 = \frac{(r_1 - r_2)^2}{4}.$$

Now observe that

$$4\mu_1^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2 + (r_1 - r_2)^2 = 2(r_1^2 + r_2^2) = 4\mu_2$$

so that

$$\sigma^2 = \mu_2 - \mu_1^2.$$

Then

$$\mathcal{F}_n(\mathcal{Q}, z) = \frac{z^{2n+1}}{(2n)!}\int_0^\infty e^{-zt}\mathcal{Q}(t)^n = \frac{z^{2n+1}}{(2n)!}\int_0^\infty e^{-z(s-\mu_1)}P(s)^n ds = e^{\mu_1 z}\mathcal{F}_n(P, z).$$

We deduce

(3.1)

$$A_0(z) = e^{\mu_1 z}, \;\; A_1(z) = -\frac{e^{\mu_1 z}}{4}(\sigma z)^2, \;\; A_2(z) = e^{\mu_1 z}\Big(-\frac{1}{8}(\sigma z)^2 + \frac{1}{2^4 4!}(\sigma z)^4\Big).$$

For the couples mixing problem we have

$$\mathcal{Q}(t) = t^2 - 4t + 2$$

so that

$$\mu_1 = -\frac{4}{2} = -2, \;\; \sigma^2 = \frac{1}{4}(r_1 - r_2)^2 = \frac{1}{4}\Big((r_1 + r_2)^2 - 4r_1 r_2\Big) = \frac{1}{4}(16 - 8) = 2,$$

and we deduce

$$\delta_n = \mathcal{F}_n(\mathcal{Q}, z = 1) = e^{-2}\Big(1 - \frac{1}{2}n^{-1} - \frac{23}{96}n^{-2} + O(n^{-3})\Big). \tag{3.2}$$

3.3. The general case. Let us determine the coefficients $A_0(z)$ and $A_1(z)$ for general degree d. We use the definition

$$A_k(z) = \sum_{b \ge 0} E_b(k)\frac{z^b}{b!}.$$

For $|\vec{\alpha}| = b$

$$W_{\vec{\alpha}}(x) = W_{b,\alpha}(x) = \prod_{i=1}^{d}\left(\prod_{j=1}^{\alpha_i - 1}(1 - jx)\right) = \prod_{i=1}^{d}\left(1 - \left(\sum_{j=1}^{\alpha_i - 1} j\right)x + \cdots\right)$$

$$= 1 - \frac{1}{2}\left(\sum_{i=1}^{d}\alpha_i(\alpha_i - 1)\right)x + \cdots .$$

$$W_b(x/d) = \prod_{k=1}^{b-1}(1 + jx/d + \cdots) = 1 + \frac{b(b-1)}{2d}x + \cdots .$$

Next, compute the expectation of $R_b(x)$

$$E_b(R_b(x)) = E_b(\rho_b) - \frac{1}{2}E_b\left(\sum_{i=1}^{d}\alpha_i(\alpha_i - 1)\vec{r}^{\vec{\alpha}}\right)x + \cdots .$$

The multinomial formula implies

$$E_b(\rho_b) = \mu_1^b.$$

Next

$$E_b\left(\sum_{i=1}^{d}\alpha_i(\alpha_i-1)\vec{r}^{\vec{\alpha}}\right)=\frac{1}{d^b}\sum_{|\vec{\alpha}|=b}\binom{b}{\vec{\alpha}}\left(\sum_{i=1}^{d}\alpha_i(\alpha_i-1)\right)\vec{r}^{\vec{\alpha}}.$$

Now consider the partial differential operator

$$\mathcal{P}=\sum_{i=1}^{d}r_i^2\frac{\partial^2}{\partial r_i^2}.$$

Observe that the monomials $\vec{r}^{\vec{\alpha}}$ are eigenvectors of $\mathcal{P}$

$$\mathcal{P}\vec{r}^{\vec{\alpha}}=\left(\sum_{i=1}^{d}\alpha_i(\alpha_i-1)\right)\vec{r}^{\vec{\alpha}}.$$

We deduce

$$E_b\left(\sum_{i=1}^{d}\alpha_i(\alpha_i-1)\vec{r}^{\vec{\alpha}}\right)=\frac{1}{2d^b}\mathcal{P}S(\vec{r})^b=\frac{1}{2}\mathcal{P}\mu_1^b.$$

Hence

$$E_b(R_b(x)=\mu_1^b-\frac{1}{2}(\mathcal{P}\mu_1^b)x+\cdots$$

and we deduce

$$\begin{aligned}E_b(x)&=\left(\mu_1^b-\frac{1}{2}(\mathcal{P}\mu_1^b)x+\cdots\right)\left(1+\frac{b(b-1)}{2d}x+\cdots\right)\\&=\mu_1^b+\frac{1}{2}\left(\frac{b(b-1)}{d}\mu_1^b-\mathcal{P}\mu_1^b\right)x+\cdots.\end{aligned}$$

We deduce $A_0(z)=e^{\mu_1 z}$

$$A_1(z)=\frac{\mu_1^2}{2d}\sum_{b=2}^{\infty}\frac{z^b}{(b-2)!}-\frac{1}{2}\mathcal{P}e^{\mu_1 z}=\frac{\mu_1^2z^2}{2d}e^{\mu_1 z}-\frac{1}{2}\mathcal{P}e^{\mu_1 z}.$$

We can simplify the answer some more.

$$\mathcal{P}\mu_1^b=\frac{1}{d^b}\mathcal{P}S(x)^b=\frac{b(b-1)}{d^b}\left(\sum_{i=1}^{d}r_i^2\right)S(x)^{b-2}=\frac{b(b-1)}{d}\mu_2\mu_1^{b-2}.$$

We conclude that

$$\mathcal{P}e^{\mu_1 z}=\frac{\mu_2z^2}{d}\sum_{b\geq 2}\frac{(\mu_1 z)^{b-2}}{(b-2)!}=\frac{\mu_2z^2}{d}e^{\mu_1 z}.$$

Hence

$$A_0(z)=e^{\mu_1 z},\ \ A_1(z)=\frac{e^{\mu_1 z}}{2d}(\mu_1^2-\mu_2)z^2. \tag{3.3}$$

For $d=2$ we recover part of the formulæ (3.1).

3.4. Proof of the structure theorem. Clearly we can assume $d > 1$. We imitate the strategy used in the case $d = 2$. Thus, after the change in variables $t \to t - \mu_1$ we can assume that $\mu_1 = 0$ so that $\mathcal{Q}(t)$ has the special form[1]

$$\mathcal{Q}(t) = t^d + a_{d-2}t^{d-2} + \cdots + a_0.$$

Set

$$T(n, b) := \frac{\mathcal{Q}(n, dn - b)}{\binom{dn}{dn-b}}.$$

This is a power series in $x = n^{-1}$,

$$T(n, b) = T_b(x)\,|_{x=n^{-1}}, \quad T_b(x) = \sum_{k\geq 0} T_b(k)x^k.$$

We have

$$A_k(z) = \sum_{b\geq 0} T_b(k)\frac{z^b}{b!},$$

and we need to prove that A_k is a polynomial for every k. We denote by $\ell(b)$ the order of the first nonzero coefficient of $T_b(x)$,

$$\ell(b) = \min\{k \geq 0;\;\; T_b(k) \neq 0\}.$$

To prove the desired conclusion it suffices to show that

$$\lim_{b\to\infty} \ell(b) = \infty. \tag{3.4}$$

For every multiindex $\vec{\beta} = (\beta_d, \beta_{d-2}, \ldots, \beta_1, \beta_0)$ we set

$$L(\vec{\beta}) = d\beta_d + (d-2)\beta_{d-2} + \cdots + \beta_1.$$

Let $\vec{a} := (1, a_{d-2}, \ldots, a_1, a_0) \in \mathbb{C}^d$ and

$$\mathcal{B}_n := \big\{\vec{\beta} \in \mathbb{Z}^d_{\geq 0};\;\; |\vec{\beta}| = n,\;\; L(\vec{\beta}) = dn - b\big\}.$$

We have

$$T(n, b) = \frac{1}{\binom{dn}{dn-b}} \cdot \sum_{\vec{\beta}\in\mathcal{B}_n} \binom{n}{\vec{\beta}}\vec{a}^{\,\vec{\beta}}. \tag{3.5}$$

Now observe that for every multiindex $\vec{\beta} \in \mathcal{B}_n$ we have

$$2\beta_{d-2} + 3\beta_{d-3} + \cdots + (d-1)\beta_1 + d\beta_0 = d|\vec{\beta}| - L(\vec{\beta}) = b.$$

In particular we deduce

$$\beta_j \leq \frac{b}{d-j} \leq \frac{b}{2}, \;\; \forall 0 \leq j \leq d-2 \tag{3.6}$$

and

$$2\beta_d + b = 2\beta_d = 2\beta_{d-2} + 3\beta_{d-3} + \cdots + (d-1)\beta_1 + d\beta_0$$

$$\geq 2\beta_d + 2\beta_{d-2} + \cdots + 2\beta_1 + 2\beta_0 = 2n$$

[1] A similar reduction trick was used in the proof of [1, Thm. 3], but there the authors follow a different approach which yields less information on the asymptotic expansion.

so that

$$n - \beta_d \leq \frac{b}{2}. \tag{3.7}$$

These simple observations have several important consequences.

First, observe that they imply that there exists an integer $N(b)$ *which depends only b and d*, such that

$$|\mathcal{B}_n| \leq N(b), \ \ \forall n > 0.$$

Thus the sum (3.5) has fewer than $N(b)$ terms.

Next, if we set $|a| := \max_{0 \leq j \leq d-2} |a_j|$ then, we deduce

$$|\vec{a}^{\,\vec{\beta}}| \leq |a|^{\beta_0 + \cdots + \beta_{d-2}} \leq |a|^{\frac{b(d-1)}{2}} = C_5(b).$$

Finally, using the identity

$$\binom{n}{\vec{\beta}} = \binom{n}{\beta_d} \cdot \binom{n-\beta_d}{\beta_{d-2}} \binom{n-\beta_d-\beta_{d-2}}{\beta_{d-3}} \cdots$$

the inequalities (3.7) and $\binom{m}{k} \leq 2^m$, $\forall m \geq k$ we deduce

$$\binom{n}{\vec{\beta}} \leq \binom{n}{\beta_d} \cdot 2^{\frac{b(d-1)}{2}} \leq 2^{\frac{b(d-1)}{2}} \binom{n}{\lfloor b/2 \rfloor + 1} \leq C_6(b) n^{\lfloor b/2 \rfloor + 1}, \ \ \forall n \gg b.$$

Hence

$$\sum_{|\vec{\beta}| = n, L(\vec{\beta}) = dn - b} \left| \binom{n}{\vec{\beta}} \vec{a}^{\,\vec{\beta}} \right| \leq N(b) C_5(b) C_6(b) n^{\lfloor b/2 \rfloor + 1} = C_7(b) n^{\lfloor b/2 \rfloor + 1}.$$

On the other hand

$$\frac{1}{\binom{dn}{dn-b}} \leq C_8(b) n^{-b}$$

so that

$$|T(n,b)| = |T_b(n^{-1})| \leq C_9(b) n^{\lfloor b/2 \rfloor + 1 - b} \leq C_9(b) n^{1 - b/2}.$$

This shows

$$T_b(k) = 0, \ \ \forall k \leq b/2 - 1$$

so that

$$\ell(b) \geq b/2 - 1 \to \infty \ \text{ as } b \to \infty.$$

□

Remark 3.1. We can say a bit more about the structure of the polynomials

$$B_k(\mu_1, \dots, \mu_d, z) \in R_d = \mathbb{C}[\mu_1, \dots, \mu_d, z], \ \ k > 0.$$

If we regard B as a polynomial in $r_1, \dots, r_d$ we see that it vanishes precisely when $r_1 = \cdots = r_d$. Note that

$$r_1 = \cdots = r_d = r \Longleftrightarrow \mathcal{Q}(t) = (t + r)^d.$$

On the other hand

$$\sum_k t^k \mu_k = \frac{1}{d} \sum_{i=1}^d \sum_{k \geq 0} (r_i t)^k = \frac{1}{d} \sum_{i=1}^d \frac{1}{1 - r_i t} \overset{(s := 1/t)}{=} \frac{s}{d} \sum_{i=1}^d \frac{1}{s + \mu_i} = \frac{s}{d} \frac{\mathcal{Q}'(s)}{\mathcal{Q}(s)}.$$

If $\mathcal{Q}(s) = (s+r)^d$ we deduce

$$\frac{s}{d}\frac{\mathcal{Q}'(s)}{\mathcal{Q}(s)} = \frac{s}{s+r} = \frac{1}{1-rt} = \sum_{k\geq 0}(rt)^k.$$

This implies that

$$r_1 = \cdots = r_d \Longleftrightarrow \mu_i^j = \mu_j^i, \ \ \forall 1 \leq i,j \leq k \Longleftrightarrow \mu_j = \mu_1^j, \ \ \forall 1 \leq j \leq d.$$

The ideal I in R_d generated by the binomials $\mu_1^j - \mu_j$ is prime since $R_d/I \cong \mathbb{C}[\mu_1, z]$. Using the Hilbert *Nullstelensatz* we deduce that B_k must belong to this ideal so that we can write

$$B_k(\mu_1, \dots, \mu_d, z) = A_{2k}(\mu, z)(\mu_1^2 - \mu_2) + \cdots + A_{dk}(\mu, z)(\mu_1^d - \mu_d).$$

□

References

[1] R. Askey, M. Ismail, T. Rashed, *A derangement problem*, Univ. of Wisconsin MRC Report # 1522, June 1975.

[2] S. Even, J. Gillis, *Derangements and Laguerre polynomials*, Math. Proc. Camb. Phil. Soc., **79**(1976), 135–143, MR 0392590 (52 #13407), Zbl 0325.05006.

[3] J. Gillis, M.E.H. Ismail, T. Offer, *An asymptotic problem in derangement theory*, Siam J. Math. Anal., **21** (1990), 262–269, MR 1032737 (91f:05004), Zbl 0748.05004.

[4] D.M. Jackson, *Laguerre polynomials and derangements*, Math. Proc. Camb. Phil. Soc., **80** (1976), 213–214, MR 0409204 (53 #12966), Zbl 0344.05005.

DEPT. OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618
nicolaescu.1@nd.edu http://www.nd.edu/~lnicolae/

This paper is available via http://nyjm.albany.edu:8000/j/2004/10-7.html.