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The double bubble problem on the cone

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Abstract. We characterize perimeter-minimizing double bubbles on a two-dimensional cone as either two concentric circles or a circle lens.

Contents

1. Introduction

The double bubble problem seeks the least-perimeter way to enclose and separate two given areas. Our Main Theorem 6.1 shows that the solution to the double bubble problem on the surface of a two-dimensional cone is either two concentric circles or a circle lens as in Figure 1, where the cone is represented as a planar wedge with sides identified. Figure 10 shows which minimizer occurs for each area ratio and cone angle. An analogous solution to the free boundary problem on a planar wedge with angle less than π follows (Corollary 6.3).

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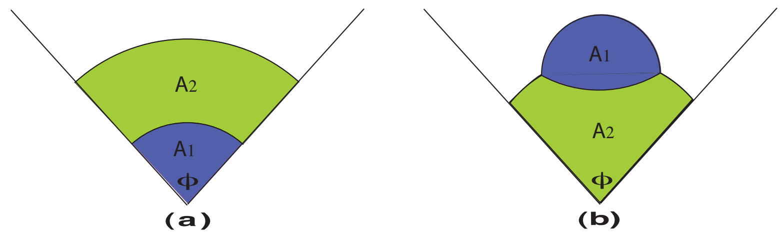

FIGURE 1. A perimeter minimizing double bubble on a cone for prescribed areas $A_1 \leq A_2$ is either: (a) two concentric circles, or (b) the circle lens. The cone is represented by a planar wedge of angle $\phi < 2\pi$ with opposite sides identified.

The proof. Following Foisy et al. [F], we consider the perimeter minimizer with at least the prescribed areas A_1, A_2, for which the exterior is connected. Lemma 4.1 shows that every boundary component encircles the vertex. Lemmas 4.3–4.6 characterize boundary components to arrive at three possibilities:

(1) two concentric circles about the vertex,
(2) one circle with an embedded lens, or
(3) two concentric circles with a lens.

Lemma 5.1 eliminates the third possibility using the equilibrium pressure equation (Proposition 3.5). These solutions must actually have the original prescribed areas, because if they had greater areas, one could reduce area and perimeter a bit and come up with a better solution.

History and recent developments. Foisy et al. [F] gave the proof of the double bubble conjecture in $\mathbf{R}^2$. Later work treated the sphere S^2 [Ma] and the torus T^2 [C]. The double bubble problem remains open on many surfaces such as the surface of the cube, for which the single bubble problem was recently solved by Cotton et al. [CF]. The free boundary problem in a planar wedge was treated in the context of connected regions by Hruska et al. [H] and is being treated in general by Foisy and Wichiramala [FW], including wedges with an angle greater than π. For recent higher-dimensional results on the double bubble problem, see [M1] and [HMRR]. Very recently, Adams et al. [ADLV] have studied single bubbles in Gauss cones.

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2. Definitions

This section lays out definitions used throughout the paper.

Definition 1. A *cone* is determined in $\mathbf{R}^3$ by $y = \lambda|x|$, for some fixed λ. The cone can be represented by a planar wedge with angle $\phi < 2\pi$ and sides identified, as shown in Figure 1.

Definition 2. A *double bubble* is a piecewise smooth enclosure of two regions of prescribed finite areas. Each region may have multiple components.

Definition 3. A *circle lens* is a circle centered on the vertex with a lens embedded in it, consisting of three constant curvature curves meeting in threes at 120 degrees (see Figure 1b). Completing the circle around the vertex yields a corresponding standard double bubble in the plane.

Definition 4. *Two concentric circles* refer to two concentric circles about the vertex of the cone as pictured in Figure 1a.

3. Existence and regularity

This section provides the basic existence and regularity properties.

Proposition 3.1 (Existence and Regularity)**.** *There exists a perimeter-minimizing double bubble of prescribed areas on the cone. Away from the vertex, it consists of smooth, constant-curvature curves meeting in threes at* 120 *degree angles.*

Proof. The proof of existence and regularity from the plane works in the cone as well [M2, Theorem 2.3]. □

Remark 3.2. Curvature has the physical interpretation of pressure difference between regions. All components of a region have the same pressure.

Lemma 3.3. *A perimeter minimizing double bubble does not pass through the vertex of the cone.* (*See Cotton et al.* [CF, Corollary 2.5]*.*)

Proof. Suppose a perimeter-minimizing double bubble passes through the vertex. The (perimeter-minimizing) tangent cone at the vertex must consist of $m \geq 2$ rays. If $m = 2$, the angle between the rays is less than π and the tangent cone is not minimizing. If $m \geq 3$, the angle between at least one pair of the rays is less than 120 degrees, and again the tangent cone is not minimizing. □

Lemma 3.4. *Given positive areas A_1, A_2, there exists a double bubble that minimizes perimeter among all double bubbles with areas at least A_1, A_2.*

Proof. The perimeter of a double bubble enclosing areas A_1, A_2 is greater than or equal to the perimeter of a single bubble enclosing the area $A_1 + A_2$. The perimeter-minimizing single bubble is a circle about the vertex of the cone [HHM, §8]. Hence as the circle gets larger, and the area it encloses $A_1 + A_2 \to \infty$, double bubble perimeter goes to infinity. For these areas, a minimizer exists by Proposition 3.1. By continuity, the least perimeter enclosure has a minimum at some $A_1' \geq A_1$, $A_2' \geq A_2$. □

Proposition 3.5. *Consider an equilibrium double bubble with perimeter P, enclosing areas A_1, A_2, with pressures p_1, p_2. Then*

$$P = 2(p_1 A_1 + p_2 A_2).$$

Proof. The properties of scaling hold on the cone, so the proof remains exactly the same as the proof in the plane (see [M1, 13.12]). □

4. Boundary components

Section 4 characterizes the boundary components of a minimizing double bubble.

Lemma 4.1. *Every component of the boundary of a minimizing double bubble must encircle the vertex.*

Proof. Assume that there is some boundary component C that does not encircle the vertex. This component C can then be translated on the cone. Upon translation, one of three violations of regularity (Proposition 3.1 and Lemma 3.3) must occur: the component C will bump another component, C will bump itself, or C will pass through the vertex. □

Lemma 4.2. *If a double bubble minimizes perimeter for areas at least A_1, A_2, then the exterior is connected.*

Proof. Assume not. Then there is at least one empty chamber (bounded component of the exterior). Removing its interface with one of the interior regions, as in Figure 2, will decrease perimeter and increase one of the areas, a contradiction. □

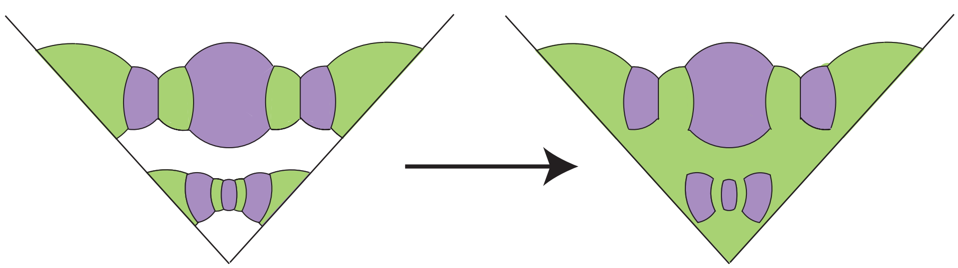

FIGURE 2. When the exterior is not connected one can always add bounded components of the exterior to the regions, increasing area and decreasing perimeter.

Lemma 4.3. *For a minimizing double bubble, if the exterior is connected, each boundary component is a simple closed curve, except that the outermost boundary component may have one or more lenses embedded in it and lenses embedded in each other.*

Proof. This is a simple combinatorial fact. See Foisy et al. [F, Lemma 2.4]. □

Lemma 4.4. *For a minimizing double bubble, if the exterior is connected then there is at most one lens in the double bubble (on the outermost boundary component).*

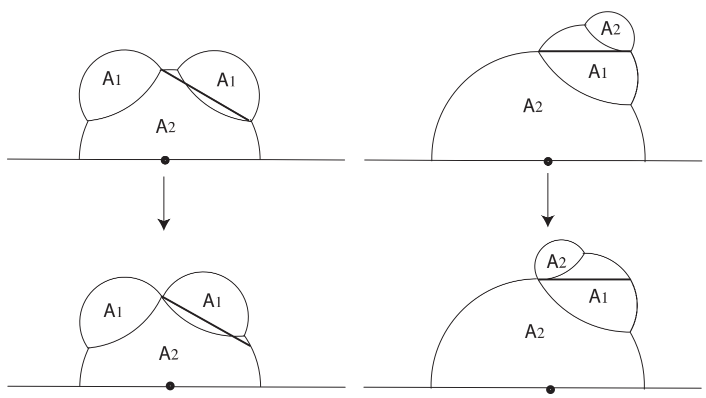

FIGURE 3. If there were several lenses one could be reflected to violate regularity.

Proof. By Lemma 4.3 we know that all but the outermost boundary component must be simple, closed curves. Assume that there is more than one lens on the outermost boundary component. As in [F, Lemma 2.4], one such lens has no lenses embedded in it. This lens and a portion of the bubble in which it is embedded may be reflected to contradict regularity (Proposition 3.1) as in Figure 3. □

Lemma 4.5. *In a minimizing double bubble, if the exterior is connected, any boundary component must be a circle about the vertex, possibly with a lens (in the outermost boundary component).*

Proof. By Proposition 3.1 the curves have constant curvature. Arcs emanating from a lens lie on a common arc (as follows e.g., from [M1, 14.1]). If this arc is not centered on the vertex then either it will not close up or it will close up at some angle less than 180 degrees, as in Figure 4. □

Lemma 4.6. *For a minimizing double bubble, if the exterior is connected, then there are at most two boundary components.*

Proof. If not, then as in Figure 5, one can remove the innermost boundary component and move the next boundary component inward, reducing perimeter. □

5. Two concentric circles with a lens

Two concentric circles with a lens is the last possible minimizer to be eliminated. Lemma 5.1 shows that any two concentric circles with a lens is not perimeter minimizing. This proof, suggested by Eric Schoenfeld, uses the pressure formula for equilibrium double bubbles (Lemma 3.5).

Lemma 5.1. *Two concentric circles with a lens is not perimeter minimizing.*

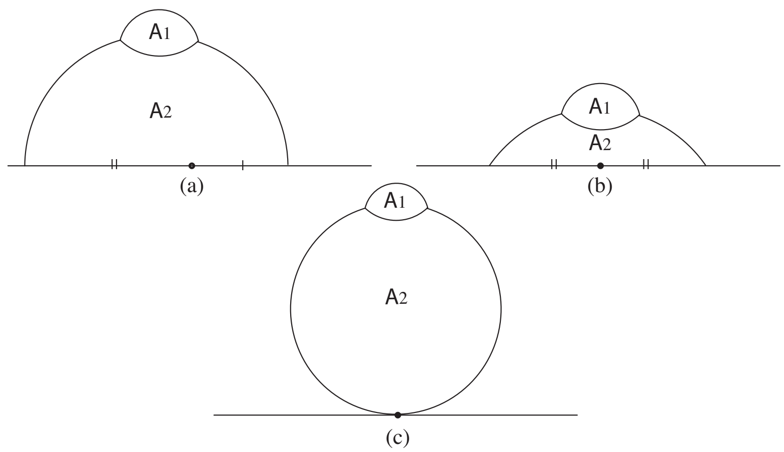

FIGURE 4. These three figures show how a boundary component that is not a circle about the vertex or such a circle with lenses embedded in it do violate regularity. In (a) the ends of the curve will not match up. In (b) the ends meet at an angle less than 180° in what should be a smooth curve, violating regularity. In (c) the curve passes through the vertex, violating regularity.

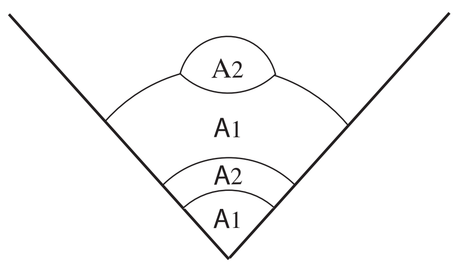

FIGURE 5. If there are multiple boundary components you can remove the component closest to the vertex and then scale the next closest boundary component back towards the vertex, reducing perimeter.

Proof. Compare (a) two concentric circles with a lens to (b) two concentric circles. As seen in Figure 6, $\kappa_2' > \kappa_2$ and $\kappa_3' > \kappa_3$, because two concentric circles have greater radii. Hence the pressures satisfy, $p_1' > p_1$ and $p_2' > p_2$. So by Proposition 3.5 the two concentric circles have less perimeter. □

6. Main result: minimizing double bubbles on the cone

The Main Theorem 6.1 characterizes the perimeter-minimizing double bubbles on the cone. Corollary 6.3 deduces a similar result for the planar wedge.

Main Theorem 6.1. *Given two prescribed areas, the least-perimeter way to enclose and separate these areas on the surface of a cone is either two concentric circles or a circle lens, as shown in Figure 1.*

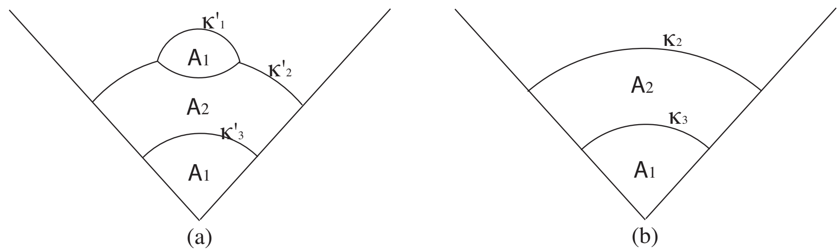

FIGURE 6. The pressure of each region is greater in the two concentric circles with a lens than in the two concentric circles, implying that two concentric circles with a lens are not perimeter minimizing.

Proof. By Lemma 3.4 there exists a minimizer among double bubbles on the cone that enclose and separate two areas that are at least as large as the two given areas A_1 and A_2. By Lemma 4.1 every boundary component encircles the vertex. By Lemma 4.2, the exterior must be connected. By Lemma 4.4 there is at most one lens in any minimizer, and it may occur only on the outermost boundary component. By Lemma 4.5, every component of the boundary must be a circle which is centered the vertex, possibly with a lens. By Lemma 4.6 any minimizer will have at most two boundary components. Therefore there are only the following three possible minimizers: (1) two concentric circles, (2) the circle lens, or (3) two concentric circles with a lens. By Lemma 5.1, two concentric circles with a lens are not perimeter minimizing, leaving only two possible minimizers, two concentric circles and the circle lens. These two possibilities must actually have the original prescribed areas because each region in the possible minimizers has positive pressure. If they had greater areas, one could reduce area and perimeter a bit, a contradiction. □

Remark 6.2. Of course, for two concentric circles the smaller area occurs in the region closest to the vertex. For the circle lens the numerical evidence given by Figure 8 shows that the smaller area occurs in the lens.

Corollary 6.3. *The perimeter-minimizing solution for a double bubble in a planar wedge with angle $\alpha < \pi$ is either two concentric circles or a circle lens as in Figure 7. Furthermore the phase diagram for the wedge of angle α corresponds to the phase diagram for a cone of angle $\phi = 2\alpha$.*

Proof. The perimeter-minimizing doubles bubbles in the cone have a plane of symmetry that bisects the cone. We claim that half the solution to the double bubble problem on the surface of the cone solves the double bubble problem in the planar wedge. Assume there is a different minimizer in the planar wedge. Then it has at most the same perimeter as half the solution to the double bubble problem in the cone. Reflection around the edge of the planar wedge would yield a different minimizer on the surface of the cone, a contradiction of Theorem 6.1. □

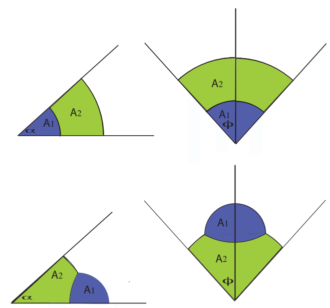

FIGURE 7. The two minimizers in a planar wedge are two concentric circles and the half circle lens as seen above, with $A_1 \leq A_2$.

7. Formulas for the minimizers

This section contains some formulas for the phase diagram (Figure 10) and information on the location of the smaller region.

Remark 7.1. In minimizing concentric circles, the smaller of the two areas must be the area enclosing the vertex. For areas $A_1 \leq A_2$, there exist two unique minimizing concentric circles. The areas A_1 and A_2 and perimeter P are given in terms of the radii of the circles and the cone angle ϕ:

$$A_1(\phi, r_1) = \frac{\phi}{2} r_1^2$$
$$A_2(\phi, r_1, r_2) = \frac{\phi}{2} r_2^2 - \frac{\phi}{2} r_1^2$$
$$P(\phi, r_1, r_2) = \phi r_1 + \phi r_2.$$

Manipulating these formulas we obtain a formula for perimeter as the function of the two areas and the cone angle:

$$P(\phi, A_1, A_2) = \sqrt{2\phi A_1} + \sqrt{2\phi(A_1 + A_2)}.$$

Remark 7.2. For a circle lens, we have not found an analytic proof of where the smaller area is located, though the numerical evidence of Figure 8 indicates that for all pairs of areas the smaller area is in the lens.

For a circle lens as in Figure 9 with interface at angle θ to a chord with length c, the radius of curvature R is given by

$$R(\theta, c) = \frac{c}{2 sin\theta}.$$

The area, A between the arc and chord is given by

$$A(\theta, c) = c^2 \frac{\theta - \sin\theta \cos\theta}{4 \sin^2 \theta},$$

and the length L of arc formed by the interface and the chord of length c is given by

$$L(\theta, c) = \frac{c\theta}{\sin\theta}.$$

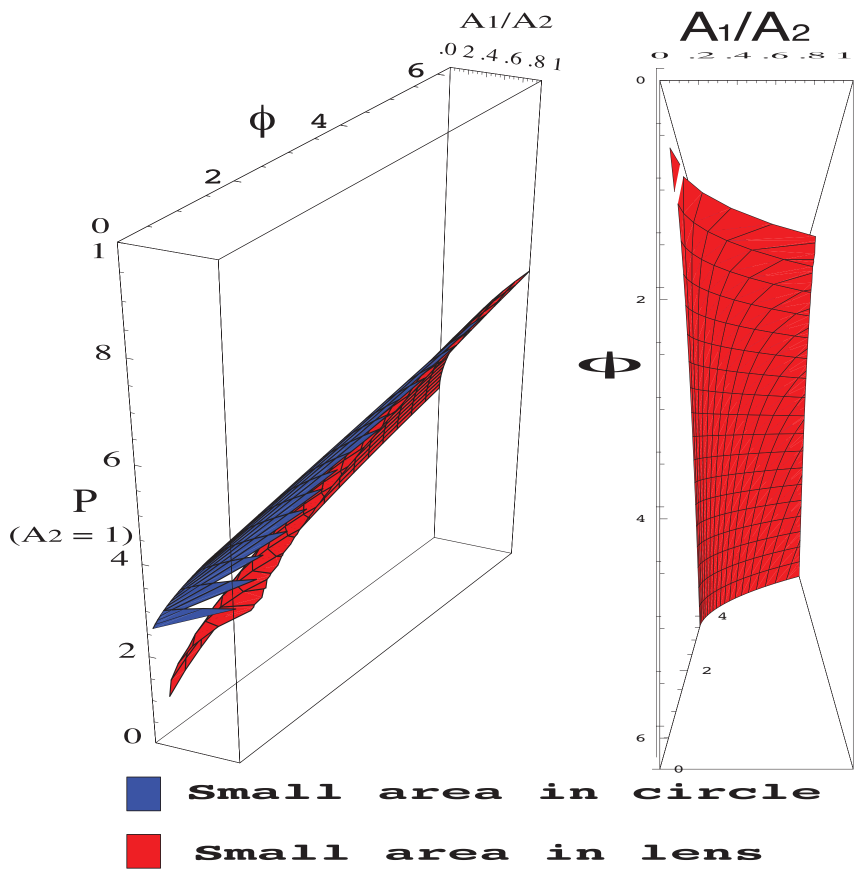

FIGURE 8. For all cone angles ϕ and all area ratios $\frac{A_1}{A_2}$, the perimeter of a circle lens is less with the smaller area in the lens. In the first picture, from the side, you can see one graph below the other. In the second picture, from below, you can see only the graph for the small area in the lens.

Hence the area of each region and the perimeter satisfy the following:

$$A_1 = A(\theta, c) + A(2\pi/3 - \theta, c),$$

$$A_2 = \frac{\phi}{2} R^2(\pi/3 - \theta, c) - A(\pi/3 - \theta, c) - A(\theta, c),$$

$$P(\theta, c, \phi) = \phi R(\pi/3 - \theta, c) - L(\pi/3 - \theta, c) + L(\theta, c) + L(2\pi/3 - \theta, c).$$

8. Phase diagram

Figure 10 shows the Mathematica plot of how the type of minimizing double bubble varies for the area A_1 (with $A_2 = 1$) and the cone angle ϕ. The two perimeters approach the same value as A_1 (with $A_2 = 1$) goes to zero. Preliminary asymptotic analysis shows that the minimizer is *always* a circle lens if and only if $\phi \geq \frac{4\pi}{3} - \sqrt{3} \approx 2.457$ radians ≈ 141 degrees. The Mathematica analysis (Figure 8)

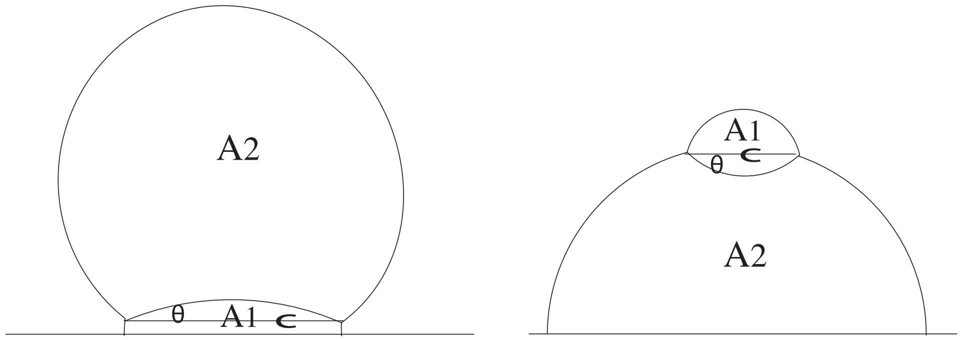

FIGURE 9. c and θ parameterize the circle lens in both cases. The more efficient and important circle lens on the right has the smaller area in the lens as shown by Figure 8.

agrees and also indicates that the minimizer is always concentric circles if and only if ϕ is less than about 1.66 radians.

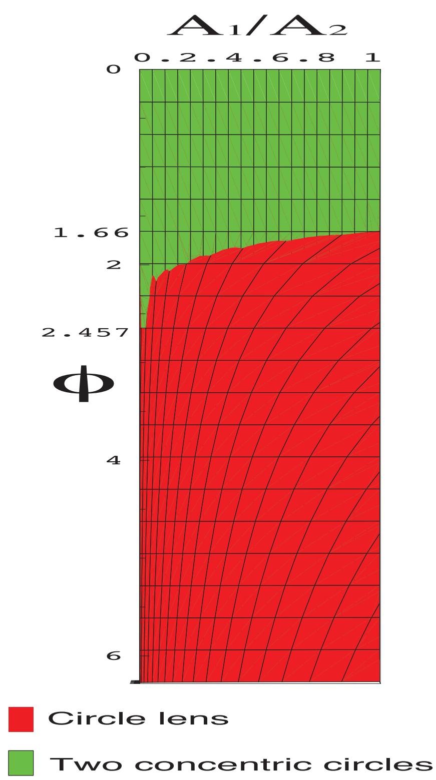

FIGURE 10. For cones with large angles the circle lens is the minimizer. As the angle of the cone decreases, two concentric circles become the minimizing double bubble on the cone.

Remark 8.1. It is easy to see that where our formulas say that the circle lens has less perimeter than two concentric circles, the lens fits on the cone, since when it bumps itself the concentric circles are better.

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