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Isomorphisms in the Farrell cohomology of $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$

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Abstract. We describe isomorphism patterns in the p-primary part of the Farrell cohomology ring $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbb{Z}[1/n]), \mathbb{Z})$ for any odd prime p and suitable integers $0 \neq n \in \mathbb{Z}$, where $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$ denotes the group of symplectic $(p-1) \times (p-1)$ matrices. Moreover, we determine the precise p-period of this ring.

Contents

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1. Introduction

The Farrell cohomology is defined for any group G with finite virtual cohomological dimension ($\operatorname{vcd} G < \infty$). It is a generalization of the Tate cohomology for finite groups. Let $\widehat{\mathrm{H}}^*(G,\mathbb{Z})$ denote the Farrell cohomology of the group G with coefficients in the ring $\mathbb{Z}$. For a prime p the p-primary part of $\widehat{\mathrm{H}}^*(G,\mathbb{Z})$ is written $\widehat{\mathrm{H}}^*(G,\mathbb{Z})_{(p)}$. We then have

$$\widehat{\mathrm{H}}^*(G,\mathbb{Z}) \cong \prod_p \widehat{\mathrm{H}}^*(G,\mathbb{Z})_{(p)},$$

where p ranges over the primes such that G has p-torsion. A group G of finite virtual cohomological dimension has periodic cohomology if for some $d \neq 0$ there is an element $u \in \widehat{\mathrm{H}}^d(G,\mathbb{Z})$ which is invertible in the ring $\widehat{\mathrm{H}}^*(G,\mathbb{Z})$. Cup product with u then gives a periodicity isomorphism

$$\widehat{\mathrm{H}}^i(G,M) \cong \widehat{\mathrm{H}}^{i+d}(G,M)$$

for any $\mathbb{Z}G$-module M and any $i \in \mathbb{Z}$. Similarly, G has p-periodic cohomology if for some $d \neq 0$ there is an element $u \in \widehat{\mathrm{H}}^d(G,\mathbb{Z})_{(p)}$ which is invertible in the ring $\widehat{\mathrm{H}}^*(G,\mathbb{Z})_{(p)}$. We recall that if G is a group with $\operatorname{vcd} G < \infty$, then G has p-periodic cohomology if and only if every elementary abelian p-subgroup of G has rank at most 1. For more details see Brown [3].

Let R be a commutative ring with 1. The general linear group $\mathrm{GL}(n,R)$ is defined to be the multiplicative group of invertible $n \times n$ matrices over R. The symplectic group $\mathrm{Sp}(2n,R)$ over the ring R is the subgroup of matrices $Y \in \mathrm{GL}(2n,R)$ that satisfy

$$Y^{\mathrm{T}} J Y = J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $n \times n$ identity matrix and Y^{T} denotes the transpose of Y. For any odd prime p the symplectic group $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$ over the ring $\mathbb{Z}[1/n]$, $0 \neq n \in \mathbb{N}$, has finite virtual cohomological dimension and p-periodic cohomology.

In Section 4 we determine isomorphisms between the cohomology groups in the p-primary part of the ring $\widehat{\mathrm{H}}^*\big(\mathrm{Sp}(p-1,\mathbb{Z}[1/n]),\mathbb{Z}\big)$.

Theorem 4.3. *Let p be an odd prime. Let n be such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains and moreover $p \mid n$. Then for any $i \in \mathbb{Z}$*

$$\widehat{\mathrm{H}}^i\big(\mathrm{Sp}(p-1,\mathbb{Z}[1/n]),\mathbb{Z}\big)_{(p)} \cong \widehat{\mathrm{H}}^{i+b}\big(\mathrm{Sp}(p-1,\mathbb{Z}[1/n]),\mathbb{Z}\big)_{(p)}$$

with $b = y$, the greatest odd divisor of $p-1$, if and only if for each $j \mid y$ a prime $q \mid n$ exists with inertia degree f_q such that $j \mid \frac{p-1}{2f_q}$. If no such q exists, then $b = 2y$.

The inertia degree f_q of a prime $q \in \mathbb{N}$ is the multiplicative order of q in the field $\mathbb{F}_p$. In Section 5 we determine the periodicity isomorphisms in the p-primary part of the cohomology.

Theorem 5.2. *Let $n \in \mathbb{Z}$ be such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains and $p \mid n$. Then the p-period of the Farrell cohomology ring*

$$\widehat{\mathrm{H}}^*\big(\mathrm{Sp}(p-1,\mathbb{Z}[1/n]),\mathbb{Z}\big)$$

equals $2y$, where y is the greatest odd divisor of $p-1$.

In fact the condition on the integer n is not very restrictive since $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains if and only if $q_1 \dots q_h$ divides n, where $q_1, \dots, q_h \in \mathbb{N}$ are primes that depend on the prime p. The integer h is the class number of $\mathbb{Q}(\xi)$, i.e., the order of the ideal class group of $\mathbb{Z}[\xi]$. For primes p with odd relative class number h^- the p-period of the Farrell cohomology ring $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1,\mathbb{Z}),\mathbb{Z})$ is $2y$, where y is odd and $p-1 = 2^r y$ for some $r > 0$, $r \in \mathbb{Z}$, see Busch [4]. The relative class number is $h^- := h/h^+$, where h, resp. h^+, denotes the class number of $\mathbb{Q}(\xi)$, resp. $\mathbb{Q}(\xi+\xi^{-1})$.

We use the following result of Brown ([3], Corollary X.7.4). Let G be a group with finite virtual cohomological dimension such that each elementary abelian p-subgroup of G has rank $\leqslant 1$. Then

$$\widehat{\mathrm{H}}^*(G,\mathbb{Z})_{(p)} \cong \prod_{P\in\mathfrak{P}} \widehat{\mathrm{H}}^*\big(N(P),\mathbb{Z}\big)_{(p)}, \tag{1.1}$$

where $\mathfrak{P}$ is a set of representatives of conjugacy classes of subgroups P of order p in G and $N(P)$ is the normalizer of P. The symplectic group $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$ that we are considering satisfies this property. Ash [2] uses the isomorphism (1.1) in order to compute the Farrell cohomology of the group $\mathrm{GL}(n,\mathbb{Z})$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$ for an odd prime p and $p-1 \leqslant n \leqslant 2p-3$. Naffah [7] considers normalizers of subgroups of prime order in $\mathrm{PSL}(2,\mathbb{Z}[1/n])$ in order to compute the Farrell cohomology of $\mathrm{PSL}(2,\mathbb{Z}[1/n])$. Glover and Mislin [6] show corresponding results for the outer automorphism group of the free group in the p-rank one case. For the case $p = 3$ see also the result of Adem and Naffah [1]: they consider the cohomology of the group $\mathrm{SL}(2,\mathbb{Z}[1/q])$, q a prime.

It is well-known that if $N(P)/C(P)$ is a finite group whose order is prime to p, then

$$\widehat{\mathrm{H}}^*\big(N(P),\mathbb{Z}\big)_{(p)} \cong \big(\widehat{\mathrm{H}}^*(C(P),\mathbb{Z})_{(p)}\big)^{N(P)/C(P)}. \tag{1.2}$$

In order to compute the p-period of $\widehat{\mathrm{H}}^*(N(P),\mathbb{Z})$, we consider the action of $N(P)/C(P)$ on the centralizer $C(P)$ and on $\widehat{\mathrm{H}}^*(C(P),\mathbb{Z})_{(p)}$. We already know the structure of $N(P)/C(P)$ and of $C(P)$, see Busch [5].

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2. Subgroups of order p in symplectic groups

2.1. Algebraic number theory. The conjugacy classes of matrices of odd prime order p in $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$ are related to some classes of ideals in $\mathbb{Z}[1/n][\xi]$, where ξ is a primitive pth root of unity.

The ring $\mathbb{Z}[\xi]$ is the ring of integers of the cyclotomic field $\mathbb{Q}(\xi)$ and $\mathbb{Z}[\xi+\xi^{-1}]$ is the ring of integers of the maximal real subfield $\mathbb{Q}(\xi+\xi^{-1})$ of $\mathbb{Q}(\xi)$. For any integer $0 \neq n \in \mathbb{Z}$ we consider the ring $\mathbb{Z}[1/n]$ and the extensions $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$. It is well-known that these are Dedekind rings. Let q be a prime in $\mathbb{Z}$. The ideal $(q) \subset \mathbb{Z}[\xi+\xi^{-1}]$ can be written as a product of prime ideals $\mathfrak{q}_1^+ \cdots \mathfrak{q}_r^+$ in $\mathbb{Z}[\xi+\xi^{-1}]$. Consider the prime ideals $\mathfrak{q} \subset \mathbb{Z}[\xi]$ over the prime ideal $\mathfrak{q}^+ \subset \mathbb{Z}[\xi+\xi^{-1}]$. The ideal $\mathfrak{q}^+$ satisfies one of the following three properties.

i) The prime $\mathfrak{q}^+$ is inert: $\mathfrak{q}^+\mathbb{Z}[\xi] = \mathfrak{q}$ is a prime ideal in $\mathbb{Z}[\xi]$ that lies over q.
ii) The prime $\mathfrak{q}^+$ splits: $\mathfrak{q}^+\mathbb{Z}[\xi] = \mathfrak{q}\bar{\mathfrak{q}}$, where $\mathfrak{q}$ is a prime ideal in $\mathbb{Z}[\xi]$ that lies over q.
iii) The ramified case: $\mathfrak{p}^+\mathbb{Z}[\xi] = \mathfrak{p}^2$, where $\mathfrak{p} := (1-\xi)$ is the only prime ideal in $\mathbb{Z}[\xi]$ that lies over p. Moreover $\mathfrak{p}^+ := ((1-\xi)(1-\xi^{-1})) = \mathfrak{p}\bar{\mathfrak{p}}$ is the only prime ideal in $\mathbb{Z}[\xi+\xi^{-1}]$ that lies over p.

The Galois group $G := \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$, resp. $G := \mathrm{Gal}(\mathbb{Q}(\xi+\xi^{-1})/\mathbb{Q})$, acts transitively on the set of prime ideals $\mathfrak{q}$, resp. $\mathfrak{q}^+$, that lie over the prime $q \in \mathbb{Z}$. Some Galois automorphisms fix the prime ideals $\mathfrak{q} \subset \mathbb{Z}[\xi]$ over q. These define the group

$$G_{\mathfrak{q}} = \{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \mid \gamma(\mathfrak{q}) = \mathfrak{q}\}.$$

The order of $G_{\mathfrak{q}}$ is f_q, the inertia degree of q. For more details see the book of Neukirch [8].

2.2. Centralizers and normalizers. From now on $C(P)$ denotes the centralizer and $N(P)$ denotes the normalizer of a subgroup P of odd prime order p in $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$.

2.2.1. The centralizer. The centralizer of a subgroup of $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$ of order p is determined by the primes that lie over the primes that divide n. Indeed we show in ([5], Theorem 4.2) that if $n \in \mathbb{Z}$ is such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains, then the centralizer $C(P)$ of a subgroup P of order p in $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$ satisfies

$$C(P) \cong \mathbb{Z}/2p\mathbb{Z} \times \mathbb{Z}^{\sigma^+}.$$

Here $\sigma^+ = \sigma$ if $p \nmid n$, $\sigma^+ = \sigma + 1$ if $p \mid n$ and σ is the number of primes in $\mathbb{Z}[\xi+\xi^{-1}]$ that split in $\mathbb{Z}[\xi]$ and lie over primes in $\mathbb{Z}$ that divide n. This result is related to the fact that the centralizer $C(P)$ of P is isomorphic to the kernel of the norm mapping

$$\begin{array}{ccc} \mathbb{Z}[1/n][\xi]^* & \longrightarrow & \mathbb{Z}[1/n][\xi+\xi^{-1}]^* \\ x & \longmapsto & x\overline{x}. \end{array}$$

Here $\overline{x}$ denotes the complex conjugate of x.

2.2.2. The quotient of the normalizer by the centralizer. Let $n \in \mathbb{Z}$ be such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains and moreover $p \mid n$. In ([5], Theorem 4.1) we see that the normalizer $N(P)$ and the centralizer $C(P)$ of a subgroup P of order p in $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$ satisfy

$$N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z},$$

where $j \mid p-1$, j odd. Moreover, for each j with $j \mid p-1$, j odd, a subgroup P of order p in $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$ exists with $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$.

3. The Farrell cohomology

Let $C(P)$ denote the centralizer and $N(P)$ the normalizer of a subgroup P of odd prime order p in $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$. In this section we consider the p-primary part of the Farrell cohomology ring $\widehat{\mathrm{H}}^*\big(N(P),\mathbb{Z}\big)$. By (1.2) we first determine the cohomology of the centralizer $C(P)$. Then we describe the invariants under the action of the quotient $N(P)/C(P)$ on $\widehat{\mathrm{H}}^*\big(C(P),\mathbb{Z}\big)_{(p)}$.

3.1. The Farrell cohomology of the centralizer. Since the centralizer $C(P) \cong (\mathbb{Z}/2p\mathbb{Z}) \times \mathbb{Z}^{\sigma^+}$, we determine the Farrell cohomology of this group. It is an exercise in the book of Brown [3] to determine a Künneth Formula for the Farrell cohomology. Here we give the formula without any proof since this can be found in the thesis of Naffah ([7], Proposition 5.5 and 5.7). Let G and G' be groups such that $\operatorname{cd} G < \infty$ and $\operatorname{vcd} G' < \infty$. Suppose that $\mathbb{Z}$ admits a projective resolution of finite type over $\mathbb{Z}G$ and that G' admits a complete resolution of finite type. Then there is an exact sequence

$$0 \longrightarrow \bigoplus_{j+k=i} \mathrm{H}^j(G,\mathbb{Z}) \otimes \widehat{\mathrm{H}}^k(G',\mathbb{Z}) \longrightarrow \widehat{\mathrm{H}}^i(G\times G',\mathbb{Z})$$
$$\longrightarrow \bigoplus_{j+k=i-1} \mathrm{Tor}_1^{\mathbb{Z}}\big(\mathrm{H}^j(G,\mathbb{Z}),\widehat{\mathrm{H}}^k(G',\mathbb{Z})\big) \longrightarrow 0.$$

Consider $\mathrm{H}^*(G,\mathbb{Z}) \otimes \widehat{\mathrm{H}}^*(G',\mathbb{Z})$ as a commutative graded ring with the product

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2) = (-1)^{\deg b_1 \deg a_2} a_1a_2 \otimes b_1b_2,$$

and denote by α_i the left map in the Künneth Formula :

$$\bigoplus_{j+k=i} \mathrm{H}^j(G,\mathbb{Z}) \otimes \widehat{\mathrm{H}}^k(G',\mathbb{Z}) \xrightarrow{\alpha_i} \widehat{\mathrm{H}}^i(G\times G',\mathbb{Z}).$$

Then the map $\alpha = (\alpha_i)_{i\in\mathbb{Z}}$

$$\mathrm{H}^*(G,\mathbb{Z}) \otimes \widehat{\mathrm{H}}^*(G',\mathbb{Z}) \xrightarrow{\alpha} \widehat{\mathrm{H}}^*(G\times G',\mathbb{Z})$$

is a ring homomorphism.

Proposition 3.1. *Choose $n \in \mathbb{Z}$ such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains and $p \mid n$. Here ξ is a primitive pth root of unity. Let σ denote the number of primes in $\mathbb{Z}[\xi+\xi^{-1}]$ that split and lie over the primes in $\mathbb{Z}$ that divide n. Then the Farrell cohomology ring of the centralizer $C(P)$ of a subgroup P of order p in $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$ is*

$$\begin{aligned}\widehat{\mathrm{H}}^*\big(C(P),\mathbb{Z}\big) &\cong \widehat{\mathrm{H}}^*\big((\mathbb{Z}/2p\mathbb{Z})\times\mathbb{Z}^{\sigma+1},\mathbb{Z}\big)\\ &\cong \mathbb{Z}/2p\mathbb{Z}[x,x^{-1}]\otimes\Lambda_{\mathbb{Z}}(e_0,\dots,e_\sigma),\end{aligned}$$

where $\deg(x)=2$ *and* $\deg(e_i)=1$, $i=0,\dots,\sigma$. *In particular*

$$\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)\cong\big(\mathbb{Z}/2p\mathbb{Z}\big)^{2^\sigma}$$

and the p-primary part is

$$\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)_{(p)}\cong\big(\mathbb{Z}/p\mathbb{Z}\big)^{2^\sigma}.$$

Proof. Applying the Künneth Formula given above we see that the entire ring of the Farrell cohomology of $(\mathbb{Z}/k\mathbb{Z})\times\mathbb{Z}^l$, $l\geqslant 1$, is

$$\widehat{\mathrm{H}}^*\big((\mathbb{Z}/k\mathbb{Z})\times\mathbb{Z}^l,\mathbb{Z}\big)\cong\mathbb{Z}/k\mathbb{Z}[x'',x''^{-1}]\otimes\Lambda_{\mathbb{Z}}(e_1,\dots,e_l),$$

where $\deg(x'')=2$ and $\deg(e_i)=1$. In particular

$$\widehat{\mathrm{H}}^i\big((\mathbb{Z}/k\mathbb{Z})\times\mathbb{Z}^l,\mathbb{Z}\big)\cong\big(\mathbb{Z}/k\mathbb{Z}\big)^{2^{l-1}}.$$

A nice proof of this result is given in the thesis of Naffah ([7], Proposition 5.10). Now the assumption follows by 2.2.1. For the p-primary part of the cohomology we have

$$\begin{aligned}\widehat{\mathrm{H}}^*\big(C(P),\mathbb{Z}\big)_{(p)} &\cong \big(\mathbb{Z}/2p\mathbb{Z}[x,x^{-1}]\otimes\Lambda_{\mathbb{Z}}(e_0,\dots,e_\sigma)\big)_{(p)}\\ &\cong \mathbb{Z}/p\mathbb{Z}[x',x'^{-1}]\otimes\Lambda_{\mathbb{Z}}(e_0,\dots,e_\sigma)\end{aligned}$$

and herewith $\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)_{(p)}\cong\big(\mathbb{Z}/p\mathbb{Z}\big)^{2^\sigma}$. □

3.1.1. The cup product. We have

$$\begin{aligned}\widehat{\mathrm{H}}^*\big(C(P),\mathbb{Z}\big)_{(p)} &\cong \mathbb{Z}/p\mathbb{Z}[x,x^{-1}]\otimes\Lambda_{\mathbb{Z}}(e_0,\dots,e_\sigma)\\ &\cong \mathbb{Z}/p\mathbb{Z}[x,x^{-1}]\otimes_{\mathbb{Z}/p\mathbb{Z}}\Lambda_{\mathbb{Z}/p\mathbb{Z}}(e_0,\dots,e_\sigma)\end{aligned}$$

and therefore we can consider the second factor to be the exterior product of a $\mathbb{Z}/p\mathbb{Z}$-vector space. We have seen that $e_i\in\mathrm{H}^1(\mathbb{Z}^{\sigma+1},\mathbb{Z})=\mathrm{Hom}(\mathbb{Z}^{\sigma+1},\mathbb{Z})$, $i=0,\dots,\sigma$, and $x\in\widehat{\mathrm{H}}^2(\mathbb{Z}/p\mathbb{Z},\mathbb{Z})$. The cup product of $x^k\otimes e$, $x^l\otimes e'\in\widehat{\mathrm{H}}^*\big(C(P),\mathbb{Z}\big)_{(p)}$ is

$$\begin{aligned}(x^k\otimes e)\cdot(x^l\otimes e') &= (-1)^{\deg x^l\deg e}x^kx^l\otimes ee'\\ &= x^{k+l}\otimes ee'\end{aligned}$$

because $\deg(x^l)$ is even. We choose the $\mathbb{Z}/p\mathbb{Z}$-basis of $\Lambda^m_{\mathbb{Z}/p\mathbb{Z}}(e_0,\dots,e_\sigma)$, $m=1,\dots,\sigma+1$, to be the set of products

$$\{e_{i_1}\dots e_{i_m} \mid 0 \leqslant i_1 < \dots < i_m \leqslant \sigma\}.$$

Each element of this basis is nilpotent. Of course each nonzero element in

$$\Lambda^0_{\mathbb{Z}/p\mathbb{Z}}(e_0,\dots,e_\sigma) \cong \mathbb{Z}/p\mathbb{Z}$$

is invertible. Let $e \in \Lambda^m_{\mathbb{Z}/p\mathbb{Z}}(e_0,\dots,e_\sigma)$, then

$$x^k \otimes e \in \widehat{\mathrm{H}}^{2k+m}\big(C(P),\mathbb{Z}\big)_{(p)}$$

is nilpotent if and only if e is nilpotent and $x^k \otimes e$ is invertible if and only if e is invertible. Therefore $\widehat{\mathrm{H}}^*(C(P),\mathbb{Z})_{(p)}$ is periodic of period 2 and the periodicity isomorphism is given by cup product with $x\otimes 1$.

3.2. An action on the Farrell cohomology of the centralizer. Let $N(P)$ be the normalizer and $C(P)$ the centralizer of a subgroup P of odd prime order p in $\mathrm{Sp}(p-1,\mathbb{Z}[1/n])$. Choose $n\in\mathbb{Z}$ such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains and moreover $p \mid n$. Here ξ denotes a primitive pth root of unity. In order to understand the action of $N(P)/C(P)$ on $\widehat{\mathrm{H}}^*(C(P),\mathbb{Z})_{(p)}$ we recall how this quotient acts on the centralizer $C(P)$. By 2.2.1 the sequence

$$C(P) \hookrightarrow \mathbb{Z}[1/n][\xi]^* \xrightarrow{N} \mathbb{Z}[1/n][\xi+\xi^{-1}]^*$$

is exact in $\mathbb{Z}[1/n][\xi]^*$. The norm N is not surjective. By 2.2.2 the group $N(P)/C(P)$ is isomorphic to a subgroup of the Galois group $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$:

$$N(P)/C(P) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\xi+\xi^{-1})/\mathbb{Q}) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}).$$

The first embedding exists because the order of $N(P)/C(P)$ is odd. Therefore the action of $N(P)/C(P)$ on the centralizer $C(P)$ is given by the action of $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ on the group of units $\mathbb{Z}[1/n][\xi]^*$ and $N(P)/C(P)$ acts faithfully on $C(P)$.

Now we determine the action of $N(P)/C(P)$ on $\widehat{\mathrm{H}}^*(C(P),\mathbb{Z})_{(p)}$. We have

$$\widehat{\mathrm{H}}^*\big(C(P),\mathbb{Z}\big)_{(p)} \cong \mathbb{Z}/p\mathbb{Z}[x,x^{-1}]\otimes\Lambda_{\mathbb{Z}/p\mathbb{Z}}(e_0,\dots,e_\sigma),$$

where $\deg(x)=2$ and $\deg(e_i)=1$, $i=0,\dots,\sigma$.

3.2.1. The action on the first factor. We know that $N(P)/C(P)$ is cyclic of order j, where $j \mid p-1$ and j is odd. Since $N(P)/C(P)$ acts faithfully on $C(P)$, the action of a generator of $N(P)/C(P)\cong\mathbb{Z}/j\mathbb{Z}$ on $\mathbb{Z}/p\mathbb{Z}[x,x^{-1}]$ is given by $x\mapsto \mu x$, where $\mu\in(\mathbb{Z}/p\mathbb{Z})^*$ is a primitive jth root of unity. Then $x^j \mapsto \mu^j x^j = x^j$ and, in particular,

$$x^j\otimes 1 \in \widehat{\mathrm{H}}^{2j}\big(C(P),\mathbb{Z}\big)^{N(P)/C(P)}_{(p)}$$

is invertible in $\widehat{\mathrm{H}}^*\big(N(P),\mathbb{Z}\big)_{(p)}$.

3.2.2. The action on the second factor. The quotient $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$ is isomorphic to a subgroup of the Galois group $\mathrm{Gal}(\mathbb{Q}(\xi+\xi^{-1})/\mathbb{Q})$ and the free abelian part of the centralizer is given by the primes $\mathfrak{p}^+, \mathfrak{q}_1^+, \ldots, \mathfrak{q}_\sigma^+$ in $\mathbb{Z}[\xi+\xi^{-1}]$ that are ramified or split in $\mathbb{Z}[\xi]$ and lie over the primes that divide n. Any element $w \in C(P)$ can be written as

$$w = w' \varepsilon_{\mathfrak{p}}^{m_0} \varepsilon_1^{m_1} \ldots \varepsilon_\sigma^{m_\sigma},$$

where w' is the torsion part and $\varepsilon_{\mathfrak{p}} \in \mathfrak{p}^+$, $\varepsilon_i \in \mathfrak{q}_i^+$, $i = 1, \ldots, \sigma$. For a given basis we define a homomorphism

$$\begin{array}{ccc} \mathfrak{p}^+\mathfrak{q}_1^+ \ldots \mathfrak{q}_\sigma^+ & \longrightarrow & \mathbb{Z}^{\sigma+1} \\ \varepsilon_{\mathfrak{p}}^{m_0}\varepsilon_1^{m_1} \ldots \varepsilon_\sigma^{m_\sigma} & \longmapsto & (m_0, m_1, \ldots, m_\sigma) \end{array}$$

and the dual elements $e_i \in \mathrm{Hom}(\mathbb{Z}^{\sigma+1}, \mathbb{Z})$, $i = 0, \ldots, \sigma$, such that

$$e_i(m_0, \ldots, m_\sigma) = \sum_{j=0}^{\sigma} \delta_{ij} m_j.$$

Let E be the $\mathbb{Z}/p\mathbb{Z}$-vector space spanned by $e_0, \ldots, e_\sigma$. The action of $\mathrm{Gal}(\mathbb{Q}(\xi+\xi^{-1})/\mathbb{Q})$ on the primes $\mathfrak{p}^+, \mathfrak{q}_1^+, \ldots, \mathfrak{q}_\sigma^+$ defines an action on E. The quotient $N(P)/C(P)$ acts as a subgroup of $\mathrm{Gal}(\mathbb{Q}(\xi+\xi^{-1})/\mathbb{Q})$. By the Herbrand unit theorem the basis $\varepsilon_{\mathfrak{p}}, \varepsilon_1, \ldots, \varepsilon_\sigma$ can be chosen such that the Galois group acts as a permutation on this basis and herewith the group also acts as a permutation on the basis $e_0, \ldots, e_\sigma$ of E. Let $\gamma \in N(P)/C(P) \subseteq G = \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ be a generator. The group $\langle\gamma\rangle \subseteq G$ permutes the primes that lie over $q \mid n$. Therefore, for each $q \mid n$ with q prime and split, we have a subspace $E_q \subseteq E$ that is invariant under the action of the Galois group. Let $q \mid n$ be a prime such that the prime $\mathfrak{q}^+ \subset \mathbb{Z}[\xi+\xi^{-1}]$ that lies over q splits, i.e., $\mathfrak{q}^+\mathbb{Z}[\xi] = \mathfrak{q}\bar{\mathfrak{q}}$. The order of γ is odd and therefore γ fixes $\mathfrak{q}$ (and $\bar{\mathfrak{q}}$) if and only if γ fixes $\mathfrak{q}^+$ and, moreover, the order f_q of the group $G_{\mathfrak{q}} = \{\widetilde{\gamma} \in G \mid \widetilde{\gamma}(\mathfrak{q}) = \mathfrak{q}\}$ satisfies $f_q \mid \frac{p-1}{2}$. The action of $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$ on E_q is a permutation of order $c_{j,q} := (\frac{p-1}{2f_q}, j)$. Therefore the eigenvalues of this action are $c_{j,q}$-th roots of unity and E_q is a direct sum of invariant subspaces $E_{j,q}$ of dimension $c_{j,q}$. The characteristic polynomial of the restriction of the action of a generator $\gamma \in N(P)/C(P)$ on the invariant subspaces $E_{j,q}$ is $1-x^{c_{j,q}}$ and the characteristic polynomial of the action of γ on E is a polynomial of the form

$$\prod_{q \mid n} (1-x^{c_{j,q}})^{d_{j,q}},$$

where $d_{j,q}$ is the number of subspaces of E_q that are isomorphic to $E_{j,q}$. By the definition of $c_{j,q}$ we have $c_{j,q} \mid j$ and herewith the eigenvalues of the action of $\gamma \in N(P)/C(P)$ on the space E are jth roots of unity μ^k, $k = 0, \ldots, j-1$. The dimension of the invariant subspace E_p is 1 and $N(P)/C(P)$ acts trivially on E_p. Therefore $c_{j,p} = 1$ and $d_{j,p} = 1$. For our purpose it is important which roots of unity occur as eigenvalues but, as soon as it is nonzero, the multiplicity of those eigenvalues is irrelevant.

3.2.3. The action on the cohomology ring. We consider the action of $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$ on the p-primary part of $\widehat{\mathrm{H}}^{2k+m}(C(P),\mathbb{Z})$. We have seen in 3.2.1 that the action of a generator γ of $N(P)/C(P)$ on $x^k \in \mathbb{Z}/p\mathbb{Z}[x,x^{-1}]$ is given by multiplication with μ^k, where $\mu \in (\mathbb{Z}/p\mathbb{Z})^*$ is a primitive jth root of unity. If the same generator γ acts on $e \in \Lambda^m E$ by multiplication with μ^l, i.e., e is an eigenvector to the eigenvalue μ^l, then γ acts on

$$x^k \otimes e \in \widehat{\mathrm{H}}^{2k+m}\big(C(P),\mathbb{Z}\big)_{(p)}$$

by multiplication with $\mu^l\mu^k$. This shows that the element $x^k \otimes e$ is an eigenvector to the eigenvalue μ^{l+k} of the action of $\gamma \in N(P)/C(P)$. Since we are interested in the $N(P)/C(P)$-invariants of the p-primary part of $\widehat{\mathrm{H}}^i(C(P),\mathbb{Z})$, we are searching for the elements to the eigenvalue $1 = \mu^{l+k}$, $l+k \in j\mathbb{Z}$.

3.3. An example. Let $p = 7$, ξ a primitive seventh root of unity and $n := 7q$, where $q \in \mathbb{Z}$ is a prime such that the primes $\mathfrak{q}_i^+ \subset \mathbb{Z}[\xi+\xi^{-1}]$, $i = 1,2,3$, that lie over q split in $\mathbb{Z}[\xi]$, i.e., $f_q = 1$. Since f_q is the smallest positive integer that satisfies $q^{f_q} \equiv 1 \bmod 7$, we immediately see that $q := 29$ satisfies our condition and we therefore choose $n := 7\cdot 29 = 203$. The ideal $\mathfrak{p}^+ = ((1-\xi)(1-\xi^{-1}))$ is the prime in $\mathbb{Z}[\xi+\xi^{-1}]$ over $p = 7$. The centralizer of a subgroup P of order 7 in $\mathrm{Sp}(6,\mathbb{Z}[1/203])$ is

$$C(P) \cong \mathbb{Z}/14\mathbb{Z} \times \mathbb{Z}^4.$$

We know that $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$, $j = 1,3$. If $j = 1$, then $N(P) = C(P)$. We assume that we have chosen P with $N(P)/C(P) \cong \mathbb{Z}/3\mathbb{Z}$, i.e., $j = 3$. Such a subgroup always exists. A generator of the quotient $N(P)/C(P)$ acts as a permutation: we choose the numbering of the $\mathfrak{q}_i$ such that $\mathfrak{q}_1^+ \mapsto \mathfrak{q}_2^+$, $\mathfrak{q}_2^+ \mapsto \mathfrak{q}_3^+$, $\mathfrak{q}_3^+ \mapsto \mathfrak{q}_1^+$. We know that $\mathfrak{p}^+ \mapsto \mathfrak{p}^+$. Then, by 3.2.2, $\varepsilon_0 \in \mathfrak{p}$, $\varepsilon_i \in \mathfrak{q}_i^+$, and $e_0, e_i \in \mathrm{Hom}(\mathbb{Z}^4,\mathbb{Z})$, $i = 1,2,3$, exist such that the generator of $N(P)/C(P)$ acts as a permutation on $E := \langle e_0,e_1,e_2,e_3\rangle$, i.e.,

$$\begin{array}{lllllll} e_0 & \longmapsto & e_0, & \qquad & e_2 & \longmapsto & e_3,\\ e_1 & \longmapsto & e_2, & & e_3 & \longmapsto & e_1. \end{array}$$

We see that $E = E_7 \oplus E_{29}$, where $E_7 = \langle e_0\rangle$ and $E_{29} = \langle e_1,e_2,e_3\rangle$ are the subspaces that are invariant under the action of $N(P)/C(P)$. We have $c_{3,29} = 3$, $d_{3,29} = 1$ and $c_{3,7} = 1$, $d_{3,7} = 1$.

Now we consider the Farrell cohomology of the centralizer:

$$\widehat{\mathrm{H}}^*\big(C(P),\mathbb{Z}\big)_{(7)} \cong \mathbb{Z}/7\mathbb{Z}[x,x^{-1}] \otimes \Lambda_{\mathbb{Z}/7\mathbb{Z}}(e_0,e_1,e_2,e_3),$$

where $x \in \widehat{\mathrm{H}}^2(\mathbb{Z}/7\mathbb{Z},\mathbb{Z})$ and $e_0, e_i \in \mathrm{H}^1(\mathbb{Z}^4,\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}^4,\mathbb{Z})$, $i = 1,2,3$. The cohomology groups are $\mathbb{Z}/7\mathbb{Z}$-vector spaces:

$$\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)_{(p)} \cong \sum_{\substack{0\leqslant m\leqslant 4\\ m\equiv i \bmod 2}} \big\langle x^{\frac{i-m}{2}}\big\rangle \otimes \Lambda^m_{\mathbb{Z}/7\mathbb{Z}}(e_0,\dots,e_3), \tag{3.1}$$

where $\big\langle x^{\frac{i-m}{2}}\big\rangle$ is the $\mathbb{Z}/7\mathbb{Z}$-vector space spanned by $x^{\frac{i-m}{2}}$. The periodicity isomorhism is given by cup product with the element $x\otimes 1 \in \widehat{\mathrm{H}}^2(C(P),\mathbb{Z})_{(7)}$.

We consider the action of $N(P)/C(P) \cong \mathbb{Z}/3\mathbb{Z}$ on $\widehat{\mathrm{H}}^{2k+m}(C(P),\mathbb{Z})_{(7)}$ since we are searching for the invariants under this action. By 3.2.3 we first determine the eigenspaces of $\Lambda^m(e_0,e_1,e_2,e_3)$, $m = 0,\dots,4$, under the action of a generator γ of $\mathbb{Z}/3\mathbb{Z}$. The eigenvalues are the third roots of unity $1,2,4 \in (\mathbb{Z}/7\mathbb{Z})^*$. Then we choose the third root of unity $\mu = 2 \in \mathbb{Z}/7\mathbb{Z}$ for the action of $\mathbb{Z}/3\mathbb{Z}$ and get a $\mathbb{Z}/7\mathbb{Z}$-basis for any cohomology group:

$$\widehat{\mathrm{H}}^0\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)} = \big\langle 1\otimes 1,\ x^{-1}\otimes(e_0e_1+4e_0e_2+2e_0e_3),\ x^{-1}\otimes(2e_1e_2+3e_1e_3+e_2e_3)\big\rangle,$$

$$\widehat{\mathrm{H}}^1\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)} = \big\langle 1\otimes e_0,\ 1\otimes(e_1+e_2+e_3),\ x^{-1}\otimes(e_0e_1e_2+5e_0e_1e_3+4e_0e_2e_3)\big\rangle,$$

$$\widehat{\mathrm{H}}^2\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)} = \big\langle 1\otimes(e_0e_1+e_0e_2+e_0e_3),\ 1\otimes(e_1e_2-e_1e_3+e_2e_3)\big\rangle,$$

$$\widehat{\mathrm{H}}^3\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)} = \big\langle x\otimes(e_1+2e_2+4e_3),\ 1\otimes(e_1e_2e_3),\ 1\otimes(e_0e_1e_2-e_0e_1e_3+e_0e_2e_3)\big\rangle,$$

$$\widehat{\mathrm{H}}^4\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)} = \big\langle x\otimes(e_0e_1+2e_0e_2+4e_0e_3),\ x\otimes(e_1e_2+3e_1e_2+2e_2e_3),\ 1\otimes e_0e_1e_2e_3\big\rangle,$$

$$\widehat{\mathrm{H}}^5\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)} = \big\langle x^2\otimes(e_1+4e_2+2e_3),\ x\otimes(e_0e_1e_2+3e_0e_1e_3+2e_0e_2e_3)\big\rangle.$$

If we choose $\mu = 4 \in \mathbb{Z}/7\mathbb{Z}$, we get other generators for the cohomology rings, but the cohomology groups are isomorphic. Cup product with the invertible element

$$x^3\otimes 1 \in \widehat{\mathrm{H}}^6\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)}$$

yields a periodicity isomorphism of degree 6. But we see that more cohomology groups are isomorphic as $\mathbb{Z}/7\mathbb{Z}$-vector spaces. Indeed

$$\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)} \cong \widehat{\mathrm{H}}^{i+3}\big(C(P),\mathbb{Z}\big)^{\mathbb{Z}/3\mathbb{Z}}_{(7)}$$

for any $i \in \mathbb{Z}$. In this article we determine under which conditions this isomorphism exists. We first explain the general discussion in this example.

The dimension of the eigenspace of $\Lambda^m E$, $m = 0, \dots, 4$, of the eigenvalue μ^l, $l = 0, 1, 2$, $\mu \in (\mathbb{Z}/7\mathbb{Z})^*$, is given by the coefficients $D_m[l]$ of $t^m \mu^l$ in the polynomial

$$\begin{aligned} L(t,\mu) &:= \sum_{m,l} D_m[l] t^m \mu^l = (1+t)(1+t\mu)(1+t\mu^2)(1+t) \\ &= 1 + 2t + t\mu + t\mu^2 + 2t^2 + 2t^2\mu + 2t^2\mu^2 + 2t^3 + t^3\mu + t^3\mu^2 + t^4. \end{aligned}$$

The variable t counts the degree of the elements. By the isomorphism (3.1) we get

$$\dim\Big(\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)_{(p)}^{N(P)/C(P)}\Big) = \sum_{\substack{0 \leqslant m \leqslant 4 \\ m \equiv i \bmod 2}} D_m\Big[\frac{m-i}{2}\Big]$$

and this formula yields the dimensions of the cohomology groups that we explicitly determined before. In the next section we make the general discussion of the arguments that we presented here.

4. Isomorphisms in the cohomology ring

Let $N(P)$ denote the normalizer and $C(P)$ the centralizer of a subgroup P of order p of $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$. Let $n \in \mathbb{Z}$ be such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi + \xi^{-1}]$ are principal ideal domains and moreover $p \mid n$.

Proposition 4.1. *Let P be such that $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$ for a fixed odd $j > 0$ with $j \mid p-1$. Then for any $i \in \mathbb{Z}$*

$$\widehat{\mathrm{H}}^i\big(N(P),\mathbb{Z}\big)_{(p)} \cong \widehat{\mathrm{H}}^{i+b_j}\big(N(P),\mathbb{Z}\big)_{(p)}$$

with $b_j = j$ if and only if a prime $q \mid n$ exists with inertia degree f_q such that $j \mid \frac{p-1}{2f_q}$. If no such q exists, then $b_j = 2j$.

Proof. We consider the action of $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$ on the $\mathbb{Z}/p\mathbb{Z}$-vector space $\widehat{\mathrm{H}}^{2k+m}(C(P),\mathbb{Z})_{(p)}$. By 3.2.3 we are searching for elements

$$x^k \otimes e \in \widehat{\mathrm{H}}^{2k+m}\big(C(P),\mathbb{Z}\big)_{(p)}$$

to the eigenvalue 1. If a generator $\gamma \in N(P)/C(P)$ acts on $x \in \mathbb{Z}/p\mathbb{Z}[x,x^{-1}]$ by multiplication with μ and on $e \in \Lambda^m E$ by multiplication with μ^l, i.e., e is an eigenvector to the eigenvalue μ^l, then γ acts on $x^k \otimes e$ by multiplication with $\mu^k \mu^l$. We first consider the eigenspaces of $\Lambda^m E$ under the action of $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$. Since this quotient acts as a permutation on the space E spanned by $\{e_0, e_1, \dots, e_\sigma\}$, the eigenvalues are jth roots of unity $\mu^l \in (\mathbb{Z}/p\mathbb{Z})^*$, $l = 0, \dots, j-1$, and a basis $\{e'_0, e'_1, \dots, e'_\sigma\}$ of eigenvectors exists for E. The elements of a basis of eigenvectors of the space $\Lambda^m E$ are the products $e'_{i_1} \cdots e'_{i_m}$, $i_1 < \cdots < i_m$, where the eigenvalue of the product equals the product of the eigenvalues. Therefore the dimension of the eigenspace of $\Lambda^m E$ to the eigenvalue μ^l is given by the coefficients $D_m[l]$ of $t^m\mu^l$ in the polynomial

$$L(t,\mu) := \sum_{m,l} D_m[l]t^m\mu^l = \prod_{\substack{q|n \text{ splits} \\ \text{or } q=p|n}} \left(\prod_{k=1}^{c_{j,q}} \left(1 + t\mu^{k\frac{j}{c_{j,q}}}\right) \right)^{d_{j,q}}.$$

The variable t in $L(t,\mu)$ counts the degree of the elements. We have

$$\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)_{(p)} = \sum_{\substack{0 \leqslant m \leqslant \sigma+1 \\ m \equiv i \bmod 2}} \left\langle x^{\frac{i-m}{2}} \right\rangle \otimes \Lambda^m(e_0,\dots,e_\sigma),$$

where $\left\langle x^{\frac{i-m}{2}} \right\rangle$ is the $\mathbb{Z}/p\mathbb{Z}$-vector space spanned by $x^{\frac{i-m}{2}}$. We get

$$\dim\left(\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)_{(p)}^{N(P)/C(P)}\right) = \sum_{\substack{0 \leqslant m \leqslant \sigma+1 \\ m \equiv i \bmod 2}} D_m\left[\frac{m-i}{2}\right].$$

Consider the polynomial

$$L(z,z^{-2}) = \sum_l a_l z^l = \prod_{\substack{q|n \text{ splits} \\ \text{or } q=p|n}} \left(\prod_{k=1}^{c_{j,q}} \left(1 + z^{1-2k\frac{j}{c_{j,q}}}\right) \right)^{d_{j,q}}.$$

Herewith we get

$$\dim\left(\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)_{(p)}^{N(P)/C(P)}\right) = \sum_{l \equiv i \bmod 2j} a_l.$$

If $q \mid n$ exists with $c_{j,q} = j$, then the product

$$\prod_{k=1}^{j} 1 + z^{1-2k}$$

is a factor of the polynomial $L(z,z^{-2})$ and for $k = \frac{j+1}{2}$ (j is odd) we get the factor $1 + z^{-j}$ of $L(z,z^{-2})$. By Lemma 4.2

$$\dim\left(\widehat{\mathrm{H}}^i\big(C(P),\mathbb{Z}\big)_{(p)}^{N(P)/C(P)}\right) = \dim\left(\widehat{\mathrm{H}}^{i+j}\big(C(P),\mathbb{Z}\big)_{(p)}^{N(P)/C(P)}\right)$$

if and only if $1 + z^{j+2kj}$ is a factor of the polynomial $L(z,z^{-2})$ for some $k \in \mathbb{Z}$. This happens if and only if q exists with $c_{j,q} = j$, i.e., if and only if q exists with inertia degree f_q such that $j \mid \frac{p-1}{2f_q}$. The cohomology groups that have the same dimension are isomorphic as $\mathbb{Z}/p\mathbb{Z}$-vector spaces, but the isomorphism is not always a periodicity isomorphism. □

Lemma 4.2. *Let $\sum_{l\in\mathbb{Z}} a_l z^l$ be a polynomial with coefficients in $\mathbb{Z}$. Then*

$$\sum_{l \equiv i \bmod 2j} a_l = \sum_{l \equiv i+j \bmod 2j} a_l$$

if and only if $1 + z^{-j}$ *(or* $1 + z^{j+2kj}$ *for some* $k \in \mathbb{Z}$*) is a factor of the polynomial* $\sum_{l\in\mathbb{Z}} a_l z^l$.

Proof. Consider the polynomial $f(z) = \sum_{l\in\mathbb{Z}} a_l z^l$, $a_l \in \mathbb{Z}$. If a polynomial $g(z) = \sum_{l\in\mathbb{Z}} a'_l z^l$, $a'_l \in \mathbb{Z}$, exists with $f(z) = (1 + z^{\pm j})g(z)$, then

$$f(z) = g(z) + z^{\pm j} g(z) = \sum_{l\in\mathbb{Z}} a'_l z^l + \sum_{l\in\mathbb{Z}} a'_l z^{l\pm j}$$

and herewith

$$\sum_{l\,\equiv\, i \bmod 2j} a_l = \sum_{l\,\equiv\, i \bmod 2j} a'_l + \sum_{l\,\equiv\, i+j \bmod 2j} a'_l = \sum_{l\,\equiv\, i \bmod j} a'_l$$
$$= \sum_{l\,\equiv\, i+j \bmod 2j} a_l.$$

For the other direction we first consider the special case

$$\sum_{l\,\equiv\, 0 \bmod 2j} a_l = \sum_{l\,\equiv\, j \bmod 2j} a_l.$$

The value of the polynomial

$$\sum_{l\,\equiv\, 0 \bmod 2j} a_l (z^j)^{l/j} + \sum_{l\,\equiv\, j \bmod 2j} a_l (z^j)^{l/j}$$

is 0 in $z^j = -1$. Therefore $(1 + z^j)$ divides the polynomial. The cases

$$\sum_{l\,\equiv\, i \bmod 2j} a_l = \sum_{l\,\equiv\, i+j \bmod 2j} a_l$$

are analogous. The assumption now follows by an addition. □

Theorem 4.3. *Let* p *be an odd prime. Let* n *be such that* $\mathbb{Z}[1/n][\xi]$ *and* $\mathbb{Z}[1/n][\xi + \xi^{-1}]$ *are principal ideal domains and moreover* $p \mid n$*. Then*

$$\widehat{\mathrm{H}}^i\big(\mathrm{Sp}(p-1, \mathbb{Z}[1/n]), \mathbb{Z}\big)_{(p)} \cong \widehat{\mathrm{H}}^{i+b}\big(\mathrm{Sp}(p-1, \mathbb{Z}[1/n]), \mathbb{Z}\big)_{(p)}$$

for any $i \in \mathbb{Z}$*, with* $b = y$*, the greatest odd divisor of* $p-1$*, if and only if for each* $j \mid y$ *a prime* $q \mid n$ *exists with inertia degree* f_q *such that* $j \mid \frac{p-1}{2f_q}$*. If no such* q *exists, then* $b = 2y$*.*

Proof. If P is a subgroup of order p in $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$ that satisfies $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$, then we know by the proof of Proposition 4.1 that

$$\widehat{\mathrm{H}}^i\big(N(P), \mathbb{Z}\big)_{(p)} \cong \widehat{\mathrm{H}}^{i+b_j}\big(N(P), \mathbb{Z}\big)_{(p)}$$

with $b_j = j$ if and only if a prime $q \mid n$ exists with inertia degree f_q such that $j \mid \frac{p-1}{2f_q}$. If no such q exists, then $b_j = 2j$. In order to determine the degree b in our assumption, we let P run through the sets of conjugacy classes of subgroups of order p in $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$. Then the order j of $N(P)/C(P)$ runs through the odd divisors of y. By the isomorphism (1.1), the degree b

is the least common multiple of the b_j and this is $b = y$ if all the b_j are odd and $b = 2y$ if one of these numbers is even. This proves the theorem. □

5. The p-periodicity

Let $N(P)$ denote the normalizer and $C(P)$ the centralizer of a subgroup P of order p of $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$. Let $n \in \mathbb{Z}$ be such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi + \xi^{-1}]$ are principal ideal domains and moreover $p \mid n$.

5.1. The p-period of $\widehat{\mathrm{H}}^*\big(N(P), \mathbb{Z}\big)$.

Proposition 5.1. *Let P be such that $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$ for a fixed odd $j > 0$ with $j \mid p-1$. Then the periodicity isomorphism in*

$$\widehat{\mathrm{H}}^*\big(N(P), \mathbb{Z}\big)_{(p)}$$

is given by cup product with $x^j \otimes 1 \in \widehat{\mathrm{H}}^{2j}\big(N(P), \mathbb{Z}\big)_{(p)}$ and the period is $2j$.

Proof. By 3.1.1 the element

$$x \otimes 1 \in \widehat{\mathrm{H}}^2\big(C(P), \mathbb{Z}\big)_{(p)}$$

is invertible in the cohomology ring and cup product with $x \otimes 1$ yields the periodicity isomorphism. We know by 3.2.1 that the action of a generator of $N(P)/C(P)$ on $x \otimes 1$ is given by multiplication with a primitive jth root of unity $\mu \in (\mathbb{Z}/p\mathbb{Z})^*$. Therefore, by 3.1.1,

$$x^j \otimes 1 \in \widehat{\mathrm{H}}^{2j}\big(C(P), \mathbb{Z}\big)_{(p)}^{N(P)/C(P)} \cong \widehat{\mathrm{H}}^{2j}\big(N(P), \mathbb{Z}\big)_{(p)}$$

is invertible and cup product with $x^j \otimes 1$ yields the periodicity isomorphism. The period is $2j$. □

5.2. The p-period of $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbb{Z}[1/n]), \mathbb{Z})$.

Theorem 5.2. *Let $n \in \mathbb{Z}$ be such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains and $p \mid n$. Then the p-period of the Farrell cohomology ring*

$$\widehat{\mathrm{H}}^*\big(\mathrm{Sp}(p-1, \mathbb{Z}[1/n]), \mathbb{Z}\big)$$

equals $2y$, where y is the greatest odd divisor of $p-1$.

Proof. If P is a subgroup of order p in $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$ that satisfies $N(P)/C(P) \cong \mathbb{Z}/j\mathbb{Z}$, then we know by the proof of Proposition 5.1 that the periodicity isomorphism of the corresponding factor

$$\widehat{\mathrm{H}}^*\big(N(P), \mathbb{Z}\big)_{(p)}$$

in (1.1) is given by $x^j \otimes 1$ and therefore the period of this factor equals $2j$. If P runs through the sets of conjugacy classes of subgroups of order p in $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$, then the order j of $N(P)/C(P)$ runs through the odd divisors of y. Therefore the least common multiple of the j is y. If $x^j \otimes 1$ is invertible in the p-primary part of $\widehat{\mathrm{H}}^*\big(N(P), \mathbb{Z}\big)$, then $x^y \otimes 1$ is also invertible, because y is a multiple of j. Now, by (1.1), the periodicity isomorphism of

$$\widehat{\mathrm{H}}^*\big(\mathrm{Sp}(p-1,\mathbb{Z}[1/n]),\mathbb{Z}\big)_{(p)}$$

is given by cup product with $x^y \otimes 1$ and therefore the p-period is $2y$. □

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