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The Morava K-Theory Eilenberg–Moore spectral sequence

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Abstract. In this article I consider the convergence of the Eilenberg–Moore spectral sequence for Morava K-theory. This spectral sequence can be constructed by applying Morava K-theory to D. L. Rector's geometric cobar construction of the Eilenberg–Moore spectral sequence. I have shown that the Eilenberg–Moore spectral sequence for Morava K-theory converges if the Eilenberg–Moore spectral sequence for ordinary homology collapses at E^2 and the homology satisfies certain finiteness conditions.

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1. Introduction

This article concerns the convergence of the Eilenberg–Moore spectral sequence for computing the Morava K-theory of a homotopy pullback. We consider the special case associated to the path-loop fibration

$$\Omega B \to PB \to B.$$

The calculation of the homology of loop spaces for generalized homology theories such as Morava K-theory has been a long-standing problem. There are two good reasons to suspect that the Eilenberg–Moore spectral sequence could be a useful tool toward this end. First it can be constructed from a tower of cofibrations and therefor can be used for any generalized homology theory. (We will make particular use of D. L. Rector's construction of the Eilenberg–Moore spectral sequence from a tower of cofibrations.) And second for a generalized homology theory with Künneth isomorphisms the Eilenberg–Moore spectral sequence has a has a tractable (in theory anyway) E^2-page, namely $E^2 = \mathrm{Cotor}^{K_*(B)}(K_*, K_*)$.

Historically, for the Eilenberg–Moore spectral sequence, good convergence results are available in the case $K_* = H_*(-, F_p)$ [6, 7, 8, 25]. For other cohomology theories with Künneth isomorphisms, namely where K_* is one of Morava's K-theories, useful general convergence criterion have proven elusive. Since the early 1980's there have been a number of convergence results for the Eilenberg–Moore spectral sequence for Morava K-theory. For example in [29], Tamaki utilizes the Snaith splitting of $\Omega^n\Sigma^n X$ to show convergence for spaces of the form $\Omega^{n-1}\Sigma^n X$. In [12], Jeanneret and Osse consider the pullback diagram

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \longrightarrow & B \end{array}$$

in which B is connected. They prove that the spectral sequence converges to $E^*(X \times_B Y)$ when p is a fibration and $E^*(\Omega B)$ is an exterior algebra on finitely many odd-degree generators. In [17], Mahowald, Ravenel, and Shick construct what they call the Thomified Eilenberg–Moore spectral sequence which is based on Rector's construction and L. Smith's related construction. In [24], Shipley considers the convergence of spectral sequence constructed from cosimplicial spaces which includes Rector's construction as a particular example. More general criteria on the base space would go a long way toward making the Eilenberg–Moore spectral sequence for Morava K-theory more useful.

We will define the Eilenberg–Moore spectral sequence for Morava K-Theory as the spectral sequence that arises from applying Morava K-theory to the tower of cofibrations due to Rector.

Theorem 1.1. *The Eilenberg–Moore spectral for Morava K-Theory comes from applying Morava K-Theory to the tower of cofibrations*

(1.1)
$$\begin{array}{ccc}
 & & P(B) \simeq \Sigma^\infty \Omega B \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
\Sigma^{-n} B^{\wedge n} & \longrightarrow & P_n(B) \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
\Sigma^{-3} B^{\wedge 3} & \longrightarrow & P_3(B) \\
 & & \downarrow \\
\Sigma^{-2} B^{\wedge 2} & \longrightarrow & P_2(B) \\
 & & \downarrow \\
 & & P_1(B) \simeq \Sigma^{-1} B.
\end{array}$$

and has

$$E^2 = \mathrm{Cotor}^{K(n)_*(B)}(K(n)_*, K(n)_*)$$

Under the assumption that the spectral sequence converges for ordinary homology with field coefficients we are justified in writing $P(B) \simeq \Sigma^\infty \Omega B$. Upon applying Morava K-theory to the tower (1.1), $\lim_n K(m)_*(P_n(B))$ is the obvious target of the spectral sequence, however convergence is by no means guaranteed.

To see that we cannot expect the $K(m)_*$-based Eilenberg–Moore spectral sequence to converge in general note that for the n^{th} Eilenberg–Maclane space, $K(n, \mathbb{Z}/p)$.

$$K(m)_*(K(n, \mathbb{Z}/p)) = \begin{cases} \neq 0 & n \leq m \\ 0 & n > m. \end{cases}$$

So, consider the $K(m)_*$-based Eilenberg–Moore spectral sequence for

$$B = K(m+1, \mathbb{Z}/p).$$

If one applies $K(m)_*(-)$ to the tower (1.1), the target of the resulting spectral sequence is the associated graded, $\mathcal{G}(-)$, of the inverse limit, i.e.,

$$\mathcal{G}(\lim K(m)_*(P_n(B))).$$

If this spectral sequence converged as hoped the target should be an associated graded group for

$$K(m)_*(\Omega K(m+1, \mathbb{Z}/p)) = K(m)_*(K(m, \mathbb{Z}/p)).$$

But this is not the case since $K(m)_*(B) = K(m)_*$ and $K(m)_*(B^{\wedge n}) = K(m)_*$ but $K(m)_*(\Omega B) \neq K(m)_*$. That is the m^{th} Morava K-theory does not 'see' $K(m+1, \mathbb{Z}/p)$ but it does 'see' $K(m, \mathbb{Z}/p)$ so all of the groups in the Eilenberg–Moore spectral sequence are trivial but the hoped for target is not. In this case the $K(m)_*$-based Eilenberg–Moore spectral sequence clearly cannot converge to $K(m)_*(\Omega K(m+1, \mathbb{Z}/p))$.

In this article I show that under some reasonable hypotheses on the base space B, that this spectral sequence calculates the Morava K-theory of ΩB. Specifically:

Theorem 1.2. *Let B be a 1-connected CW complex with finitely many cells in each dimension such that the Eilenberg–Moore spectral sequence for mod p homology collapses at the E^2 page and*

$$\mathrm{Rank}\left(\bigoplus_i H_{j-i}(\Omega B, K(m)_i)\right)$$

is not infinite for adjacent j. Then the $K(m)_*$-based Eilenberg–Moore spectral sequence for B converges to $K(m)_*(\Omega B)$.

Here convergence is convergence in the sense of Boardman [4, Definition 5.2]. That is:

(1) The filtration exhausts G and there are isomorphisms $E_s^\infty \cong F_s/F_{s+1}$ for all s. (This alone is weak convergence.)
(2) The filtration of G is Hausdorff, i.e., $\bigcap_s F_s = \{0\}$.

The criteria for 1.2 come quite naturally from the method of proof which might be of interest in its own right.

To get at the convergence question we apply the Atiyah–Hirzebruch spectral sequence for $K(n)_*$ to all of the spaces in Rector's tower. We are motivated here by the fact that the Atiyah–Hirzebruch spectral sequence for $K(n)_*$ has E^∞ term $K(n)_*(B)$, so the inverse limit of the E^∞ terms has the same form as the Eilenberg–Moore spectral sequence for Morava K-theory. This process gives rise to a tower of spectral sequences which gives rise to some obvious questions:

(1) Let E be the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P(B))$, and $E(n)$ the Atiyah–Hirzebruch sequence for $K(m)_*(P_n(B))$. Is the spectral sequence E equal to $\lim E(n)$, the inverse limit of the spectral sequences $E(n)$?
(2) If the answer to question one is yes, what relationship holds between $\lim(K(m)_*(P_n(B)))$ and $K(m)_*(P(B))$?
(3) If we understand the answers to questions one and two, what does this say about the convergence of the $K(m)_*$-based Eilenberg–Moore spectral sequence for ΩB?

Notice that if the answer to the first of these questions is yes then, assuming the Atiyah–Hirzebruch spectral sequence for all of the spaces in the tower converge, we get a close relationship between, $\lim E(n)^\infty$, an associated graded group for $\lim(K(m)_*(P_n(B)))$ and, E^∞, an associated graded group for $K(m)_*(P(B)) = K(m)_*(\Sigma^\infty \Omega B)$. The criteria of Theorem 1.2 are precisely the criteria that guarantee that $\lim E(n)^\infty = E^\infty$ as graded groups, and they arise from a careful analysis of how differentials can lift in a tower of spectral sequences. Once we have this result we can compare the Eilenberg–Moore spectral sequence for Morava K-theory with the spectral sequence that results from filtering $\lim E(n)^\infty$ (an associated graded group for $K(m)_*(P(B)) = K(m)_*(\Sigma^\infty \Omega B)$) by the groups $E(n)^\infty$ which are associated graded groups for $K(m)_*(P_n(B))$. This leads to our result and practical criteria for the computation of the Morava K-theory of at least some spaces.

1.1. A computation. It is important to note that the assumptions of 1.2 apply to some interesting spaces. For example this spectral sequence will converge for any space where the loopspace has finite mod-p, homology or has only even or only odd-dimensional homology in mod-p homology. Examples include $\mathrm{SU}(m)/\mathrm{SU}(n)$, $B\mathrm{U}$, a product of odd spheres, any H-space with homology or cohomology an exterior algebra etc. In all of these cases the ordinary Eilenberg–Moore spectral sequence will collapse at E^2 since the differential lowers degree by 1 and $\bigoplus_i H_{j-i}(P(B), K(m)_i)$ will be concentrated in all even dimensions.

Example 1.3. As an example of a space whose Morava K-theory can be calculated with the $K(m)_*$-based Eilenberg–Moore spectral sequence consider $B_{m,n} = \mathrm{SU}(m)/\mathrm{SU}(n)$. To see that Theorem 1.2 applies recall that

$$H_*(B_{m,n}) = E(b_{2n+1}, b_{2n+3}, \ldots, b_{2m+1})$$

where $|b_{2n+1}| = 2n+1$. That is the homology of $B_{m,n}$ is an exterior algebra on generators in odd dimension. This implies that the Eilenberg–Moore spectral sequence for ordinary homology is concentrated in even degrees and hence collapses. Finally notice that $H_{p,q}(\Omega B_{m,n})$ is concentrated in even dimensions and so is not infinite in adjacent dimensions, so that Theorem 1.2 applies.

It is known that for any multiplicative homology theory

$$h_*(B_{m,n}) = E(b_{2n+1}, b_{2n+3}, \ldots, b_{2m+1}),$$

so it is a primitively generated exterior algebra on odd-dimensional generators (see [2]). We show in this case Cotor is a polynomial algebra. To see this first notice that although Cotor is defined as the dual of Tor, it can be identified with Ext in this case. To simplify notation here let $k = K(m)_*$ i.e., k is a graded field.

Lemma 1.4. *If P_* is a free resolution of M such that A and M are of finite type, then*

$$\mathrm{Cotor}_A(M, k) \cong \mathrm{Ext}_{A^*}(M^*, k).$$

Proof. Since A and M are of finite type, we can choose a free A-resolution P_* of finite type. Since linear duality of finite-dimensional vector spaces is an exact functor, P^* is a free A^*-resolution of M^*. Then

$$\begin{aligned}\mathrm{Ext}_{A^*}(M^*, k) &= H_*(\mathrm{hom}_{A^*}(P^*, k)) = H_*(\mathrm{hom}_{A^*}(P^*, \mathrm{hom}_k(k, k)))\\ &\cong H_*(\mathrm{hom}_k(P^* \otimes_{A^*} k, k)) \cong H_*((P \square_A k)^{**})\\ &\cong H_*(P \square_A k) = \mathrm{Cotor}_A(M, k). \qquad \square\end{aligned}$$

A standard Ext calculation then gives the following [13, 14]:

Lemma 1.5. *If E is a finitely generated primitive exterior algebra with generators x_i then*

$$\mathrm{Cotor}_E(k, k) = k(y_i)$$

where $\deg(y_i) = (1, \deg(x_i))$.

The E^2 page of the $K(m)_*$-based spectral sequence for computing $\Omega B_{m,n}$ is

$$\mathrm{Cotor}_{K(m)_*(B_{m,n})}(K(m)_*, K(m)_*) = k(y_i)$$
$$\deg(y_i) = (1, b_i).$$

Since $K(m)_*(B_{m,n})$ is concentrated in even degrees $E^2 = E^\infty$. So

$$\mathcal{G}(K(m)_*(B_{m,n})) = k(y_i).$$

Since it is not well-known, we begin by recalling Rector's construction of the Eilenberg–Moore spectral sequence.

1.2. Rector's construction. The original motivation for Rector's cosimplicial construction for the Eilenberg–Moore spectral sequence ([22]) was to show how the Steenrod operations interact with the spectral sequence structure. But his construction also allows one to construct the Eilenberg–Moore spectral sequence for generalized homology theories. Rector began with the cosimplicial space construction associated to a pullback. His construction proceeded as follows.

Assume B is a simply connected pointed space and start with a pullback diagram,

$$\begin{array}{ccc} E_f & \longrightarrow & E \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B. \end{array}$$

Let

$$\Delta : B \to B \times B$$

be the diagonal map. This is cocommutative, coassociative, and has a counit. We can thus create a cosimplicial space $G_*(X, B, E)$.

$$G_0(X, B, E) \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} G_1(X, B, E) \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} G_2(X, B, E) \cdots$$

where

$$G_n(E,B,X) = E \times B^n \times X,$$

$\delta^t : G_{s-1} \to G_s$ for $0 \leqq t \leqq s$ is given by

$$\delta^t(e, b_1, b_2, \dots, b_{s-1}, x) = \begin{cases} (e, f(e), b_1, b_2, \dots, b_{s-1}, x) & \text{if } t = 0 \\ (e, b_1, b_2, \dots, b_t, b_t, b_{t+1}, \dots, b_{s-1}, x) & \text{if } 1 \leqq t \leqq s-1 \\ (e, b_1, b_2, \dots, b_{s-1}, p(x), x) & \text{if } t = s, \end{cases}$$

and

$$\sigma^t(e, b_1, \dots, b_n, x) = (e, b_1, \dots, b_{t-1}, b_{t+1}, \dots, b_n, x).$$

For the pullback diagram above we have a natural weak homotopy equivalence [8, 9.2]

$$E_f \simeq_w \mathrm{Tot}(G_*(X,B,E))$$

From this he defined a sequence of cofibrations,

$$L_0(B) \to G_1^N(E,B,X) \to L_1(B)$$
$$L_1(B) \to G_2^N(E,B,X) \to L_2(B)$$
$$L_2(B) \to G_3^N(E,B,X) \to L_3(B)$$
$$\vdots$$
$$L_{n-1}(B) \to G_n^N(E,B,X) \to L_n(B).$$

Here $G_*^N(E,B,X)$ is the normalized cosimplicial space (see [32, Definition 8.3.6] example) and $L_{n-1}(B)$ is the cofiber of the induced map. This sequence of cofibrations induces an exact couple

$$D^1_{-p,q} = H_q(L_p(B))$$
$$E^1_{-p,q} = H_q(G_p^N(E,B,X)).$$

Rector then showed that the spectral sequence associated to this exact couple is isomorphic to the Eilenberg–Moore spectral sequence. (A good source for this is [30].)

In the language of spectra let $\overline{G}_n^N(E,B,X)$ and $\overline{L}_n(B)$ be the suspension spectra associated to these spaces. Define the spectra

$$E_s := \Sigma^{-s}\overline{G}_s^N(E,B,X)$$
$$P_s(B) := \Sigma^{-s}\overline{L}_s(B).$$

At the level of spectra for $s \geqq 0$, δ^0 induces a map $P_s(B) \to E_s$ giving a cofiber sequence $P_s(B) \xrightarrow{h_s} E_s \xrightarrow{\partial_s} \Sigma P_{s+1}(B)$, where ∂_s is the projection from the topological quotient of G_s by one subspace to the quotient by a bigger subspace. One can obtain an exact couple and hence a spectral sequence by applying any homology theory to this sequence of cofibrations.

For the constructions in this paper we will specialize to the path-loop fibration, i.e.,

$$\begin{array}{ccc} \Omega B & \longrightarrow & PB \\ \downarrow & & \downarrow \\ * & \longrightarrow & B. \end{array}$$

For the path-loop fibration, E^s and $P_s(B)$ have a particularly nice form. For $s \geq 0$ there is a homology isomorphism [17]

$$H_*(E_s) = \Sigma^{-s} H_*(*) \otimes \overline{H}_*(B^{\wedge s}) = \Sigma^{-s} \overline{H}_*(B^{\wedge s})$$

where $\overline{H}$ denotes reduced homology. Since B is simply connected, the connectivity of E^s is at least $s-1$. The theorem below is a translation of Rector's theorem [22].

Theorem 1.6. *The Eilenberg–Moore spectral sequence comes from applying homology to the tower of cofibrations*

$$\begin{array}{ccc} & & P(B) \simeq \Sigma^\infty \Omega B \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ \Sigma^{-n} B^{\wedge n} & \longrightarrow & P_n(B) \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ \Sigma^{-3} B^{\wedge 3} & \longrightarrow & P_3(B) \\ & & \downarrow \\ \Sigma^{-2} B^{\wedge 2} & \longrightarrow & P_2(B) \\ & & \downarrow \\ & & P_1(B) \simeq \Sigma^{-1} B \end{array} \tag{1.2}$$

and has

$$E^2 = \mathrm{Cotor}^{H_*(B)}(k,k) \Rightarrow H_*(\Omega B).$$

Remark 1.7. In this formulation of the Eilenberg–Moore spectral sequence, convergence is transparent when $\pi_1(B) = 0$. Formally we know that a spectral sequence of this form converges to $\lim_n H_*(P_n(B))$. In this case since the maps $P_n(B) \to P_{n-1}(B)$ are increasingly connective, the maps $H_*(P_n(B)) \to H_*(P_{n-1}(B))$ are isomorphisms over a greater range and so the tower of homology groups is Mittag-Leffler. Therefore $\lim_n H_*(P_n(B)) = 0$, so $\lim_n H_*(P_n(B)) \cong H_*(P(B))$. Though convergence is transparent, one still has to compare it to Eilenberg and Moore's original approach to prove that it abuts to $H_*(\Omega X)$.

For any generalized homology theory we can use the tower (1.1) to define a spectral sequence.

Definition 1.8. The h_*-based Eilenberg–Moore spectral sequence is the spectral sequence that results from applying $h_*(-)$ to Rector's tower of cofibrations, (1.1). Furthermore, if $h_*(-)$ is a multiplicative homology theory with a perfect Künneth theorem for B, then

$$E^2 = \mathrm{Cotor}^{h_*(B)}(h_*, h_*).$$

Clearly this process will give rise to a spectral sequence defined by the exact couple $h_q(P_l(B)) = D_{l,q}$ and $h_q(\Sigma^{-l} B^{\wedge l}) = E_{l,q}$. This defines a right half-plane spectral sequence with exiting differentials. Since colim $D_{l,*} = 0$ if the spectral sequence converges, it converges strongly to $\lim D_{l,*}$ [4, 6.1b].

1.3. $K(m)_*$-based Eilenberg–Moore spectral sequence. We review the properties of Morava K-theory, $K(m)_*$, which are used in this paper. Fix a prime p. For each m there is a spectrum $K(m)$ with

$$\pi_*(K(0)) = K(0)_* = \mathbb{Q}$$

concentrated in degree 0, and

$$\pi_*(K(m)) = K(m)_* = \mathbb{Z}/p[v_m, v_m^{-1}]$$

with $|v_m| = 2(p^m - 1)$ for $m > 0$.

In addition $K(m)_*(-)$ has a perfect Künneth theorem; that is

$$K(m)_*(X \times Y) = K(m)_*(X) \otimes K(m)_*(Y).$$

The spectrum $K(m)$ is derived from complex cobordism and is characterized by the formal group of height m. For a definition and more complete discussion of the properties of $K(m)_*$ see [34].

Lemma 1.9. *If the $K(m)_*$-based Eilenberg–Moore spectral sequence converges for a CW spectrum B, it converges strongly to* $\lim K(m)_q(P_l(B))$.

Because the Morava K-theories have a perfect Künneth theorem

$$E^2 = \mathrm{Cotor}^{K(m)_*(B)}(K(m)_*, K(m)_*),$$

which in some cases will be amenable to calculation.

We would like this spectral sequence to converge to $K(m)_*(\Omega B)$ but it is unclear when it does so. The main subject of this paper is to establish some criteria which guarantee convergence to $K(m)_*(\Omega B)$.

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2. Analysis of the spectral sequence

Our goal is to give convergence conditions for the $K(m)_*$-based Eilenberg–Moore spectral sequence. One major tool I will use for the analysis of the Eilenberg–Moore spectral sequence is the Atiyah–Hirzebruch spectral sequence. The Atiyah–Hirzebruch spectral sequence is a spectral sequence used to calculate generalized homology theories in terms of ordinary homology. For a CW complex X, and a generalized homology theory $h_*(-)$, this spectral sequence has $E^2_{p,q} = H_p(X, h_q(*)) \Rightarrow h_{p+q}(X)$. Similarly if X is a spectrum and $h_*(-)$ a generalized homology theory, the Atiyah–Hirzebruch spectral sequence has $E^2_{p,q} = H_p(X, h_q(*)) \Rightarrow h_{p+q}(X)$. See, for example, [28, Chapter 15].

The Eilenberg–Moore spectral sequence for $K(m)_*$ comes from applying $K(m)_*(-)$ to a tower of cofibrations of spectra (from (1.1)). To analyze this spectral sequence we consider the Atiyah–Hirzebruch spectral sequence for each $P_n(B)$ with

$$E^2_{p,q} = H_p(P_n(B), K(m)_q) \Rightarrow K(m)_*(P_n(B)).$$

This yields a tower of spectral sequences. We will use this tower to understand $K(m)_*(\Omega B)$. To get information from this tower of spectral sequences there are several things we would like to know.

(1) Let E be the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P(B))$, and $E(n)$ the Atiyah–Hirzebruch sequence for $K(m)_*(P_n(B))$. Is the spectral sequence E equal to $\lim E(n)$, the inverse limit of the spectral sequences $E(n)$?

(2) If the answer to question one is yes, what relationship holds between $\lim(K(m)_*(P_n(B)))$ and $K(m)_*(P(B))$?

(3) If we understand the answers to questions one and two, what does this say about the convergence of the $K(m)_*$-based Eilenberg–Moore spectral sequence for ΩB?

3. Inverse limits of spectral sequences

In this section we will show that the inverse limit (over n) of the Atiyah–Hirzebruch spectral sequences for $K(m)_*(P_n(B))$ is the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P(B))$. First I will describe more specifically what I mean by the inverse limit of the Atiyah–Hirzebruch spectral sequences for $K(m)_*(P_n(B))$.

Let $\{E(n)\}$ be a sequence of spectral sequences with maps of spectral sequences $E(n) \to E(n-1)$. We form the tri-graded abelian groups $E^r_{p,q} = \lim E^r_{p,q}(P_n(B))$, and define differentials by taking the inverse limit of the differentials in the $E(n)$. That is if $d^r(n)$ is the r^{th} differential in $E(n)$ and $(x^n) \in E^r_{p,q}$ then $d^r((x^n)) = (d^r(n)(x^n))$. The resulting object is a spectral sequence provided that $H(E^r_{p,q}, d^r) = E^{r+1}_{p,q}$. This is equivalent to showing that $H(\lim_n E(n)^r_{p,q}, d^r) = \lim_n H(E(n)^r_{p,q}, d^r)$. To see when

$H(\lim_n E(n)^r_{p,q}, d^r) = \lim_n H(E(n)^r_{p,q}, d^r)$, notice that $(E^r_{p,q}, d^r)$ is a chain complex. Thus we can apply the following theorem due to Milnor. See [32, 3.5.8] for details.

Theorem 3.1. *Let $\cdots C_2 \to C_1 \to C_0$ be a tower of chain complexes of abelian groups satisfying the Mittag-Leffler condition and let $C = \lim_i C_i$. Then there is an exact sequence for each q:*

$$0 \to \lim{}^1 H_{q+1}(C_i) \to H_q(C) \to \lim H_q(C_i) \to 0.$$

So if $\lim^1 H_{q+1}(C_i) = 0$, the inverse limit of spectral sequences forms a spectral sequence. If, for example, the $E^r_{p,q}$ are all finite-dimensional vector spaces, then the hypothesis of Theorem 3.1. will hold for all $(E^r_{p,q}, d^r)$ and all $\lim^1$ terms of the homology will be 0.

Let $E(n)$ be the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P_n(B))$. It is not clear *a priori* that the inverse limit of Atiyah–Hirzebruch spectral sequence is the Atiyah–Hirzebruch spectral sequence of the inverse limit.

Lemma 3.2. *Let B be a 1-connected CW complex with finitely many cells in each dimension. Take $E(n)$ to be the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P_n(B))$ Then:*

(1) $\lim E(n)$ *is a spectral sequence.*
(2) *The canonical map from the Atiyah–Hirzebruch spectral sequence for $K(m)_*(\Omega B)$ to $\lim E(n)$ is an isomorphism.*

Proof. The idea is to use the fact that the Eilenberg–Moore spectral sequence converges for ordinary homology to identify the E^2 terms of these spectral sequences. First, since B has only finitely many cells in each dimension, the $E^r_{p,q}$ are all finite-dimensional vector spaces, and $\lim E(n)$ is a spectral sequence. By construction we have a map from $P(B)$ to each of the $P_n(B)$ that is compatible with the maps in the tower. This gives us a map from the Atiyah–Hirzebruch spectral sequence for $P(B)$ to the Atiyah–Hirzebruch spectral sequence for each $P_n(B)$ that is compatible with the maps in the tower. By the universal property of the inverse limit we get a map Φ from the Atiyah–Hirzebruch spectral sequence for $P(B)$ to the inverse limit of the Atiyah–Hirzebruch spectral sequences for the $P_n(B)$.

Since B is 1-connected, the Eilenberg–Moore spectral sequence for ΩB converges. This implies that $\lim H_*(P_n(B)) = H_*(P(B))$. Recall that $E(n)^2_{p,q} = H_p(P_n(B), K(m)_q)$ and that $P(B) = \Sigma^\infty \Omega B$. So

$$\begin{aligned}\lim E(n)^2_{p,q} &= \lim H_p(P_n(B), K(m)_q)\\ &= H_p(P(B), K(m)_q)\\ &= H_p(\Omega B, K(m)_q).\end{aligned}$$

Since the Atiyah–Hirzebruch spectral sequence for $K(m)_*(\Omega B)$ has

$$E^2_{p,q} = H_p(\Omega B, K(m)_q)$$

this implies that Φ is an isomorphism at the E^2 page. The isomorphism follows from the standard comparison theorem for spectral sequences. See [4, 5.3] for a precise statement. □

4. The E^∞ term of the inverse limit spectral sequence

We would like to prove that $\lim K(m)_*(P_n(B)) = K(m)_*(P(B))$. This would give strong convergence on the $K(m)_*$-based Eilenberg–Moore spectral sequence. However $\lim K(m)_*(P_n(B)) \neq K(m)_*(P(B))$ even under some very restrictive conditions. So instead we will show that

$$\lim \mathcal{G}(K(m)_*(P_n(B))) = \mathcal{G}(K(m)_*(P(B))).$$

First we must show that $\lim(E(n)^\infty)$ is the same as $(\lim E(n))^\infty$. Recall that for each $P_n(B)$ there is an Atiyah–Hirzebruch spectral sequence that converges strongly to $K(m)_*(P_n(B))$. In other words $E(n)^\infty$ is the associated graded to $K(m)_*(P_n(B))$. It seems natural to guess that the E^∞ term of a spectral sequence defined as the inverse limit of spectral sequences would have as its E^∞ term the inverse limit of the E^∞ terms. We shall see that this is not necessarily the case. First we must analyze $\lim(E(n)^\infty)$ and $(\lim E(n))^\infty$ and determine how they are related.

Proposition 4.1. *Let B be a CW complex with finitely many cells in each dimension. The canonical map $\Phi : (\lim E(n))^\infty \to \lim (E(n)^\infty)$ is onto.*

This map is not one-to-one in general (see below).

Proof. Let $(\overline{x}_n)$ be a class in $\lim (E(n)^\infty)$ with $\overline{x}_n \in E(n)^\infty$. Define $M_n \in E(n)^2$ as $M_n = \{x : \overline{x} = \overline{x}_n \text{ in } E(n)^\infty\}$. Since B has only finitely many cells in each dimension, this is a finite-dimensional vector space. We now have a tower

$$\begin{array}{c} \downarrow \\ \vdots \\ \downarrow \\ M_{k+2} \\ \downarrow \\ M_{k+1} \\ \downarrow \\ M_k. \end{array}$$

Since this is a tower of finite-dimensional vector spaces, $\lim M_n \neq 0$. This gives an element $(x_n) \in (\lim E(n))_2 = (\lim E(n)_2)$ such that

$$\Phi(\overline{(x_n)}) = (\overline{x}_n). \qquad \square$$

We now show that $\Phi : (\lim E(n))^\infty \to \lim (E(n)^\infty)$ may not be one-to-one. Let $x_n \in E^2(n)$ with $x_n \mapsto x_{n-1}$. Suppose that $x_n \in \operatorname{im}(d^{r_n})$ and $\lim_{n\to\infty} r_n = \infty$. Then (x_n) is not a boundary in $\lim (E(n))$ so $(x_n) \neq 0$ in $(\lim E(n))^\infty$ even though $x_n = 0$ in $E(n)^\infty$ for each n (see Figure 1). Thus $\overline{(x_n)} \neq 0$ in $(\lim E(n))^\infty$ but $\Phi(\overline{x}_n) = (\overline{x}_n) = 0$.

(4.1)

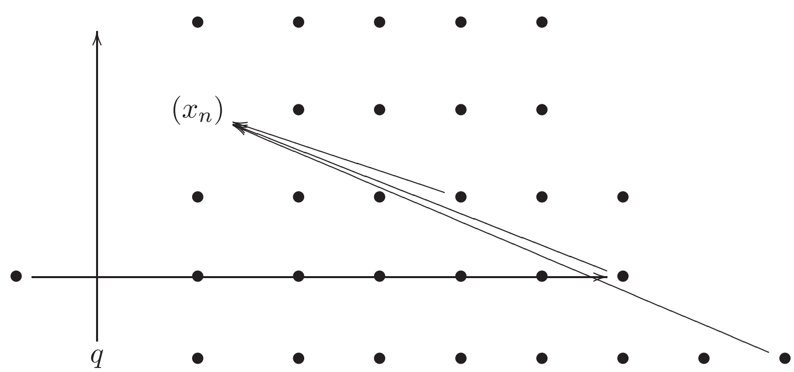

FIGURE 1. This diagram is meant to represent the superimposed pages of the Atiyah–Hirzebruch spectral sequence for three of the $P_n(B)$. The arrows shown are the differential hitting x_n in $P_n(B)$. This illustrates that as n increases so may the length of the differential hitting x_n. If this trend holds for all n then in the inverse limit spectral sequence there will be no differential hitting (x_n). This phenomenon is central in both our convergence results and nonconvergence examples.

5. The E^∞ term of the Eilenberg–Moore spectral sequence for Morava K-theory

Recall the $K(m)_*$-based Eilenberg–Moore spectral sequence results from applying $K(m)_*(-)$ to Rector's tower of cofibrations (1.1). Recall also that if the spectral sequence converges, it has E^∞ term [4, 6.1b]

$$\mathcal{G}(\lim D^{l,*}) = \mathcal{G}(\lim K(m)_*(P_n(B))).$$

Under certain conditions we will show that $\mathcal{G}(\lim D^{l,*})$ is an associated graded of $K(m)_*(P(B))$ with an exhaustive Hausdorff filtration.

Lemma 5.1. *If B has only finitely many cells in each dimension and*

$$\lim(E(n)^\infty) = (\lim E(n))^\infty$$

then:

(1) *The canonical map* $\Phi : K(m)_*(P(B)) \to \lim K(m)_*(P_n(B))$ *induces a morphism of filtered groups which is an isomorphism of their respective associated graded groups. That is*

$$\mathcal{G}(\lim K(m)_*(P_n(B))) \cong \mathcal{G}(K(m)_*(P(B))).$$

The filtration on $K(m)_*(P(B))$ *is exhaustive and Hausdorff.*

(2) *The canonical map* $K(m)_*(P(B)) \to \lim K(m)_*(P_n(B))$ *is one-to-one.*

Proof. First recall the E^∞ term of the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P_n(B))$ is $\mathcal{G}(K(m)_*(P_n(B)))$. The filtration for $\lim(E(n)^\infty)$ is the inverse limit of the filtrations for $E(n)^\infty$, that is

$$F_k = \lim_n \left(\operatorname{Im}\left(K(m)_*((P_n(B))^{(k)}) \xrightarrow{i} K(m)_*(P_n(B))\right)\right).$$

This is an increasing filtration on $\lim_n K(m)_*(P_n(B))$. In Lemma 3.2 we proved that for B a simply connected CW complex with finitely many cells in each dimension, the inverse limit spectral sequence is isomorphic to the Atiyah–Hirzebruch spectral sequence for $P(B)$. The latter spectral sequence has E^∞ term $\mathcal{G}(K(m)_*(P(B)))$, so we have a homomorphism of filtered groups that is an isomorphism at the level of associated graded groups. Finally the filtration on $\mathcal{G}(K(m)_*(P(B)))$ is the filtration from the Atiyah–Hirzebruch spectral sequence for $P_n(B)$ which is both exhaustive and Hausdorff [4, 12.6]. □

Corollary 5.2. *If* B *has only finitely many cells in each dimension and*

$$\lim(E(n)^\infty) = (\lim E(n))^\infty,$$

then the $K(m)_*$*-based Eilenberg–Moore spectral sequence for* B *converges.*

Proof. Recall that convergence of a spectral sequence to a group G requires that [4, Definition 5.2]:

(1) The filtration exhausts G and there are isomorphisms $E_s^\infty \cong F_s/F_{s+1}$ for all s. (This alone is weak convergence.)

(2) The filtration of G is Hausdorff.

Let F_n be the filtration for the $K(m)_*$-based Eilenberg–Moore spectral sequence. This spectral sequence formally converges to $\lim K(m)_*(P_n(X))$ and has a canonical filtration

$$F_n = \operatorname{Ker}\left(K(m)_*(\lim P_n(X)) \to K(m)_*(P_n(X)\right).$$

F_n is a complete, exhaustive filtration for $(\lim K(m)_*(P_n(X))$ [4, 5.4]. The map Φ induces a filtration, F', on $K(m)_*(\Omega B)$ with

$$F'_n = F_n \cap K(m)_*(\Omega B).$$

Since Φ is one-to-one, $\cap_n F'_n = \cap_n F_n = 0$, so F'_n is a Hausdorff filtration. Also $K(m)_*(\Omega B) = F'_{-1}$ so F'_n is exhaustive as well. Finally to see that

$F_n/F_{n+1} = F'_n/F'_{n+1}$ note that we have a commutative diagram,

(5.1)
$$\begin{array}{ccc} K(m)_*(\Omega B) & \longrightarrow & K(m)_*(P_k B) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \cong} \\ \lim_n K(m)_*(P_n B) & \xrightarrow{i_k} & K(m)_*(P_k B) \end{array}$$

where Φ is one-to-one. And thus, in Boardman's sense, the spectral sequence converges to $K(m)_*(P(X))$. □

6. Results

6.1. Main theorem.

Theorem 6.1. *Let B be a 1-connected space defined by a CW complex with finitely many cells in each degree such that the Eilenberg–Moore spectral sequence for ordinary homology collapses at the E^2 page and*

$$\mathrm{Rank}\left(\bigoplus_i H_{j-i}(\Omega B, K(m)_i)\right)$$

is not infinite for consecutive values of j. Then the Morava K-theory Eilenberg–Moore spectral sequence for B converges to $K(m)_*(\Omega B)$.

Recall that $\bigoplus_i H_{j-i}(\Omega B, K(m)_i)$ consists of all elements of total degree j on the E^2 page of the Atiyah–Hirzebruch spectral sequence for ΩB.

Proof. We consider the unraveled exact couple that forms the Eilenberg–Moore spectral sequence for ordinary homology.

(6.1)
$$\begin{array}{ccccc} & & \vdots & & \\ & & \downarrow{\scriptstyle i} & & \\ H_*(\Sigma^{-n}B^{\wedge n}) & \xrightarrow[j_n]{} & H_*(P_n(B)) & \xrightarrow[k_n]{} & H_*(\Sigma^{-n}B^{\wedge n+1}) \\ & & \downarrow{\scriptstyle i_n} & & \\ H_*(\Sigma^{-n+1}B^{\wedge n-1}) & \xrightarrow[j_{n-1}]{} & H_*(P_{n-1}(B)) & \xrightarrow[k_{n-1}]{} & H_*(\Sigma^{-n+1}B^{\wedge n}) \\ & & \downarrow{\scriptstyle i_{n-1}} & & \\ H_*(\Sigma^{-n+2}B^{\wedge n-2}) & \xrightarrow[j_{n-2}]{} & H_*(P_{n-2}(B)) & \xrightarrow[k_{n-2}]{} & H_*(\Sigma^{-n+2}B^{\wedge n-1}) \\ & & \downarrow{\scriptstyle i_{n-2}} & & \\ & & \vdots & & \end{array}$$

Recall that $d^1 = k \circ j$ and d^n is constructed by applying j, lifting through i_{n-1}, and applying k.

Represent a permanent cycle in the Atiyah–Hirzebruch spectral sequence for $K(m)_*(\Omega B)$ by (x_n) where $x_n \in E(n)^2$. We are using Lemma 3.2 which identifies the Atiyah–Hirzebruch spectral sequence for $K(m)_*(\Omega B)$ with the inverse limit of Atiyah–Hirzebruch spectral sequences for $K(m)_*(P_n B)$.

We wish to construct our proof by showing that under our hypothesis, the hypotheses of Corollary 5.2 hold. We need to show that under the above assumptions if for all n, $x_n = 0 \in E(n)^\infty$ then $(x_n) = 0 \in (\lim E(n))^\infty$. By abuse x_n represents both an element of the E^2 page which is a permanent cycle, and its class on the E^∞ page. In short we need to rule out something like the situation described by Diagram (4.1). The key fact here is that, by assumption, the only nontrivial differential in the Eilenberg–Moore spectral sequence is d^1.

Suppose $d^{i_n}(y_n) = x_n$ in the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P_n B)$ where the i_n are increasing monotonically. If we can show that $\lim_{n\to\infty} i_n$ is finite, then we will have shown that $(x_n) = 0$ in $(\lim E(n))^\infty$. We investigate what circumstances can lead to $i_{n+1} > i_n$. Since we are really worried about $i_{n+1} > i_n$, for infinitely many n, we may begin by assuming n is large. In particular, we take advantage of the fact that our hypotheses guarantee that the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P_n B)$ stabilizes in any fixed dimension for n sufficiently large. So with out loss of generality, assume n is large enough so that $H_*(P_{n+1}) \to H_*(P_n)$ is an isomorphism in the dimension of x_n.

There are two possible ways to have $i_{n+1} > i_n$: either y_n does not lift to y_{n+1}, and x_{n+1} is in the image of a longer differential, or there is a lift of y_n, say $\widetilde{y}_n$, but $\widetilde{y}_n$ supports a shorter differential than d^{i_n}. These two situations are summarized graphically in Diagram (6.2) and Diagram (6.3) in Figures 2 and 3, respectively.

The situation described under (6.2) does not happen under our hypotheses. To see why, assume $y_n \in E(n)^2$ does not lift to a $y_{n+1} \in E(n+1)^2$. This implies $k_n(y_n) = b \neq 0 \in H_*(\Sigma^{-n} B^{\wedge n+1})$. Let y_{n-s} be the image of y_n in $H_*(P(n-s))$. Choose s so that $y_{n-s} \neq 0$ in $H_*(P(n-s))$ but $y_{n-s-1} = 0$ in $H_*(P(n-s-1))$. Then $y_{n-s} = j_{n-s}(\alpha)$. This shows $d^{s+1}(\alpha) = b$ in the Eilenberg–Moore spectral sequence for ΩB. But since d^1 is the only nontrivial differential in that Eilenberg–Moore spectral sequence, $d^{s+1} = 0$ if $s > 0$. So b must be 0 on the E^{s+1} page and so it must be 0 on the E^2 page of the Eilenberg–Moore spectral sequence. Choose a so that $d^1(a) = b$, and let $y'_n = j_n(a)$. Let $\lambda_n = y_n - y'_n$, and notice that $k(\lambda_n) = 0$ so λ_n lifts to $\lambda_{n+1} \in P(n+1)$ with $i(\lambda_{n+1}) = \lambda_n$. Finally

$$d^{i_n}(\lambda_n) = d^{i_n}(y_n - y'_n) = d^{i_n}(y_n) - d^{i_n}(j_n(a)) = d^{i_n}(y_n) = x_n,$$

and so $d^{i_n}(\lambda_{n+1}) = x_{n+1}$ Here we are using the assumption that n is large enough so that x_{n+1} is determined by x_n. So the situation described in Diagram (6.2) cannot happen.

(6.2)

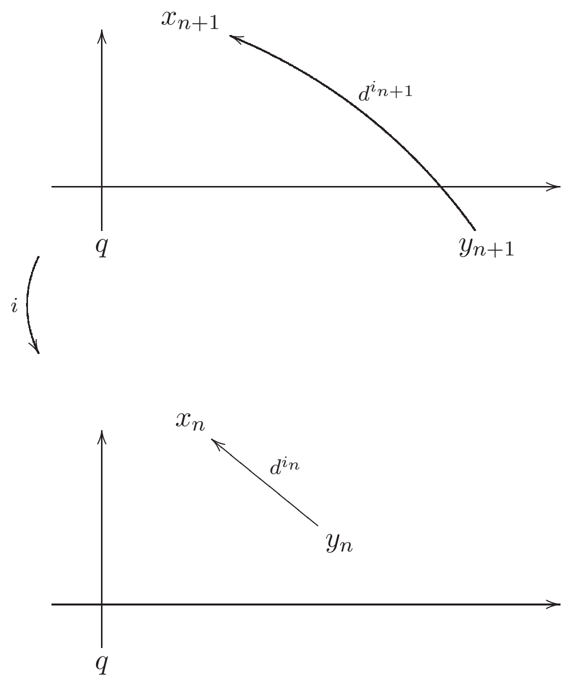

FIGURE 2. This is an illustration of elements on the E^2 pages of the Atiyah–Hirzebruch spectral sequences for $K(m)_*(P_{n+1}(B))$ and $K(m)_*(P_n(B))$ with higher differentials superimposed. In this diagram y_n does not lift and $y_n, x_n \in E(n)^2$ while $x_{n+1}, y_{n+1} \in E(n+1)^2$.

The situation described in Diagram (6.3) can happen. For (x_n) to be nonzero in $(\lim E(n))^\infty$ it would have to happen for infinitely many n. Suppose this is the case: let $(x_n) \in (\lim E(n))^2$ such that $x_n = 0 \in E(n)^\infty$ for all n with $(x_n) \neq 0 \in (\lim E(n))^\infty$. Also let $x_n \in H_{j-i}(P_n(B), K(m)_i)$. Recall that in the Atiyah–Hirzebruch spectral sequence for homology the differential lowers total degree by one, so that all elements that can support a differential with image x_n are in $\bigoplus_{p+q=j+1} H_p(P_n(B), K(m)_q)$. In the above proof for the impossibility of the scenario in Diagram (6.2) we show that if an element y_n on the E^2 page of the Atiyah–Hirzebruch spectral sequence for $P_n(B)$ supports a differential, there is an element on the E^2 page of the Atiyah–Hirzebruch spectral sequence for $P_{n+1}(B)$ that supports a differential with image x_{n+1}. As a consequence each $y_n \in E(n)$ gives rise to an element $(y_n) \in (\lim E(n))^2$. This means that under the assumption that the situation described in Diagram (6.3) happens an infinite number of times the rank of $\bigoplus_{p+q=j+1} H_p(P(B), K(m)_q)$ is infinite. Furthermore since each y_n supports a differential, and the i_n are increasingly connective,

(6.3)

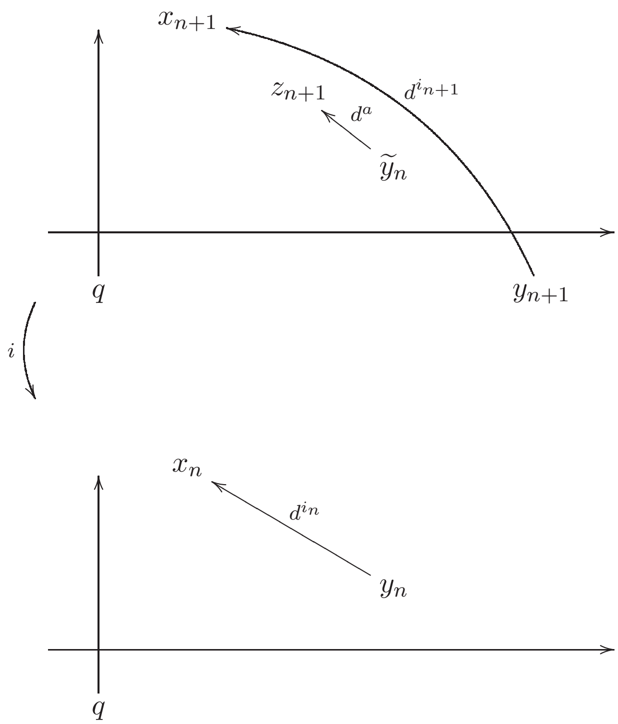

FIGURE 3. This is an illustration similar to Diagram 6.2, but in this case y_n does lift but y_{n+1} is 0 on the $E(n)^{i_n}$ page of the Atiyah–Hirzebruch spectral sequence for $K(m)_*(P_{n+1}B)$. In this diagram $y_n, x_n \in E(n)^2$ while $x_{n+1}, y_{n+1}, \widetilde{y_n}, z_{n+1} \in E(n+1)^2$.

(y_n) supports a differential in $\lim(E(n))$. One consequence of this is that for each $y_n \in (\lim E(n))^2$ there is an $x_n \in (\lim E(n))^2$. Since the differential lowers dimension by one, if y_n is in total degree $j+1$, x_n is in total degree j. As a result the rank of $\bigoplus_{p+q=j} H_p(P(B), K(m)_q)$ would also be infinite. This contradicts the assumptions of Theorem 6.1 so the situation described in Diagram (6.3) can happen at most a finite number of times.

We have shown that under the assumptions of Theorem 6.1,

$$\mathcal{G}(\lim D^{l,*}) = \mathcal{G}(\lim K(m)_*(P_n(B)))$$

so by Lemma 5.1 the $K(m)_*$-based Eilenberg–Moore spectral sequence converges. □

6.2. Secondary result.

Theorem 6.2. *Let B be a space defined by a finite CW complex such that the Eilenberg–Moore spectral sequence for ordinary homology collapses at the E^2*

page and the Atiyah–Hirzebruch spectral sequence for Morava K-theory for $P_n(B)$ collapses at the E_k page for all n and some fixed k. Then the Morava K-theory Eilenberg–Moore spectral sequence for B converges to $K(m)_*(\Omega B)$.

We use the following lemma. (The proof is not given but hopefully obvious.)

Lemma 6.3. *For any space B, if the $K(m)_*$ Atiyah–Hirzebruch spectral sequence for B collapses then the Atiyah–Hirzebruch spectral sequence for $B^{\wedge n}$ also collapses.*

Proof. As we have seen, the $K(m)_*$-based Eilenberg–Moore spectral sequence converges under any conditions which rule out the scenarios described by Diagrams (6.2) and (6.3) occurring an infinite number of times. Since the Eilenberg–Moore spectral sequence for ordinary homology collapses at E^2, We know that Diagram (6.2) cannot occur by our previous argument. It suffices to prove that the situation described by Diagram (6.3) can occur at most a finite number of times, but this is clear since we know that the longest differential in the Atiyah–Hirzebruch spectral sequence for any $P_n(B)$ is of length $k-1$. So we have shown that under the assumptions of Theorem 6.1,

$$\mathcal{G}(\lim D^{l,*}) = \mathcal{G}(\lim K(m)_*(P_n(B)))$$

so by Lemma 5.1 the $K(m)_*$-based Eilenberg–Moore spectral sequence converges. □

Remark 6.4. In practice it would be very hard to verify that the Atiyah–Hirzebruch spectral sequence for all $P_n(B)$ collapses at E^k for a given k. As we will show in a sequel to this paper even if the Atiyah–Hirzebruch spectral sequence for B has only 1 differential the Atiyah–Hirzebruch spectral sequence for $P_n(B)$ can have many nontrivial differentials.

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