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New binary and ternary digit extraction (BBP-type) formulas for trilogarithm constants

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Abstract. Not many degree-3 digit extraction (BBP-type) formulas are proved in literature. In this paper we present two binary and one ternary new digit extraction formulas, together with their proofs, for trilogarithm constants.

Contents

1. Introduction

The discovery and study of digit extraction formulas, especially BBP-type formulas, for mathematical constants have continued to receive much attention.

Apart from digit extraction, another reason the study of BBP-type formulas has continued to attract attention is that BBP-type constants are conjectured to be either rational or normal to base b [5, 7, 10, 3], that is their base-b digits are randomly distributed.

David Bailey maintains a Compendium of BBP-type formulas for Mathematical constants on his website [3]. A nice collection of such formulas may

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also be found in MathWorld [14] while there is a nice article on the subject in Wikipedia [15].

Experimentally, BBP-type formulas are usually discovered through computer searches, especially by using Bailey and Ferguson's PSLQ (Partial Sum of Least Squares) algorithm [11] or its variations. A downside is that PSLQ and other integer relation finding schemes typically do not suggest proofs [7, 4]. Formal proofs must be sought after the discovery of the formulas. There have been attempts in the past to give general formulas which include the proofs [6, 8, 9, 1, 5, 2, 12].

In the Compendium, only one degree 3 BBP-type formula is listed as having been proved, with the remaining formulas waiting to be proved. In this paper we give two identities which generate some degree 3 BBP-type formulas.

2. Generators of degree 3 BBP-type formulas

The trilogarithm function of the complex argument z, for $|z| < 1$, is defined by

$$\mathrm{Li}_3(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^3}\,.$$

Choosing $z = p \exp ix$, x, p real and $|p| < 1$, the real and imaginary parts of the trilogarithm function can be expressed as

$$\text{Re Li}_3(pe^{ix}) = \sum_{k=1}^{\infty} \frac{p^k \cos(kx)}{k^3} \tag{1}$$

and

$$\text{Im Li}_3(pe^{ix}) = \sum_{k=1}^{\infty} \frac{p^k \sin(kx)}{k^3}. \tag{2}$$

Setting $p = \sin\theta$ and $x = \theta - \pi/2$, Equation (1) can be written

$$\text{Re Li}_3\left[\sin\theta e^{i(\theta-\pi/2)}\right] = \sum_{k=1}^{\infty} \frac{\sin^k\theta \cos\left[k(\theta-\pi/2)\right]}{k^3}. \tag{3}$$

The left hand side of Equation (3) can be evaluated (see reference [13]), giving

$$\begin{aligned}\text{Re Li}_3\left[\sin\theta e^{i(\theta-\pi/2)}\right] = {} & \frac{7}{16}\zeta(3) + \frac{1}{8}\mathrm{Li}_3(\sin^2\theta) + \frac{1}{2}\theta^2 \ln\sin\theta \\ & - \frac{1}{4}\mathrm{Cl}_3(2\theta) + \frac{1}{4}\mathrm{Cl}_3(\pi - 2\theta),\end{aligned} \tag{4}$$

where Cl_3 is a generalized Clausen integral defined by

$$\mathrm{Cl}_3(y) = \zeta(3) - \int_0^y \mathrm{Cl}_2(x)\mathrm{d}x$$

with ζ the Riemann Zeta function and Cl_2 the Clausen integral defined by

$$\mathrm{Cl}_2(y) = -\int_0^y \ln|2\sin(x/2)|\,\mathrm{d}x.$$

Combining Equation (3) and Equation (4), we obtain the following generator of degree 3 BBP-type formulas:

$$\frac{7}{16}\zeta(3) + \frac{1}{8}\mathrm{Li}_3(\sin^2\theta) + \frac{1}{2}\theta^2\ln\sin\theta - \frac{1}{4}\mathrm{Cl}_3(2\theta) + \frac{1}{4}\mathrm{Cl}_3(\pi - 2\theta) = \sum_{k=1}^{\infty}\frac{\sin^k\theta\cos[k(\theta-\pi/2)]}{k^3}. \tag{5}$$

Explicit BBP-type formulas from Equation (5) will be discussed in Section 3.

Setting $p = \tan\theta$ and $x = \pi/2 - 2\theta$ Equation (1) can be written

$$\mathrm{Re}\ \mathrm{Li}_3\left[\tan\theta e^{i(\pi/2-2\theta)}\right] = \sum_{k=1}^{\infty}\frac{\tan^k\theta\cos[k(\pi/2-2\theta)]}{k^3}. \tag{6}$$

Again the left hand side of Equation (6) can be evaluated [13] thus

$$\mathrm{Re}\ \mathrm{Li}_3\left[\tan\theta e^{i(\pi/2-2\theta)}\right] = \frac{5}{16}\zeta(3) + \frac{1}{4}\mathrm{Li}_3(\tan^2\theta) - \frac{1}{8}\mathrm{Li}_3(-\tan^2\theta) + \theta^2\ln\tan\theta + \frac{1}{4}\mathrm{Cl}_3(\pi-4\theta) - \frac{1}{8}\mathrm{Cl}_3(4\theta). \tag{7}$$

Combining Equation (6) and Equation (7), we obtain yet another generator of degree 3 BBP-type formulas:

$$\frac{5}{16}\zeta(3) + \frac{1}{4}\mathrm{Li}_3(\tan^2\theta) - \frac{1}{8}\mathrm{Li}_3(-\tan^2\theta) + \theta^2\ln\tan\theta + \frac{1}{4}\mathrm{Cl}_3(\pi-4\theta) - \frac{1}{8}\mathrm{Cl}_3(4\theta) = \sum_{k=1}^{\infty}\frac{\tan^k\theta\cos[k(\pi/2-2\theta)]}{k^3}. \tag{8}$$

The explicit BBP-type formulas from Equation (8) will be discussed in Section 4.

3. BBP-type formulas generated by Equation (5)

3.1. $\theta = \pi/4$ in Equation (5). Plugging $\theta = \pi/4$ in Equation (5) gives

$$\frac{1}{48}\ln^3 2 - \frac{5\pi^2}{192}\ln 2 + \frac{35}{64}\zeta(3) = \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k\frac{\cos(k\pi/4)}{k^3}. \tag{9}$$

By noting that

$$\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k\frac{\cos(k\pi/4)}{k^3} = \frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{16^k}\left[\frac{8}{(8k+1)^3}-\frac{4}{(8k+3)^3}-\frac{4}{(8k+4)^3} - \frac{2}{(8k+5)^3}+\frac{1}{(8k+7)^3}+\frac{1}{(8k+8)^3}\right], \tag{10}$$

and using this in Equation (9) we obtain the following binary BBP-type formula:

$$\frac{1}{3}\ln^3 2-\frac{5\pi^2}{12}\ln 2+\frac{35}{4}\zeta(3) = \sum_{k=0}^{\infty}\frac{1}{16^k}\left[\frac{8}{(8k+1)^3}-\frac{4}{(8k+3)^3}-\frac{4}{(8k+4)^3} -\frac{2}{(8k+5)^3}+\frac{1}{(8k+7)^3}+\frac{1}{(8k+8)^3}\right]. \tag{11}$$

3.2. $\theta = \pi/6$ in Equation (5). Inserting $\theta = \pi/6$ in Equation (5) we have

$$\frac{1}{8}\mathrm{Li}_3\left(\frac{1}{4}\right)-\frac{\pi^2}{72}\ln 2+\frac{35}{144}\zeta(3)=\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\frac{\cos(k\pi/3)}{k^3}. \tag{12}$$

In obtaining Equation (12) we used the known values [13] $\mathrm{Cl}_3(\pi/3) = \zeta(3)/3$ and $\mathrm{Cl}_3(2\pi/3) = -4\zeta(3)/9$. By definition

$$\mathrm{Li}_3\left(\frac{1}{4}\right)=\sum_{k=1}^{\infty}\frac{1}{4^k}\frac{1}{k^3} = \frac{1}{64}\sum_{k=0}^{\infty}\frac{1}{64^k}\left[\frac{16}{(3k+1)^3}+\frac{4}{(3k+2)^3}+\frac{1}{(3k+3)^3}\right]. \tag{13}$$

We also note that

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\frac{\cos(k\pi/3)}{k^3} = \frac{1}{64}\sum_{k=0}^{\infty}\frac{1}{64^k}\left[\frac{16}{(6k+1)^3}-\frac{8}{(6k+2)^3}-\frac{8}{(6k+3)^3} - \frac{2}{(6k+4)^3}+\frac{1}{(6k+5)^3}+\frac{1}{(6k+6)^3}\right]. \tag{14}$$

Equation (13) and Equation (14) in Equation (12) yields the following binary digit extraction formula:

$$
\begin{aligned}
&35\zeta(3)-2\pi^2\ln 2\\
&\quad=\frac{9}{32}\sum_{k=0}^{\infty}\frac{1}{64^k}\left[\frac{128}{(6k+1)^3}-\frac{64}{(6k+2)^3}-\frac{64}{(6k+3)^3}-\frac{16}{(6k+4)^3}\right.\\
&\qquad\left.+\frac{8}{(6k+5)^3}+\frac{8}{(6k+6)^3}-\frac{16}{(3k+1)^3}-\frac{4}{(3k+2)^3}-\frac{1}{(3k+3)^3}\right].
\end{aligned}
$$

The above can be put in the standard BBP-type form:

$$
\begin{aligned}
&35\zeta(3)-2\pi^2\ln 2\\
&\quad=\frac{9}{4}\sum_{k=0}^{\infty}\frac{1}{64^k}\left[\frac{16}{(6k+1)^3}-\frac{24}{(6k+2)^3}-\frac{8}{(6k+3)^3}\right.\\
&\qquad\left.-\frac{6}{(6k+4)^3}+\frac{1}{(6k+5)^3}\right].
\end{aligned}
\tag{15}
$$

4. BBP-type formula generated by Equation (8)

4.1. $\theta=\pi/6$ in Equation (8). Putting $\theta=\pi/6$ in Equation (8), we have

$$
\frac{13}{18}\zeta(3)+\frac{\ln^3 3}{48}-\frac{5\pi^2}{144}\ln 3=\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^k\frac{\cos(k\pi/6)}{k^3}.
\tag{16}
$$

In obtaining Equation (16), we made use of the identity [12]

$$
\mathrm{Li}_3\left(\frac{1}{3}\right)-\frac{1}{2}\mathrm{Li}_3\left(-\frac{1}{3}\right)=\frac{13\zeta(3)-\pi^2\ln 3+\ln^3 3}{12}.
$$

We also used the known values [13]

$$
\mathrm{Cl}_3(\pi/3)=\zeta(3)/3\quad\text{and}\quad\mathrm{Cl}_3(2\pi/3)=-4\zeta(3)/9.
$$

By noting that

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^k\frac{\cos(k\pi/6)}{k^3}\\
&\quad=\frac{1}{1458}\sum_{k=0}\frac{1}{729^k}\left[\frac{729}{(12k+1)^3}+\frac{243}{(12k+2)^3}-\frac{81}{(12k+4)^3}\right.\\
&\qquad-\frac{81}{(12k+5)^3}-\frac{54}{(12k+6)^3}-\frac{27}{(12k+7)^3}-\frac{9}{(12k+8)^3}\\
&\qquad\left.+\frac{3}{(12k+10)^3}+\frac{3}{(12k+11)^3}+\frac{2}{(12k+12)^3}\right].
\end{aligned}
\tag{17}
$$

and using this in Equation (16) we obtain the following ternary (base 3) BBP-type formula

$$
\begin{aligned}
(18)\qquad & \frac{13}{9}\zeta(3) + \frac{\ln^3 3}{24} - \frac{5\pi^2 \ln 3}{72} \\
&= \frac{1}{729}\sum_{k=0} \frac{1}{729^k}\left[\frac{729}{(12k+1)^3} + \frac{243}{(12k+2)^3} - \frac{81}{(12k+4)^3}\right. \\
&\quad - \frac{81}{(12k+5)^3} - \frac{54}{(12k+6)^3} - \frac{27}{(12k+7)^3} - \frac{9}{(12k+8)^3} \\
&\quad \left. + \frac{3}{(12k+10)^3} + \frac{3}{(12k+11)^3} + \frac{2}{(12k+12)^3}\right].
\end{aligned}
$$

5. Conclusion

Using straightforward, elementary techniques and without doing any computer searches, we have proved three digit extraction formulas for trilogarithm constants.

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