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Directed algebraic topology and higher dimensional transition systems

Philippe Gaucher

Abstract. Cattani–Sassone's notion of higher dimensional transition system is interpreted as a small-orthogonality class of a locally finitely presentable topological category of weak higher dimensional transition systems. In particular, the higher dimensional transition system associated with the labelled n-cube turns out to be the free higher dimensional transition system generated by one n-dimensional transition. As a first application of this construction, it is proved that a localization of the category of higher dimensional transition systems is equivalent to a locally finitely presentable reflective full subcategory of the category of labelled symmetric precubical sets. A second application is to Milner's calculus of communicating systems (CCS): the mapping taking process names in CCS to flows is factorized through the category of higher dimensional transition systems. The method also applies to other process algebras and to topological models of concurrency other than flows.

Contents

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1. Introduction

Presentation of the results. In directed algebraic topology, the concurrent execution of n actions is modelled by a full n-cube, each coordinate corresponding to one of the n actions. In this setting, a general concurrent process is modelled by a gluing of n-cubes modelling the execution paths and the higher dimensional homotopies between them. Various topological models are being studied: in alphabetic and non-chronological order, d-space [Gra03], d-space generated by cubes [FR08], flow [Gau03], globular complex [GG03], local po-space [FGR98], locally preordered space [Kri09], multipointed d-space [Gau09], and more [Gou03] (this list is probably not complete, indeed). The combinatorial model of labelled (symmetric) precubical set is also of interest because, with such a model, it is exactly known where the cubes are located in the geometry of the object. It was introduced for the first time in [Gou02] [Wor04], following ideas from [Dij68] [Pra91] [Gun94] [Gun01] [VG06] (the last paper is a recent survey containing references to older papers), and improved in [Gau08] [Gau10] in relation with the study of process algebras. The paper [Gau08] treated the case of labelled precubical sets, and the paper [Gau10] the more general cases of labelled symmetric precubical sets and labelled symmetric transverse precubical sets.

An apparently different philosophy is the one of higher dimensional transition system. This notion, introduced in [CS96], models the concurrent execution of n actions by a transition between two states labelled by a multiset of n actions. A multiset is a set with possible repetition of some elements (e.g., $\{0,0,2,3,3,3\}$). It is usually modelled by an object of $\mathbf{Set}{\downarrow}\mathbb{N}^*$, i.e., by a set map $N : X \to \mathbb{N}^*$ where X is the underlying set of the multiset N in which $x \in X$ appears $N(x) > 0$ times. A higher dimensional transition system must satisfy several natural axioms CSA1, CSA2 and CSA3 (cf. Definition 4). This notion is a generalization of the 1-dimensional notion of transition system in which transitions between states are labelled by one action (e.g., [WN95, Section 2.1]). The latter 1-dimensional notion cannot of course model concurrency.

One of the purposes of this paper is to make precise the link between process algebras modelled as labelled symmetric precubical sets, as higher dimensional transition systems, and as flows, by introducing the notion of weak higher dimensional transition system. The only process algebras treated are the ones in Milner's calculus of communicating systems (CCS) [WN95] [Mil89]. And only the topological model of flows introduced in [Gau03] is used. Similar results can easily be obtained for other process algebras and for topological models of concurrency other than flows. For other synchronization algebras, one needs only to change the set of synchronizations in Definition 13.1. For other topological models of concurrency one needs only to change the realization of the full n-cube $[n]^{\mathrm{cof}}$ in Definition 12.5. These modifications do not affect the mathematical results of the paper. The first main result can then be stated as follows:

Theorem (Theorem 9.2, Theorem 9.5, Theorem 12.7 and Corollary 13.7)**.** *The mapping defined in* [Gau08] *and* [Gau10] *taking each CCS process name to the geometric realization as flow* $|\square_S[\![P]\!]|_{\mathrm{flow}}$ *of the labelled symmetric precubical set* $\square_S[\![P]\!]$ *factors through Cattani–Sassone's category of higher dimensional transition systems.*

In fact, the functorial factorization $|\mathbb{T}(K)| \cong |K|_{\mathrm{flow}}$ exists as soon as K satisfies the HDA paradigm and $\mathbb{T}(K)$ the Unique intermediate state axiom (Every K satisfying the latter condition is called a strong labelled symmetric precubical set).

Let us recall for the reader that the semantics of process algebras used in this paper in Section 13 is the one of [Gau10]. This semantics is nothing else but the labelled free symmetric precubical set generated by the labelled precubical set given in [Gau08]. The reason for working with labelled symmetric precubical sets in this paper is that this category is closely related to the category of (weak) higher dimensional transition systems by Theorem 8.5: the full subcategories in the two categories generated by the labelled n-cubes for all $n \geqslant 0$ are isomorphic.

The interest of the combinatorial model of (weak) higher dimensional transition systems is that the HDA paradigm (cf. Section 7) is automatically satisfied. That is to say, the concurrent execution of n actions (with $n \geqslant 2$) always assembles to exactly one n-cube in a (weak) higher dimensional transition system. Indeed, the realization functor $\mathbb{T}$ from labelled symmetric precubical sets to weak higher dimensional transition systems factors through the category of labelled symmetric precubical sets satisfying the HDA paradigm by Theorem 9.5. On the contrary, as already explained in [Gau08] and in [Gau10], there exist labelled (symmetric) precubical sets containing n-tuples of actions running concurrently which assemble to several different n-cubes. Let us explain this phenomenon for the case of the square. Consider the concurrent execution of two actions a and b as depicted in Figure 2. Let $S = \{0,1\} \times \{0,1\}$ be the set of states. Let $L = \{a, b\}$ be the set of actions with $a \neq b$. The boundary of the square is modelled by adding to the set of states S the four 1-transitions $((0,0), a, (1,0))$, $((0,1), a, (1,1))$, $((0,0), b, (0,1))$ and $((1,0), b, (1,1))$. The concurrent execution of a and b is modelled by adding the 2-transitions

$((0,0),a,b,(1,1))$ and $((0,0),b,a,(1,1))$. Adding one more time the two 2-transitions $((0,0),a,b,(1,1))$ and $((0,0),b,a,(1,1))$ does not change anything to the object since the set of transitions remains equal to

$$\{((0,0),a,(1,0)),((0,1),a,(1,1)),((0,0),b,(0,1)),((1,0),b,(1,1)),\\ ((0,0),a,b,(1,1)),((0,0),b,a,(1,1))\}.$$

On the contrary, the labelled symmetric precubical set $\square_S[a,b] \sqcup_{\partial\square_S[a,b]} \square_S[a,b]$ contains two different labelled squares $\square_S[a,b]$ modelling the concurrent execution of a and b, obtaining this way a geometric object homotopy equivalent to a 2-dimensional sphere (see Proposition 9.3). This is meaningless from a computer scientific point of view. Indeed, either the two actions a and b run sequentially, and the square must remain empty, or the two actions a and b run concurrently and the square must be filled by exactly one square modelling concurrency. The topological hole created by the presence of two squares as in $\square_S[a,b] \sqcup_{\partial\square_S[a,b]} \square_S[a,b]$ does not have any computer scientific interpretation. The concurrent execution of two actions (and more generally of n actions) must be modelled by a contractible object.

The factorization of $\mathbb{T}$ even yields a faithful functor $\overline{\mathbb{T}}$ from labelled symmetric precubical sets satisfying the HDA paradigm to weak higher dimensional transition systems by Corollary 10.2. However, the functor $\overline{\mathbb{T}}$ is not full by Proposition 10.3. It only induces an equivalence of categories by restricting to a full subcategory:

Theorem (Theorem 11.6)**.** *The localization of the category of higher dimensional transition systems by the cubification functor is equivalent to a locally finitely presentable reflective full subcategory of the category of labelled symmetric precubical sets. In this localization, two higher dimensional transition systems are isomorphic if they have the same cubes and they only differ by their set of actions.*

We must introduce the technical notion of weak higher dimensional transition system since there exist labelled symmetric precubical sets K such that $\mathbb{T}(K)$ is not a higher dimensional transition system by Proposition 9.7. It is of course not difficult to find a labelled symmetric precubical set contradicting CSA1 of Definition 4.1 (e.g., Figure 1). It is also possible to find counterexamples for the other axioms CSA2 and CSA3 of higher dimensional transition system. This matters: if a labelled symmetric precubical set K is such that $\mathbb{T}(K)$ is not a higher dimensional transition system, then it cannot be constructed from a process algebra.

Organization of the paper. Section 3 expounds the notion of weak higher dimensional transition system. The notion of multiset recalled in the introduction is replaced by the Multiset axiom on tuples to make the categorical treatment easier. Logical tools are used to prove that the category of weak higher dimensional transition systems is locally finitely presentable and topological. Section 4 recalls Cattani–Sassone's notion of higher dimensional transition system. It is proved that every higher dimensional transition system is a weak one. The notion of higher dimensional transition system is also reformulated to make it easier to use. The Unique intermediate state axiom is introduced for that purpose. It is also proved in the same section that the set of transitions of any reasonable colimit is the union of the transitions of the components (Theorem 4.7). It is proved in Section 5 that higher dimensional transition systems assemble to a small-orthogonality class of the category of weak higher dimensional transition systems (Corollary 5.7). This implies that the category of higher dimensional transition systems is a full reflective locally finitely presentable category of the category of weak higher dimensional transition systems. Section 6 recalls the notion of labelled symmetric precubical set. This section collects information scattered between [Gau08] and [Gau10]. Section 7 defines the paradigm of higher dimensional automata (HDA paradigm). It is the adaptation to the setting of labelled symmetric precubical sets of the analogous definition presented in [Gau08] for labelled precubical sets. A labelled symmetric precubical set satisfies the HDA paradigm if every labelled p-shell with $p \geqslant 1$ can be filled by at most one labelled $(p+1)$-cube. This notion is a technical tool for various proofs of this paper. It is proved in the same section that the full subcategory of labelled symmetric precubical sets satisfying the HDA paradigm is a full reflective subcategory of the category of labelled symmetric precubical sets by proving that it is a small-orthogonality class as well. It is also checked in the same section that the full labelled n-cube satisfies the HDA paradigm (this trivial point is fundamental!). Section 8 establishes that the full subcategory of labelled n-cubes of the category of labelled symmetric precubical sets is isomorphic to the full subcategory of labelled n-cubes of the category of (weak) higher dimensional transition systems (Theorem 8.5). The proof is of combinatorial nature. Section 9 constructs the realization functor from labelled symmetric precubical sets to weak higher dimensional transition systems. And it is proved that this functor factors through the full subcategory of labelled symmetric precubical sets satisfying the HDA paradigm. The two functors, the realization functor and its factorization are left adjoints (Theorem 9.2 and Theorem 9.5). Section 10 studies when these latter functors are faithful and full. It is proved that the HDA paradigm is related to faithfulness and that the combination of the HDA paradigm together and the Unique intermediate state axiom is related to fullness. Section 11 uses all previous results to compare the two settings of higher dimensional transition systems and labelled symmetric precubical sets. Section 12 is a straightforward but crucial application of the previous results. It is proved in Theorem 12.7 that the geometric realization as flow of a labelled symmetric precubical set K is the geometric realization as flow of its realization as weak higher dimensional transition system provided that K is strong and satisfies the HDA paradigm. The purpose of Section 13 is to prove that these conditions are satisfied by the labelled symmetric precubical sets coming from process algebras. Hence we obtain the second application stated in Corollary 13.7.

2. Prerequisites

The notations used in this paper are standard. A small class is called a set. All categories are locally small. The set of morphisms from X to Y in a category $\mathcal{C}$ is denoted by $\mathcal{C}(X,Y)$. The identity of X is denoted by Id_X. Colimits are denoted by $\varinjlim$ and limits by $\varprojlim$.

The reading of this paper requires general knowledge in category theory [ML98], in particular in presheaf theory [MLM94], but also a good understanding of the theory of locally presentable categories [AR94] and of the theory of topological categories [AHS06]. A few model category techniques are also used [DS95] [Hov99] [Hir03] in the proof of Theorem 9.4 and in Section 12.

A short introduction to process algebra can be found in [WN95]. An introduction to CCS (Milner's calculus of communicating systems [Mil89]) for mathematician is available in [Gau08] and in [Gau10]. Hardly any knowledge of process algebra is needed to read Section 13 of the paper. In fact, the paper [Gau08] can be taken as a starting point.

Some salient mathematical facts are collected in this section. Of course, this section does not intend to be an introduction to these notions. It will only help the reader to understand what kinds of mathematical tools are used in this work.

Let λ be a regular cardinal (see for example [HJ99, p 160]). When $\lambda = \aleph_0$, the word "$\lambda-$" is replaced by the word "finitely". An object C of a category $\mathcal{C}$ is λ-presentable when the functor $\mathcal{C}(C,-)$ preserves λ-directed colimits. Practically, that means that every map $C \to \varinjlim C_i$ factors as a composite $C \to C_i \to \varinjlim C_i$ when the colimit is λ-directed. A λ-accessible category is a category having λ-directed colimits such that each object is generated (in some strong sense) by a set of λ-presentable objects. For example, each object is a λ-directed colimit of a subset of a given set of λ-presentable objects. If moreover the category is cocomplete, it is called a locally λ-presentable category. We use at several places of the paper a logical characterization of accessible and locally presentable categories which are axiomatized by theories with set of sorts $\{s\} \cup \Sigma$, s being the sort of states and Σ a nonempty fixed set of labels. Another kind of locally presentable category is a category of presheaves, and any comma category constructed from it. Every locally presentable category has a set of generators, is complete, cocomplete, wellpowered and co-wellpowered. The Special Adjoint Functor Theorem SAFT is then usable to prove the existence of right adjoints. A functor between locally λ-presentable category is λ-accessible if it preserves λ-directed colimits (or equivalently λ-filtered colimits). Another important fact is that a functor between locally presentable categories is a right adjoint if and only if it is accessible and limit-preserving.

An object C is orthogonal to a map $X \to Y$ if every map $X \to C$ factors uniquely as a composite $X \to Y \to C$. A full subcategory of a given category is reflective if the inclusion functor is a right adjoint. The left adjoint to the inclusion is called the reflection. In a locally presentable category, the full subcategory of objects orthogonal to a given set of morphisms is an example of a reflective subcategory. Such a category, called a small-orthogonality class, is locally presentable. And the inclusion functor is of course accessible and limit-preserving.

The paradigm of topological category over the category of **Set** is the one of general topological spaces with the notions of initial topology and final topology. More precisely, a functor $\omega : \mathcal{C} \to \mathcal{D}$ is topological if each cone $(f_i : X \to \omega A_i)_{i \in I}$ where I is a class has a unique ω-initial lift (the initial structure) $(\overline{f}_i : A \to A_i)_{i \in I}$, i.e.:

(1) $\omega A = X$ and $\omega \overline{f}_i = f_i$ for each $i \in I$.
(2) Given $h : \omega B \to X$ with $f_i h = \omega \overline{h}_i$, $\overline{h}_i : B \to A_i$ for each $i \in I$, then $h = \omega \overline{h}$ for a unique $\overline{h} : B \to A$.

Topological functors can be characterized as functors such that each cocone $(f_i : \omega A_i \to X)_{i \in I}$ where I is a class has a unique ω-final lift (the final structure) $\overline{f}_i : A_i \to A$, i.e.:

(1) $\omega A = X$ and $\omega \overline{f}_i = f_i$ for each $i \in I$.
(2) Given $h : X \to \omega B$ with $h f_i = \omega \overline{h}_i$, $\overline{h}_i : A_i \to B$ for each $i \in I$, then $h = \omega \overline{h}$ for a unique $\overline{h} : A \to B$.

Let us suppose $\mathcal{D}$ complete and cocomplete. A limit (resp. colimit) in $\mathcal{C}$ is calculated by taking the limit (resp. colimit) in $\mathcal{D}$, and by endowing it with the initial (resp. final) structure. In this work, a topological category is a topological category over the category $\mathbf{Set}^{\{s\} \cup \Sigma}$ where $\{s\} \cup \Sigma$ is as above the set of sorts.

Let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be maps in a category $\mathcal{C}$. Then i has the left lifting property (LLP) with respect to p (or p has the right lifting property (RLP) with respect to i) if for every commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

there exists a lift g making both triangles commutative.

Let $\mathcal{C}$ be a cocomplete category. If K is a set of morphisms of $\mathcal{C}$, then the class of morphisms of $\mathcal{C}$ that satisfy the RLP with respect to every morphism of K is denoted by $\mathbf{inj}(K)$ and the class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of K is denoted by $\mathbf{cell}(K)$. Denote by $\mathbf{cof}(K)$ the class of morphisms of $\mathcal{C}$ that satisfy the LLP with respect to the morphisms of $\mathbf{inj}(K)$. It is a purely categorical fact that $\mathbf{cell}(K) \subset \mathbf{cof}(K)$. Moreover, every morphism of $\mathbf{cof}(K)$ is a retract of a morphism of $\mathbf{cell}(K)$ as soon as the domains of K are small relative to $\mathbf{cell}(K)$ [Hov99, Corollary 2.1.15]. An element of $\mathbf{cell}(K)$ is called a relative K-cell complex. If X is an object of $\mathcal{C}$, and if the canonical morphism $\varnothing \longrightarrow X$ is a relative K-cell complex, then the object X is called a K-cell complex.

Let $\mathcal{C}$ be a category. A weak factorization system is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that the class $\mathcal{L}$ is the class of morphisms having the LLP with respect to $\mathcal{R}$, such that the class $\mathcal{R}$ is the class of morphisms having the RLP with respect to $\mathcal{L}$ and such that every morphism of $\mathcal{C}$ factors as a composite $r\circ\ell$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$. The weak factorization system is functorial if the factorization $r \circ \ell$ is a functorial factorization. It is cofibrantly generated if it is of the form $(\mathbf{cof}(K), \mathbf{inj}(K))$ for some set of maps K.

A model category is a complete cocomplete category equipped with a model structure consisting of three classes of morphisms Cof, Fib, $\mathcal{W}$ respectively called cofibration, fibration and weak equivalence such that the pairs of classes of morphisms $(\mathrm{Cof}, \mathrm{Fib}\cap\mathcal{W})$ and $(\mathrm{Cof}\cap\mathcal{W}, \mathrm{Fib})$ are weak factorization systems and such that if two of the three morphisms $f, g, g\circ f$ are weak equivalences, then so is the third one. This model structure is cofibrantly generated provided that the two weak factorization systems $(\mathrm{Cof}, \mathrm{Fib}\cap\mathcal{W})$ and $(\mathrm{Cof}\cap\mathcal{W}, \mathrm{Fib})$ are cofibrantly generated. The only model category used in this paper is the one of flows. We need only in fact the notion of cofibrant replacement. For an object X of a model category, the canonical map $\varnothing \to X$ factors as a composite $\varnothing \to X^{\mathrm{cof}} \to X$ where the left-hand map is a cofibration and the right-hand map a trivial fibration, i.e., an element of $\mathrm{Fib}\cap\mathcal{W}$. The object X^{cof} is called a cofibrant replacement of X.

The proof of Theorem 9.4 uses the fact that for every set of morphisms K in a locally presentable category, a map $X \to Y$ always factors as a composite $X \to Z \to Y$ where the left-hand map is an object of $\mathbf{cell}(K)$ and the right-hand map an object of $\mathbf{inj}(K)$.

Beware of the fact that the word "model" has three different meanings in this paper, a logical one, a homotopical one, and also a non-mathematical one like in the sentence "the n-cube models the concurrent execution of n actions".

3. Weak higher dimensional transition systems

The formalism of multiset as used in [CS96] is not easy to handle. In this paper, an n-transition between two states α and β (or from α to β) modelling the concurrent execution of n actions $u_1, \dots, u_n$ with $n \geqslant 1$ is modelled by an $(n+2)$-tuple $(\alpha, u_1, \dots, u_n, \beta)$ satisfying the new Multiset axiom: for every permutation σ of $\{1,\dots,n\}$, $(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta)$ is an n-transition.

Notation 3.1. We fix a nonempty set of labels Σ. We suppose that Σ always contains a distinguished element denoted by τ.

Definition 3.2. A weak higher dimensional transition system consists of a triple

$$\left(S, \mu : L \to \Sigma, T = \bigcup_{n\geqslant 1} T_n\right)$$

where S is a set of states, where L is a set of actions, where $\mu : L \to \Sigma$ is a set map called the labelling map, and finally where $T_n \subset S \times L^n \times S$ for $n \geqslant 1$ is a set of n-transitions or n-dimensional transitions such that one has:

- (Multiset axiom) For every permutation σ of $\{1,\dots,n\}$ with $n \geqslant 2$, if $(\alpha, u_1, \dots, u_n, \beta)$ is a transition, then $(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta)$ is a transition as well.
- (Coherence axiom) For every $(n+2)$-tuple $(\alpha, u_1, \dots, u_n, \beta)$ with $n \geqslant 3$, for every $p, q \geqslant 1$ with $p + q < n$, if the five tuples

  $$(\alpha, u_1, \dots, u_n, \beta),\ (\alpha, u_1, \dots, u_p, \nu_1),\ (\nu_1, u_{p+1}, \dots, u_n, \beta),$$
  $$(\alpha, u_1, \dots, u_{p+q}, \nu_2),\ (\nu_2, u_{p+q+1}, \dots, u_n, \beta),$$

  are transitions, then the $(q+2)$-tuple $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$ is a transition as well.

A map of weak higher dimensional transition systems

$$f : (S, \mu : L \to \Sigma, (T_n)_{n\geqslant 1}) \to (S', \mu' : L' \to \Sigma, (T'_n)_{n\geqslant 1})$$

consists of a set map $f_0 : S \to S'$, a commutative square

$$\begin{tikzcd} L \arrow[r, "\mu"] \arrow[d, "\widetilde{f}"'] & \Sigma \arrow[d, equal] \\ L' \arrow[r, "\mu'"'] & \Sigma \end{tikzcd}$$

such that if $(\alpha, u_1, \dots, u_n, \beta)$ is a transition, then

$$(f_0(\alpha), \widetilde{f}(u_1), \dots, \widetilde{f}(u_n), f_0(\beta))$$

is a transition. The corresponding category is denoted by **WHDTS**. The n-transition $(\alpha, u_1, \dots, u_n, \beta)$ is also called a transition from α to β.

Notation 3.3. A transition $(\alpha, u_1, \dots, u_n, \beta)$ will be also denoted by

$$\alpha \stackrel{u_1,\dots,u_n}{\longrightarrow} \beta.$$

Theorem 3.4. *The category* **WHDTS** *is locally finitely presentable. The functor*

$$\omega : \mathbf{WHDTS} \longrightarrow \mathbf{Set}^{\{s\}\cup\Sigma}$$

taking the weak higher dimensional transition system $(S, \mu : L \to \Sigma, (T_n)_{n\geqslant 1})$ *to the* $(\{s\}\cup\Sigma)$*-tuple of sets* $(S, (\mu^{-1}(x))_{x\in\Sigma}) \in \mathbf{Set}^{\{s\}\cup\Sigma}$ *is topological.*

Proof. Let $(f_i : \omega X_i \to (S, (L_x)_{x\in\Sigma}))_{i\in I}$ be a cocone where I is a class with $X_i = (S_i, \mu_i : L_i \to \Sigma, T^i = \bigcup_{n\geqslant 1} T^i_n)$. The closure by the Multiset axiom and the Coherence axiom of the union of the images of the T^i in $\bigcup_{n\geqslant 1}(S \times L^n \times S)$ with $L = \bigsqcup_{x\in\Sigma} L_x$ gives the final structure. Hence, the functor ω is topological.

We use the terminology of [AR94, Chapter 5]. Let us consider the theory $\mathcal{T}$ in finitary first-order logic defined by the set of sorts $\{s\}\cup\Sigma$, by a relational symbol $T_{x_1,\dots,x_n}$ of arity $s \times x_1 \times \dots \times x_n \times s$ for every $n \geqslant 1$ and every $(x_1, \dots, x_n) \in \Sigma^n$, and by the axioms:

- For all $x_1, \dots, x_n \in \Sigma$, for all $n \geqslant 2$ and for all permutations σ of $\{1, \dots, n\}$:

$$(\forall \alpha, u_1 \dots, u_n, \beta), T_{x_1,\dots,x_n}(\alpha, u_1, \dots, u_n, \beta) \Rightarrow T_{x_{\sigma(1)},\dots,x_{\sigma(n)}}(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta).$$

- For all $x_1, \dots, x_n \in \Sigma$, for all $n \geqslant 3$, for all $p, q \geqslant 1$ with $p + q < n$,

$$\begin{aligned}&(\forall \alpha, u_1 \dots, u_n, \beta, \nu_1, \nu_2)(T_{x_1,\dots,x_n}(\alpha, u_1, \dots, u_n, \beta)\\ &\wedge T_{x_1,\dots,x_p}(\alpha, u_1, \dots, u_p, \nu_1) \wedge T_{x_{p+1},\dots,x_n}(\nu_1, u_{p+1}, \dots, u_n, \beta)\\ &\wedge T_{x_1,\dots,x_{p+q}}(\alpha, u_1, \dots, u_{p+q}, \nu_2) \wedge T_{x_{p+q+1},\dots,x_n}(\nu_2, u_{p+q+1}, \dots, u_n, \beta))\\ &\Rightarrow T_{x_{p+1},\dots,x_{p+q}}(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2).\end{aligned}$$

Since the axioms are of the form $(\forall x), \phi(x) \Rightarrow (\exists! y\ \psi(x, y))$ (with no y) where ϕ and ψ are conjunctions of atomic formulas with a finite number of arguments, the category $\mathrm{Mod}(\mathcal{T})$ of models of $\mathcal{T}$ in $\mathbf{Set}^{\{s\}\cup\Sigma}$ is locally finitely presentable by [AR94, Theorem 5.30]. It remains to observe that there is an isomorphism of categories $\mathrm{Mod}(\mathcal{T}) \cong \mathbf{WHDTS}$ to complete the proof. □

Note that the category **WHDTS** is axiomatized by a universal strict Horn theory without equality, i.e., by statements of the form $(\forall x), \phi(x) \Rightarrow \psi(x)$ where ϕ and ψ are conjunctions of atomic formulas without equalities. So [Ros81, Theorem 5.3] provides another argument to prove that the functor ω above is topological.

Let us conclude this section by some additional comments about colimits in **WHDTS**. We will come back to this question in Theorem 4.7.

Proposition 3.5. *Let* $X = \varinjlim X_i$ *be a colimit of weak higher dimensional transition systems with* $X_i = (S_i, \mu_i : L_i \to \Sigma, T^i = \bigcup_{n\geqslant 1} T^i_n)$ *and* $X = (S, \mu : L \to \Sigma, T = \bigcup_{n\geqslant 1} T_n)$*. Then:*

(1) $S = \varinjlim S_i$, $L = \varinjlim L_i$, $\mu = \varinjlim \mu_i$.
(2) *The union* $\bigcup_i T^i$ *of the image of the* T^i *in* $\bigcup_{n\geqslant 1}(S\times L^n\times S)$ *satisfies the Multiset axiom.*
(3) T *is the closure of* $\bigcup_i T^i$ *under the Coherence axiom.*
(4) *When the union* $\bigcup_i T^i$ *is already closed under the Coherence axiom, this union is the final structure.*

Proof. By [AHS06, Proposition 21.15], (1) is a consequence of the fact that the category **WHDTS** is topological over $\mathbf{Set}^{\{s\}\cup\Sigma}$. (2) comes from the fact that each T_i satisfies the Multiset axiom. (4) is a consequence of (2) . It remains to prove (3) . Let $G_0(\bigcup_i T^i) = \bigcup_i T^i$. Let us define $G_\alpha(\bigcup_i T^i)$ by induction on the transfinite ordinal $\alpha \geqslant 0$ by $G_\alpha(\bigcup_i T^i) = \bigcup_{\beta<\alpha} G_\beta(\bigcup_i T^i)$ for every limit ordinal α and $G_{\alpha+1}(\bigcup_i T^i)$ is obtained from $G_\alpha(\bigcup_i T^i)$ by adding to $G_\alpha(\bigcup_i T^i)$ all $(q+2)$-tuples $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$ such that there exist five tuples $(\alpha, u_1, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_{p+q}, \nu_2)$ and $(\nu_2, u_{p+q+1}, \dots, u_n, \beta)$ of the set $G_\alpha(\bigcup_i T^i)$. Hence we have the inclusions $G_\alpha(\bigcup_i T^i) \subset G_{\alpha+1}(\bigcup_i T^i) \subset \bigcup_{n\geqslant 1}(S\times L^n \times S)$ for all $\alpha \geqslant 0$. For cardinality reason, there exists an ordinal α_0 such that for every $\alpha \geqslant \alpha_0$, one has $G_\alpha(\bigcup_i T^i) = G_{\alpha_0}(\bigcup_i T^i)$. By transfinite induction on $\alpha \geqslant 0$, one sees that $G_\alpha(\bigcup_i T^i)$ satisfies the Multiset axiom. So the closure $G_{\alpha_0}(\bigcup_i T^i)$ of $\bigcup_i T^i$ under the Coherence axiom is the final structure and $G_{\alpha_0}(\bigcup_i T^i) = T$. $\square$

4. Higher dimensional transition systems

Let us now propose our (slightly revised) version of higher dimensional transition system.

Definition 4.1. A higher dimensional transition system is a triple

$$\left(S, \mu : L \to \Sigma, T = \bigcup_{n\geqslant 1} T_n\right)$$

where S is a set of states, where L is a set of actions, where $\mu : L \to \Sigma$ is a set map called the labelling map, and finally where $T_n \subset S \times L^n \times S$ is a set of n-transitions or n-dimensional transitions such that one has:

(1) (Multiset axiom) For every permutation σ of $\{1, \dots, n\}$ with $n \geqslant 2$, if $(\alpha, u_1, \dots, u_n, \beta)$ is a transition, then $(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta)$ is a transition as well.
(2) (First Cattani–Sassone axiom CSA1) If (α, u, β) and (α, u', β) are two transitions such that $\mu(u) = \mu(u')$, then $u = u'$.
(3) (Second Cattani–Sassone axiom CSA2) For every $n \geqslant 2$, every p with $1 \leqslant p < n$, and every transition $(\alpha, u_1, \dots, u_n, \beta)$, there exists a unique state ν_1 and a unique state ν_2 such that $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\alpha, u_{p+1}, \dots, u_n, \nu_2)$ and $(\nu_2, u_1, \dots, u_p, \beta)$ are transitions.

(4) (Third Cattani–Sassone axiom CSA3) For every state
$$\alpha, \beta, \nu_1, \nu_2, \nu_1', \nu_2'$$
and every action $u_1, \dots, u_n$, with $p, q \geqslant 1$ and $p + q < n$, if the nine tuples
$$(\alpha, u_1, \dots, u_n, \beta), (\alpha, u_1, \dots, u_p, \nu_1), (\nu_1, u_{p+1}, \dots, u_n, \beta),$$
$$(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2), (\nu_2, u_{p+q+1}, \dots, u_n, \beta), (\alpha, u_1, \dots, u_{p+q}, \nu_2'),$$
$$(\nu_2', u_{p+q+1}, \dots, u_n, \beta), (\alpha, u_1, \dots, u_p, \nu_1'), (\nu_1', u_{p+1}, \dots, u_{p+q}, \nu_2'),$$
are transitions, then $\nu_1 = \nu_1'$ and $\nu_2 = \nu_2'$.

Note that our notion of morphism of higher dimensional transition systems differs from Cattani–Sassone's one: we take only the morphisms between the underlying sets of states and actions preserving the structure. This is necessary to develop the theory presented in this paper. So it becomes false that two general higher dimensional transition systems differing only by the set of actions are isomorphic. However, this latter fact is true in some appropriate categorical localization (see the very end of Section 11). We also have something similar for (weak) higher dimensional transition systems coming from strong labelled symmetric precubical sets by Corollary 10.6, that is to say from any labelled symmetric precubical set coming from process algebras by Theorem 13.6.

Let us cite [CS96]: "CSA1 in the above definition simply guarantees that there are no two transitions between the same states carrying the same multiset of labels. CSA2 guarantees that all the interleaving of a transition $\alpha \xrightarrow{u_1,\dots,u_n} \beta$ are present as paths from α to β, whilst CSA3 ensures that such paths glue together properly: it corresponds to the cubical laws of higher dimensional automata".

Proposition 4.2. *Every higher dimensional transition system is a weak higher dimensional transition system.*

Proof. Let $X = (S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n)$ be a higher dimensional transition system. Let $(\alpha, u_1, \dots, u_n, \beta)$ be a transition with $n \geqslant 3$. Let $p, q \geqslant 1$ with $p + q < n$. Suppose that the five tuples
$$(\alpha, u_1, \dots, u_n, \beta),\ (\alpha, u_1, \dots, u_p, \nu_1),\ (\nu_1, u_{p+1}, \dots, u_n, \beta),$$
$$(\alpha, u_1, \dots, u_{p+q}, \nu_2),\ (\nu_2, u_{p+q+1}, \dots, u_n, \beta),$$
are transitions. Let ν_1' be the (unique) state of X such that $(\alpha, u_1, \dots, u_p, \nu_1')$ and $(\nu_1', u_{p+1}, \dots, u_{p+q}, \nu_2)$ are transitions of X. Let ν_2' be the (unique) state of X such that $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2')$ and $(\nu_2', u_{p+q+1}, \dots, u_n, \beta)$ are transitions of X. Then $\nu_1 = \nu_1'$ and $\nu_2 = \nu_2'$ by CSA3. Therefore the Coherence axiom is satisfied. □

Notation 4.3. The full subcategory of higher dimensional transition systems is denoted by **HDTS**. So one has the inclusion **HDTS** $\subset$ **WHDTS**.

Definition 4.4. A weak higher dimensional transition system satisfies the Unique Intermediate state axiom if for every $n \geqslant 2$, every p with $1 \leqslant p < n$ and every transition $(\alpha, u_1, \dots, u_n, \beta)$, there exists a unique state ν such that both the tuples $(\alpha, u_1, \dots, u_p, \nu)$ and $(\nu, u_{p+1}, \dots, u_n, \beta)$ are transitions.

Proposition 4.5. *A weak higher dimensional transition system satisfies the second and third Cattani–Sassone axioms if and only if it satisfies the Unique intermediate state axiom.*

Proof. A weak higher dimensional transition system satisfying CSA2 and CSA3 clearly satisfies the Unique intermediate state axiom. Conversely, if a weak higher dimensional transition system satisfies the Unique intermediate state axiom, it clearly satisfies CSA2. Let $\alpha, \beta, \nu_1, \nu_2, \nu'_1, \nu'_2$ be states and let $u_1, \dots, u_n$ be actions with $n \geqslant 3$. Let $p, q \geqslant 1$ with $p + q < n$. Suppose that

$$\begin{gathered}(\alpha, u_1, \dots, u_n, \beta), (\alpha, u_1, \dots, u_p, \nu_1), (\nu_1, u_{p+1}, \dots, u_n, \beta),\\ (\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2), (\nu_2, u_{p+q+1}, \dots, u_n, \beta), (\alpha, u_1, \dots, u_{p+q}, \nu'_2),\\ (\nu'_2, u_{p+q+1}, \dots, u_n, \beta), (\alpha, u_1, \dots, u_p, \nu'_1), (\nu'_1, u_{p+1}, \dots, u_{p+q}, \nu'_2)\end{gathered}$$

are transitions. By the Coherence axiom, the tuple $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu'_2)$ is a transition. By the Unique intermediate state axiom, one obtains $\nu_1 = \nu'_1$ and $\nu_2 = \nu'_2$. So CSA3 is satisfied too. $\square$

One obtains a new formulation of the notion of higher dimensional transition system:

Proposition 4.6. *A higher dimensional transition system is a weak higher dimensional transition system satisfying CSA1 and the Unique intermediate state axiom.*

Let us conclude this section by an important remark about colimits of weak higher dimensional transition systems satisfying the Unique intermediate state axiom, so in particular about colimits of higher dimensional transition systems.

Theorem 4.7. *Let $X = \varinjlim X_i$ be a colimit of weak higher dimensional transition systems such that every weak higher dimensional transition system X_i satisfies the Unique intermediate state axiom. Let*

$$X_i = \left(S_i, \mu_i : L_i \to \Sigma, T^i = \bigcup_{n \geqslant 1} T^i_n\right) \quad \textit{and}$$

$$X = \left(S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n\right).$$

Denote by $\bigcup_i T^i$ the union of the images by the map $X_i \to X$ of the sets of transitions of the X_i for i running over the set of objects of the base category of the diagram $i \mapsto X_i$. Then the following conditions are equivalent:

(1) *X satisfies the Unique intermediate state axiom.*
(2) *The set of transitions $\bigcup_i T^i$ satisfies the Unique intermediate state axiom.*
(3) *The set of transitions $\bigcup_i T^i$ satisfies the Multiset axiom, the Coherence axiom and the Unique intermediate state axiom.*

Whenever one of the preceding three conditions is satisfied, the set of transitions $\bigcup_i T^i$ is the final structure.

Proof. The set of transitions $\bigcup_i T^i$ always satisfies the Multiset axiom by Proposition 3.5.

(1) $\Rightarrow$ (2). The set of transitions of X is the closure under the Coherence axiom of $\bigcup_i T^i$ by Proposition 3.5. So $\bigcup_i T^i \subset T$.

(2) $\Rightarrow$ (3). Let $n \geqslant 3$. Let $(\alpha, u_1, \dots, u_n, \beta)$ be a transition of $\bigcup_i T^i$. Let $p, q \geqslant 1$ with $p + q < n$. Let $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_{p+q}, \nu_2)$ and $(\nu_2, u_{p+q+1}, \dots, u_n, \beta)$ be four transitions of $\bigcup_i T^i$. Let i such that there exists a transition $(\alpha^i, u^i_1, \dots, u^i_n, \beta^i)$ of X_i taken by the canonical map $X_i \to X$ to $(\alpha, u_1, \dots, u_n, \beta)$. Since X_i satisfies the Unique intermediate state axiom, there exists a (unique) state ν^i_1 and a (unique) state ν^i_2 of X_i such that $(\alpha^i, u^i_1, \dots, u^i_p, \nu^i_1)$, $(\nu^i_1, u^i_{p+1}, \dots, u^i_n, \beta^i)$, $(\alpha^i, u^i_1, \dots, u^i_{p+q}, \nu^i_2)$ and $(\nu^i_2, u^i_{p+q+1}, \dots, u^i_n, \beta)$ are four transitions of X_i. Since $\bigcup_i T^i$ satisfies the Unique intermediate state axiom as well, the map $X_i \to X$ takes ν^i_1 to ν_1 and ν^i_2 to ν_2. By the Coherence axiom applied to X_i, the tuple $(\nu^i_1, u^i_{p+1}, \dots, u^i_{p+q}, \nu^i_2)$ is a transition of X_i. So the union $\bigcup_i T^i$ is closed under the Coherence axiom.

(3) $\Rightarrow$ (1). If (3) holds, then the inclusion $\bigcup_i T^i \subset T$ is an equality by Proposition 3.5. Therefore the weak higher dimensional transition system X satisfies the Unique intermediate state axiom.

The last assertion is then clear. $\square$

5. Higher dimensional transition systems as a small-orthogonality class

Notation 5.1. Let $[0] = \{()\}$ and $[n] = \{0,1\}^n$ for $n \geqslant 1$. By convention, one has $\{0,1\}^0 = [0] = \{()\}$. The set $[n]$ is equipped with the product ordering $\{0 < 1\}^n$.

Let us now describe the higher dimensional transition system associated with the n-cube for $n \geqslant 0$.

Proposition 5.2. *Let $n \geqslant 0$ and $a_1, \dots, a_n \in \Sigma$. Let*

$$T_d \subset \{0,1\}^n \times \{(a_1, 1), \dots, (a_n, n)\}^d \times \{0,1\}^n$$

(with $d \geqslant 1$) be the subset of $(d+2)$-tuples

$$((\epsilon_1, \dots, \epsilon_n), (a_{i_1}, i_1), \dots, (a_{i_d}, i_d), (\epsilon'_1, \dots, \epsilon'_n))$$

such that:

- *$i_m = i_n$ implies $m = n$, i.e., there are no repetitions in the list*
$$(a_{i_1}, i_1), \dots, (a_{i_d}, i_d).$$
- *For all i, $\epsilon_i \leqslant \epsilon'_i$.*
- *$\epsilon_i \neq \epsilon'_i$ if and only if $i \in \{i_1, \dots, i_d\}$.*

Let $\mu : \{(a_1, 1), \dots, (a_n, n)\} \to \Sigma$ be the set map defined by $\mu(a_i, i) = a_i$. Then

$$C_n[a_1, \dots, a_n] = (\{0, 1\}^n, \mu : \{(a_1, 1), \dots, (a_n, n)\} \to \Sigma, (T_d)_{d \geqslant 1})$$

is a well-defined higher dimensional transition system.

Note that for $n = 0$, $C_0[]$, also denoted by C_0, is nothing else but the higher dimensional transition system $(\{()\}, \mu : \varnothing \to \Sigma, \varnothing)$.

Proof. There is nothing to prove for $n = 0, 1$. So one can suppose that $n \geqslant 2$. We use the characterization of Proposition 4.6. CSA1 and the Multiset axiom are obviously satisfied. Let

$$((\epsilon_1, \dots, \epsilon_n), (a_{i_1}, i_1), \dots, (a_{i_m}, i_m), (\epsilon'_1, \dots, \epsilon'_n))$$

be a transition of $C_n[a_1, \dots, a_n]$. By construction of $C_n[a_1, \dots, a_n]$, the unique state

$$(\epsilon''_1, \dots, \epsilon''_n) \in [n]$$

such that the $(p + 2)$-tuple

$$((\epsilon_1, \dots, \epsilon_n), (a_{i_1}, i_1), \dots, (a_{i_p}, i_p), (\epsilon''_1, \dots, \epsilon''_n))$$

and the $(m - p + 2)$-tuple

$$((\epsilon''_1, \dots, \epsilon''_n), (a_{i_{p+1}}, i_{p+1}), \dots, (a_{i_m}, i_m), (\epsilon'_1, \dots, \epsilon'_n))$$

are transitions of $C_n[a_1, \dots, a_n]$ is the one satisfying $\epsilon_i \leqslant \epsilon''_i \leqslant \epsilon'_i$ for all $i \in \{1, \dots, n\}$ and $\epsilon_i \neq \epsilon''_i$ if and only if $i \in \{i_1, \dots, i_p\}$. So the Unique intermediate state axiom is satisfied. The Coherence axiom can be checked in a similar way. ☐

Note that for every permutation σ of $\{1, \dots, n\}$, one has the isomorphism of weak higher dimensional transition systems

$$C_n[a_1, \dots, a_n] \cong C_n[a_{\sigma(1)}, \dots, a_{\sigma(n)}].$$

We must introduce n distinct actions $(a_1, 1), \dots, (a_n, n)$ as in [CS96] otherwise an object like $C_2[a, a]$ would not satisfy the Unique intermediate state axiom.

Notation 5.3. For $n \geqslant 1$, let $0_n = (0, \dots, 0)$ (n-times) and $1_n = (1, \dots, 1)$ (n-times). By convention, let $0_0 = 1_0 = ()$.

Notation 5.4. For $n \geqslant 0$, let $C_n[a_1,\dots,a_n]^{\mathrm{ext}}$ be the weak higher dimensional transition system with set of states $\{0_n, 1_n\}$, with set of actions $\{(a_1,1),\dots,(a_n,n)\}$ and with transitions the $(n+2)$-tuples

$$(0_n, (a_{\sigma(1)}, \sigma(1)), \dots, (a_{\sigma(n)}, \sigma(n)), 1_n)$$

for σ running over the set of permutations of the set $\{1,\dots,n\}$.

Proposition 5.5. *Let $n \geqslant 0$ and $a_1,\dots,a_n \in \Sigma$. Let*

$$X = \left(S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n\right)$$

be a weak higher dimensional transition system. Let $f_0 : \{0,1\}^n \to S$ and $\widetilde{f} : \{(a_1,1),\dots,(a_n,n)\} \to L$ be two set maps. Then the following conditions are equivalent:

1. *The pair $(f_0, \widetilde{f})$ induces a map of weak higher dimensional transition systems from $C_n[a_1,\dots,a_n]$ to X.*
2. *For every transition*
$$((\epsilon_1,\dots,\epsilon_n), (a_{i_1}, i_1), \dots, (a_{i_r}, i_r), (\epsilon'_1,\dots,\epsilon'_n))$$
of $C_n[a_1,\dots,a_n]$ with $(\epsilon_1,\dots,\epsilon_n) = 0_n$ or $(\epsilon'_1,\dots,\epsilon'_n) = 1_n$, the tuple
$$(f_0(\epsilon_1,\dots,\epsilon_n), \widetilde{f}(a_{i_1}, i_1), \dots, \widetilde{f}(a_{i_r}, i_r), f_0(\epsilon'_1,\dots,\epsilon'_n))$$
is a transition of X.

Note that the Coherence axiom plays a crucial role in the proof.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Suppose that (2) holds. Let

$$((\epsilon_1,\dots,\epsilon_n), (a_{i_{r+1}}, i_{r+1}), \dots, (a_{i_{r+s}}, i_{r+s}), (\epsilon'_1,\dots,\epsilon'_n))$$

be a transition of $C_n[a_1,\dots,a_n]$ with

$$\begin{aligned}(\epsilon_1,\dots,\epsilon_n) &\in [n]\backslash\{0_n\},\\ (\epsilon'_1,\dots,\epsilon'_n) &\in [n]\backslash\{1_n\}.\end{aligned}$$

There exists a transition

$$(0_n, (a_{i_1}, i_1), \dots, (a_{i_r}, i_r), (\epsilon_1,\dots,\epsilon_n))$$

in $C_n[a_1,\dots,a_n]$ from 0_n to $(\epsilon_1,\dots,\epsilon_n)$. And there exists a transition

$$((\epsilon'_1,\dots,\epsilon'_n), (a_{i_{r+s+1}}, i_{r+s+1}), \dots, (a_{i_n}, i_n), 1_n)$$

from $(\epsilon'_1,\dots,\epsilon'_n)$ to 1_n in $C_n[a_1,\dots,a_n]$. By construction of $C_n[a_1,\dots,a_n]$, the two tuples

$$(0_n, (a_{i_1}, i_1), \dots, (a_{i_{r+s}}, i_{r+s}), (\epsilon'_1,\dots,\epsilon'_n))$$

and

$$((\epsilon_1,\dots,\epsilon_n), (a_{i_{r+1}}, i_{r+1}), \dots, (a_{i_n}, i_n), 1_n)$$

are two transitions of $C_n[a_1,\dots,a_n]$ as well. Thus, the transition

$$((\epsilon_1,\dots,\epsilon_n), (a_{i_{r+1}}, i_{r+1}), \dots, (a_{i_{r+s}}, i_{r+s}), (\epsilon'_1,\dots,\epsilon'_n))$$

is in the closure in $\bigcup_{d\geqslant 1}\{0,1\}^n \times \{(a_1,1),\dots,(a_n,n)\}^d \times \{0,1\}^n$ under the Coherence axiom of the subset of transitions of $C_n[a_1,\dots,a_n]$ of the form

$$(0_n,(a_{i_1},i_1),\dots,(a_{i_r},i_r),(\epsilon'_1,\dots,\epsilon'_n))$$

or $((\epsilon_1,\dots,\epsilon_n),(a_{i_1},i_1),\dots,(a_{i_r},i_r),1_n)$ with $(\epsilon_1,\dots,\epsilon_n),(\epsilon'_1,\dots,\epsilon'_n) \in [n]$. Hence, one obtains (2) $\Rightarrow$ (1). ☐

Theorem 5.6. *A weak higher dimensional transition system satisfies the Unique intermediate state axiom if and only if it is orthogonal to the set of inclusions*

$$\{C_n[a_1,\dots,a_n]^{\mathrm{ext}} \subset C_n[a_1,\dots,a_n], n \geqslant 0 \text{ and } a_1,\dots,a_n \in \Sigma\}.$$

Proof. Only if part. Let $X = (S, \mu : L \to \Sigma, T = \bigcup_{n\geqslant 1} T_n)$ be a weak higher dimensional transition system satisfying the Unique intermediate state axiom. Let $n \geqslant 0$ and $a_1,\dots,a_n \in \Sigma$. We have to prove that the inclusion of weak higher dimensional transition systems $C_n[a_1,\dots,a_n]^{\mathrm{ext}} \subset C_n[a_1,\dots,a_n]$ induces a bijection

$$\mathbf{WHDTS}(C_n[a_1,\dots,a_n],X) \xrightarrow{\cong} \mathbf{WHDTS}(C_n[a_1,\dots,a_n]^{\mathrm{ext}},X).$$

This fact is trivial for $n=0$ and $n=1$ since the inclusion $C_n[a_1,\dots,a_n]^{\mathrm{ext}} \subset C_n[a_1,\dots,a_n]$ is an equality. Suppose now that $n \geqslant 2$. Let

$$f,g \in \mathbf{WHDTS}(C_n[a_1,\dots,a_n],X)$$

having the same restriction to $C_n[a_1,\dots,a_n]^{\mathrm{ext}}$. So there is the equality $\widetilde{f} = \widetilde{g} : \{(a_1,1),\dots,(a_n,n)\} \to L$ as set map. Moreover, one has $f_0(0_n) = g_0(0_n)$ and $f_0(1_n) = g_0(1_n)$. Let $(\epsilon_1,\dots,\epsilon_n) \in [n]$ be a state of $C_n[a_1,\dots,a_n]$ different from 0_n and 1_n. Then there exist (at least) two transitions

$$(0_n,(a_{i_1},i_1),\dots,(a_{i_r},i_r),(\epsilon_1,\dots,\epsilon_n))$$

and

$$((\epsilon_1,\dots,\epsilon_n),(a_{i_{r+1}},i_{r+1}),\dots,(a_{i_{r+s}},i_{r+s}),1_n)$$

of $C_n[a_1,\dots,a_n]$ with $r,s \geqslant 1$. So the four tuples

$$(f_0(0_n),\widetilde{f}(a_{i_1},i_1),\dots,\widetilde{f}(a_{i_r},i_r),f_0(\epsilon_1,\dots,\epsilon_n)),$$

$$(f_0(\epsilon_1,\dots,\epsilon_n),\widetilde{f}(a_{i_{r+1}},i_{r+1}),\dots,\widetilde{f}(a_{i_{r+s}},i_{r+s}),f_0(1_n)),$$

$$(g_0(0_n),\widetilde{g}(a_{i_1},i_1),\dots,\widetilde{g}(a_{i_r},i_r),g_0(\epsilon_1,\dots,\epsilon_n))$$

and

$$(g_0(\epsilon_1,\dots,\epsilon_n),\widetilde{g}(a_{i_{r+1}},i_{r+1}),\dots,\widetilde{g}(a_{i_{r+s}},i_{r+s}),g_0(1_n))$$

are four transitions of X. Since X satisfies the Unique intermediate state axiom, one obtains $f_0(\epsilon_1,\dots,\epsilon_n) = g_0(\epsilon_1,\dots,\epsilon_n)$. Thus $f = g$ and the set map

$$\mathbf{WHDTS}(C_n[a_1,\dots,a_n],X) \longrightarrow \mathbf{WHDTS}(C_n[a_1,\dots,a_n]^{\mathrm{ext}},X)$$

is one-to-one. Let $f : C_n[a_1,\dots,a_n]^{\mathrm{ext}} \to X$ be a map of weak higher dimensional transition systems. The map f induces a set map $f_0 : \{0_n,1_n\} \to S$

and a set map $\widetilde{f} : \{(a_1, 1), \dots, (a_n, n)\} \to L$. Let $(\epsilon_1, \dots, \epsilon_n) \in [n]$ be a state of $C_n[a_1, \dots, a_n]$ different from 0_n and 1_n. Then there exist (at least) two transitions

$$(0_n, (a_{i_1}, i_1), \dots, (a_{i_r}, i_r), (\epsilon_1, \dots, \epsilon_n))$$

and

$$((\epsilon_1, \dots, \epsilon_n), (a_{i_{r+1}}, i_{r+1}), \dots, (a_{i_{r+s}}, i_{r+s}), 1_n)$$

of $C_n[a_1, \dots, a_n]$ with $r, s \geqslant 1$. Let us denote by $f_0(\epsilon_1, \dots, \epsilon_n)$ the unique state of X such that

$$(f_0(0_n), \widetilde{f}(a_{i_1}, i_1), \dots, \widetilde{f}(a_{i_r}, i_r), f_0(\epsilon_1, \dots, \epsilon_n))$$

and

$$(f_0(\epsilon_1, \dots, \epsilon_n), \widetilde{f}(a_{i_{r+1}}, i_{r+1}), \dots, \widetilde{f}(a_{i_{r+s}}, i_{r+s}), f_0(1_n))$$

are two transitions of X. Since every transition from 0_n to $(\epsilon_1, \dots, \epsilon_n)$ is of the form

$$(0_n, (a_{i_{\sigma(1)}}, i_{\sigma(1)}), \dots, (a_{i_{\sigma(r)}}, i_{\sigma(r)}), (\epsilon_1, \dots, \epsilon_n))$$

where σ is a permutation of $\{1, \dots, r\}$ and since every transition from $(\epsilon_1, \dots, \epsilon_n)$ to 1_n is of the form

$$((\epsilon_1, \dots, \epsilon_n), (a_{i_{\sigma'(r+1)}}, i_{\sigma'(r+1)}), \dots, (a_{i_{\sigma'(r+s)}}, i_{\sigma'(r+s)}), 1_n)$$

where σ' is a permutation of $\{r+1, \dots, r+s\}$, one obtains a well-defined set map $f_0 : [n] \to S$. The pair of set maps $(f_0, \widetilde{f})$ induces a well-defined map of weak higher dimensional transition systems by Proposition 5.5. Therefore the set map

$$\mathbf{WHDTS}(C_n[a_1, \dots, a_n], X) \longrightarrow \mathbf{WHDTS}(C_n[a_1, \dots, a_n]^{\mathrm{ext}}, X)$$

is onto.

If part. Conversely, let $X = (S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n)$ be a weak higher dimensional transition system orthogonal to the set of inclusions

$$\{C_n[a_1, \dots, a_n]^{\mathrm{ext}} \subset C_n[a_1, \dots, a_n], n \geqslant 0 \text{ and } a_1, \dots, a_n \in \Sigma\}.$$

Let $(\alpha, u_1, \dots, u_n, \beta)$ be a transition of X with $n \geqslant 2$. Then there exists a (unique) map $C_n[\mu(u_1), \dots, \mu(u_n)]^{\mathrm{ext}} \longrightarrow X$ taking the transition

$$(0_n, (\mu(u_1), 1), \dots, (\mu(u_n), n), 1_n)$$

to the transition $(\alpha, u_1, \dots, u_n, \beta)$. By hypothesis, this map factors uniquely as a composite

$$C_n[\mu(u_1), \dots, \mu(u_n)]^{\mathrm{ext}} \subset C_n[\mu(u_1), \dots, \mu(u_n)] \xrightarrow{g} X.$$

Let $1 \leqslant p < n$. There exists a (unique) state ν of $C_n[\mu(u_1), \dots, \mu(u_n)]$ such that the tuples

$$(0_n, (\mu(u_1), 1), \dots, (\mu(u_p), p), \nu),$$
$$(\nu, (\mu(u_{p+1}), p+1), \dots, (\mu(u_n), n), 1_n),$$

are two transitions of $C_n[\mu(u_1), \dots, \mu(u_n)]$ by Proposition 5.2. Hence the existence of a state $\nu_1 = g_0(\nu)$ of X such that the tuples $(\alpha, u_1, \dots, u_p, \nu_1)$

$$\begin{tikzcd} 0 \arrow[r, bend left, "{(a,1)}"] \arrow[r, bend right, "{(a,2)}"'] & 1 \end{tikzcd}$$

FIGURE 1. The higher dimensional transition system $D[a]$

and $(\nu_1, u_{p+1}, \ldots, u_n, \beta)$ are two transitions of X. Suppose that ν_2 is another state of X such that $(\alpha, u_1, \ldots, u_p, \nu_2)$ and $(\nu_2, u_{p+1}, \ldots, u_n, \beta)$ are two transitions of X. Let $\widetilde{h} = \widetilde{g} : \{(\mu(u_1), 1), \ldots, (\mu(u_n), n)\} \longrightarrow L$ be defined by $\widetilde{h}(\mu(u_i), i) = u_i$. Let $h_0 : [n] \to S$ be defined by $h_0(\nu') = g_0(\nu')$ for $\nu' \neq \nu$ and $h_0(\nu) = \nu_2$ (instead of ν_1). By Proposition 5.5, the pair of set maps $(h_0, \widetilde{h})$ yields a well-defined map of weak higher dimensional transition systems $h : C_n[\mu(u_1), \ldots, \mu(u_n)] \longrightarrow X$. So by orthogonality, one obtains $h = g$, and therefore $\nu_1 = \nu_2$. Thus, the weak higher dimensional transition system X satisfies the Unique intermediate state axiom. □

Corollary 5.7. *The full subcategory* **HDTS** *of higher dimensional transition systems is a small-orthogonality class of the category* **WHDTS** *of weak higher dimensional transition systems. More precisely, it is the full subcategory of objects orthogonal to the (unique) morphisms $D[a] \to C_1[a]$ for $a \in \Sigma$ and to the inclusions $C_n[a_1, \ldots, a_n]^{\mathrm{ext}} \subset C_n[a_1, \ldots, a_n]$ for $n \geqslant 2$ and $a_1, \ldots, a_n \in \Sigma$ where $D[a]$ is the higher dimensional transition system with set of states $\{0, 1\}$, with set of labels $\{(a, 1), (a, 2)\}$, with labelling maps $\mu(a, i) = a$, and containing the two 1-transitions $(0, (a, 1), 1)$ and $(0, (a, 2), 1)$ (see Figure 1).*

Proof. This is a consequence of Theorem 5.6 and Proposition 4.6. □

Corollary 5.8. *The full subcategory of higher dimensional transition systems is a full reflective locally finitely presentable subcategory of the category of weak higher dimensional transition systems. In particular, the inclusion functor* **HDTS** $\subset$ **WHDTS** *is limit-preserving and accessible.*

Proof. That **HDTS** is a full reflective locally presentable subcategory of **WHDTS** is a consequence of [AR94, Theorem 1.39]. Unfortunately, [AR94, Theorem 1.39] may be false for $\lambda = \aleph_0$. It only enables us to conclude that the category **HDTS** is locally $\aleph_1$-presentable. To prove that **HDTS** is locally finitely presentable, we observe, thanks to Proposition 4.6, that the notion of higher dimensional transition system is axiomatized by the axioms of weak higher dimensional transition system and by the two additional families of axioms: $(\forall \alpha, u, \beta), T_x(\alpha, u, \beta) \Rightarrow (\exists! u') T_x(\alpha, u', \beta)$ for $x \in \Sigma$ and

$$\begin{aligned}(\forall \alpha, u_1, \ldots, u_n, \beta), &T_{x_1,\ldots,x_n}(\alpha, u_1, \ldots, u_n, \beta) \Rightarrow \\ &(\exists! \nu)(T_{x_1,\ldots,x_p}(\alpha, u_1, \ldots, u_p, \nu) \wedge T_{x_{p+1},\ldots,x_n}(\nu, u_{p+1}, \ldots, u_n, \beta))\end{aligned}$$

for $n \geqslant 2$, $1 \leqslant p < n$ and $x_1, \dots, x_n \in \Sigma$. So the notion of higher dimensional transition system is axiomatized by a limit theory, i.e., by axioms of the form $(\forall x), \phi(x) \Rightarrow (\exists! y\ \psi(x,y))$ where ϕ and ψ are conjunctions of atomic formulas. Moreover, each symbol contains a finite number of arguments. Hence the result as in Theorem 3.4. $\square$

In fact, one can easily prove that the inclusion functor

$$\mathbf{HDTS} \subset \mathbf{WHDTS}$$

is finitely accessible. Let $X : I \to \mathbf{HDTS}$ be a directed diagram of higher dimensional transition systems. Let $X_i = (S_i, \mu_i : L_i \to \Sigma, T^i = \bigcup_{n \geqslant 1} T^i_n)$ and $X = (S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n)$. The weak higher dimensional transition system X remains orthogonal to the maps $D[a] \to C_1[a]$ for every $a \in \Sigma$ since this property is axiomatized by the sentences $(\forall \alpha, u, \beta), T_x(\alpha, u, \beta) \Rightarrow (\exists! u') T_x(\alpha, u', \beta)$ for $x \in \Sigma$. Since $\mathbf{WHDTS}$ is topological over $\mathbf{Set}^{\{s\} \cup \Sigma}$ by Theorem 3.4, the colimit $\varinjlim X$ in $\mathbf{WHDTS}$ is the weak higher dimensional transition system having as set of states the colimit $S = \varinjlim S_i$, as set of actions the colimit $L = \varinjlim L_i$, as labelling map the colimit $\mu = \varinjlim \mu_i$ and equipped with the final structure of weak higher dimensional transition system. The final structure is the set of transitions obtained by taking the closure under the Coherence axiom of the union $\bigcup_i T^i$ of the image of the T^i in $\bigcup_{n \geqslant 1}(S \times L^n \times S)$. Let $(\alpha, u_1, \dots, u_n, \beta)$ be a transition of $\bigcup_i T^i$ with $n \geqslant 2$. Let $1 \leqslant p < n$. There exists $i \in I$ such that the map $X_i \to \varinjlim X$ takes $(\alpha^i, u^i_1, \dots, u^i_n, \beta^i)$ to $(\alpha, u_1, \dots, u_n, \beta)$. By hypothesis, there exists a state ν^i of X_i such that $(\alpha^i, u^i_1, \dots, u^i_p, \nu^i)$ and $(\nu^i, u^i_{p+1}, \dots, u^i_n, \beta^i)$ are transitions of X_i. So the map $X_i \to \varinjlim X$ takes ν^i to a state ν of $\varinjlim X$ such that $(\alpha, u_1, \dots, u_p, \nu)$ and $(\nu, u_{p+1}, \dots, u_n, \beta)$ are transitions of $\bigcup_i T^i$. Let ν_1 and ν_2 be two states of $\varinjlim X$ such that $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_p, \nu_2)$ and $(\nu_2, u_{p+1}, \dots, u_n, \beta)$ are transitions of $\bigcup_i T^i$. Since the diagram X is directed, these four transitions come from four transitions of some X_j. So $\nu_1 = \nu_2$ since X_j satisfies the Unique intermediate state axiom. Thus, the set of transitions $\bigcup_i T^i$ satisfies the Unique intermediate state axiom. So by Theorem 4.7, the set of transitions $\bigcup_i T^i$ is the final structure and X satisfies the Unique intermediate state axiom. Therefore the inclusion functor $\mathbf{HDTS} \subset \mathbf{WHDTS}$ is finitely accessible.

6. Labelled symmetric precubical sets

The category of partially ordered sets or posets together with the strictly increasing maps ($x < y$ implies $f(x) < f(y)$) is denoted by $\mathbf{PoSet}$.

Let $\delta^\alpha_i : [n-1] \to [n]$ be the set map defined for $1 \leqslant i \leqslant n$ and $\alpha \in \{0,1\}$ by $\delta^\alpha_i(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$. These maps are called the *face maps*. The *reduced box category*, denoted by $\square$, is the subcategory of $\mathbf{PoSet}$ with the set of objects $\{[n], n \geqslant 0\}$ and generated by the morphisms δ^α_i. They satisfy the cocubical relations $\delta^\beta_j \delta^\alpha_i = \delta^\alpha_i \delta^\beta_{j-1}$ for $i < j$ and for all $(\alpha,\beta) \in \{0,1\}^2$. In fact, these algebraic relations give a presentation by generators and relations of $\square$.

Proposition 6.1 ([Gau10, Proposition 3.1])**.** *Let $n \geqslant 1$. Let $(\epsilon_1,\dots,\epsilon_n)$ and $(\epsilon'_1,\dots,\epsilon'_n)$ be two elements of the poset $[n]$ with $(\epsilon_1,\dots,\epsilon_n) \leqslant (\epsilon'_1,\dots,\epsilon'_n)$. Then there exist $i_1 > \dots > i_{n-r}$ and $\alpha_1,\dots,\alpha_{n-r} \in \{0,1\}$ such that $(\epsilon_1,\dots,\epsilon_n) = \delta_{i_1}^{\alpha_1}\dots\delta_{i_{n-r}}^{\alpha_{n-r}}(0\dots0)$ and $(\epsilon'_1,\dots,\epsilon'_n) = \delta_{i_1}^{\alpha_1}\dots\delta_{i_{n-r}}^{\alpha_{n-r}}(1\dots1)$ where $r \geqslant 0$ is the number of 0 and 1 in the arguments $0\dots0$ and $1\dots1$. In other terms, $(\epsilon_1,\dots,\epsilon_n)$ is the bottom element and $(\epsilon'_1,\dots,\epsilon'_n)$ the top element of a r-dimensional subcube of $[n]$.*

Definition 6.2. Let $n \geqslant 1$. Let $(\epsilon_1,\dots,\epsilon_n)$ and $(\epsilon'_1,\dots,\epsilon'_n)$ be two elements of the poset $[n]$. The integer r of Proposition 6.1 is called the distance between $(\epsilon_1,\dots,\epsilon_n)$ and $(\epsilon'_1,\dots,\epsilon'_n)$. Let us denote this situation by $r = d((\epsilon_1,\dots,\epsilon_n),(\epsilon'_1,\dots,\epsilon'_n))$. By definition, one has

$$r = \sum_{i=1}^{i=n} |\epsilon_i - \epsilon'_i|.$$

Definition 6.3. A set map $f : [m] \to [n]$ is adjacency-preserving if it is strictly increasing and if $d((\epsilon_1,\dots,\epsilon_m),(\epsilon'_1,\dots,\epsilon'_m)) = 1$ implies

$$d(f(\epsilon_1,\dots,\epsilon_m), f(\epsilon'_1,\dots,\epsilon'_m)) = 1.$$

The subcategory of **PoSet** with set of objects $\{[n], n \geqslant 0\}$ generated by the adjacency-preserving maps is denoted by $\widehat{\square}$.

Let $\sigma_i : [n] \to [n]$ be the set map defined for $1 \leqslant i \leqslant n-1$ and $n \geqslant 2$ by $\sigma_i(\epsilon_1,\dots,\epsilon_n) = (\epsilon_1,\dots,\epsilon_{i-1},\epsilon_{i+1},\epsilon_i,\epsilon_{i+2},\dots,\epsilon_n)$. These maps are called the *symmetry maps*. The face maps and the symmetry maps are examples of adjacency-preserving maps.

Proposition 6.4 ([Gau10, Proposition A.3])**.** *Let $f : [m] \to [n]$ be an adjacency-preserving map. The following conditions are equivalent:*

(1) *The map f is a composite of face maps and symmetry maps.*
(2) *The map f is one-to-one.*

Notation 6.5. The subcategory of $\widehat{\square}$ generated by the one-to-one adjacency-preserving maps is denoted by $\square_S$. In particular, one has the inclusions of categories

$$\square \subset \square_S \subset \widehat{\square}.$$

By [GM03, Theorem 8.1], the category $\square_S$ is the quotient of the free category generated by the face maps δ_i^α and symmetry maps σ_i, by the following algebraic relations:

- the cocubical relations: $\delta_j^\beta\delta_i^\alpha = \delta_i^\alpha\delta_{j-1}^\beta$ for $i < j$ and for all $(\alpha,\beta) \in \{0,1\}^2$;
- the Moore relations for symmetry operators: $\sigma_i\sigma_i = \mathrm{Id}$, $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $i = j-1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i < j-1$;

- the relations $\sigma_i\delta_j^\alpha = \delta_j^\alpha\sigma_{i-1}$ for $j < i$, $\sigma_i\delta_j^\alpha = \delta_{i+1}^\alpha$ for $j = i$, $\sigma_i\delta_j^\alpha = \delta_i^\alpha$ for $j = i+1$ and $\sigma_i\delta_j^\alpha = \delta_j^\alpha\sigma_i$ for $j > i+1$.

Definition 6.6. A symmetric precubical set is a presheaf over $\square_S$. The corresponding category is denoted by $\square_S^{op}\mathbf{Set}$. If K is a symmetric precubical set, then let $K_n := K([n])$ and for every set map $f : [m] \to [n]$ of $\square_S$, denote by $f^* : K_n \to K_m$ the corresponding set map.

Let $\square_S[n] := \square_S(-, [n])$. It is called the *$n$-dimensional (symmetric) cube.* By the Yoneda lemma, one has the natural bijection of sets

$$\square_S^{op}\mathbf{Set}(\square_S[n], K) \cong K_n$$

for every precubical set K. The *boundary* of $\square_S[n]$ is the symmetric precubical set denoted by $\partial\square_S[n]$ defined by removing the interior of $\square_S[n]$: $(\partial\square_S[n])_k := (\square_S[n])_k$ for $k < n$ and $(\partial\square_S[n])_k = \varnothing$ for $k \geqslant n$. In particular, one has $\partial\square_S[0] = \varnothing$. An *$n$-dimensional* symmetric precubical set K is a symmetric precubical set such that $K_p = \varnothing$ for $p > n$ and $K_n \neq \varnothing$. The labelled at most n-dimensional symmetric precubical set $K_{\leqslant n}$ denotes the labelled symmetric precubical set defined by $(K_{\leqslant n})_p = K_p$ for $p \leqslant n$ and $(K_{\leqslant n})_p = \varnothing$ for $p > n$.

Notation 6.7. Let $f : K \to L$ be a morphism of symmetric precubical sets. Let $n \geqslant 0$. The set map from K_n to L_n induced by f will be sometimes denoted by f_n.

Notation 6.8. Let $\partial_i^\alpha = (\delta_i^\alpha)^*$. And let $s_i = (\sigma_i)^*$.

Proposition 6.9 ([Gau10, Proposition A.4])**.** *The following data define a symmetric precubical set denoted by* $!^S\Sigma$:

- $(!^S\Sigma)_0 = \{()\}$ *(the empty word).*
- *For* $n \geqslant 1$, $(!^S\Sigma)_n = \Sigma^n$.
- $\partial_i^0(a_1, \dots, a_n) = \partial_i^1(a_1, \dots, a_n) = (a_1, \dots, \widehat{a_i}, \dots, a_n)$ *where the notation* $\widehat{a_i}$ *means that* a_i *is removed.*
- $s_i(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n)$ *for* $1 \leqslant i \leqslant n$.

Moreover, the symmetric precubical set $!^S\Sigma$ *is orthogonal to the set of morphisms*

$$\{\square_S[n] \sqcup_{\partial\square_S[n]} \square_S[n] \longrightarrow \square_S[n], n \geqslant 2\}.$$

Definition 6.10. A labelled symmetric precubical set (over Σ) is an object of the comma category $\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$. That is, an object is a map of symmetric precubical sets $\ell : K \to !^S\Sigma$ and a morphism is a commutative diagram

$$\begin{tikzcd} K \arrow[rr] \arrow[dr] & & L \arrow[dl] \\ & !^S\Sigma. & \end{tikzcd}$$

$$\begin{tikzcd}
& () \arrow[dr, "(b)"] & \\
() \arrow[ur, "(a)"] \arrow[dr, "(b)"'] & (a,b) & () \\
& () \arrow[ur, "(a)"'] &
\end{tikzcd}$$

FIGURE 2. Concurrent execution of a and b

The map ℓ is called the labelling map. The symmetric precubical set K is sometimes called the underlying symmetric precubical set of the labelled symmetric precubical set. A labelled symmetric precubical set $K \to !^S\Sigma$ is sometimes denoted by K without explicitly mentioning the labelling map.

Notation 6.11. Let $n \geqslant 1$. Let $a_1, \dots, a_n$ be labels of Σ. Let us denote by $\square_S[a_1, \dots, a_n] : \square_S[n] \to !^S\Sigma$ the labelled symmetric precubical set defined by

$$\square_S[a_1, \dots, a_n](f) = f^*(a_1, \dots, a_n).$$

And let us denote by $\partial\square_S[a_1, \dots, a_n] : \partial\square_S[n] \to !^S\Sigma$ the labelled symmetric precubical set defined as the composite

$$\partial\square_S[a_1, \dots, a_n] : \partial\square_S[n] \subset \square_S[n] \xrightarrow{\square_S[a_1,\dots,a_n]} !^S\Sigma.$$

Figure 2 gives the example of the labelled 2-cube $\square_S[a, b]$. It represents the concurrent execution of a and b. It is important to notice that two opposite faces of Figure 2 have the same label.

Since colimits are calculated objectwise for presheaves, the n-cubes are finitely accessible. Since the set of cubes is a dense (and hence strong) generator, the category of labelled symmetric precubical sets is locally finitely presentable by [AR94, Theorem 1.20 and Proposition 1.57]. When the set of labels Σ is the singleton $\{\tau\}$, the category $\square_S^{op}\mathbf{Set}{\downarrow}!^S\{\tau\}$ is isomorphic to the category of (unlabelled) symmetric precubical sets since $!^S\{\tau\}$ is the terminal symmetric precubical set.

7. The higher dimensional automata paradigm

Definition 7.1. A labelled symmetric precubical set K satisfies the paradigm of higher dimensional automata (HDA paradigm) if for every $p \geqslant 2$, every commutative square of solid arrows (called a labelled p-shell or labelled

p-dimensional shell)

$$\begin{tikzcd} \partial\square_S[p] \arrow[r] \arrow[d] & K \arrow[d] \\ \square_S[p] \arrow[r] \arrow[ur, dashed, "k"] & !^S\Sigma \end{tikzcd}$$

admits at most one lift k (i.e., a map k making the two triangles commutative).

By Definition 6.10, a commutative square consisting of solid arrows as in

$$\begin{tikzcd} \partial\square_S[p] \arrow[r] \arrow[d] & K \arrow[d] \\ \square_S[p] \arrow[r] \arrow[ur, dashed, "k"] & !^S\Sigma \end{tikzcd}$$

is equivalent to a diagram of labelled symmetric precubical sets consisting of the solid arrows in

$$\begin{tikzcd} \partial\square_S[a_1,\dots,a_p] \arrow[r] \arrow[d] & (K \to !^S\Sigma) \\ \square_S[a_1,\dots,a_p], \arrow[ur, dashed, "k"] & \end{tikzcd}$$

where $(a_1,\dots,a_p)$ is the image of $\mathrm{Id}_{[p]}$ under $\square_S[p] \to !^S\Sigma$. For the same reason, the existence of the lift k in the former diagram is equivalent to the existence of the lift k in the latter diagram.

Proposition 7.2. *Let $n \geqslant 0$ and $a_1,\dots,a_n \in \Sigma$. The labelled n-cube $\square_S[a_1,\dots,a_n]$ satisfies the HDA paradigm.*

Proof. Consider a commutative diagram of solid arrows of the form

$$\begin{tikzcd} \partial\square_S[p] \arrow[r, "f"] \arrow[d] & \square_S[n] \arrow[d, "\mathrm{Id}_{[n]}\mapsto(a_1,\dots,a_n)"] \\ \square_S[p] \arrow[r] \arrow[ur, dashed, "k"] & !^S\Sigma \end{tikzcd}$$

with $p \geqslant 2$. Then $f_0 = k_0$ as set map from $[p]$ to $[n]$. By the Yoneda lemma, there is a bijection $\square_S^{op}\mathbf{Set}(\square_S[p], \square_S[n]) \cong \square_S([p],[n])$ induced by the mapping $g \mapsto g_0$. So there exists at most one such lift k. $\square$

Proposition 7.3. *For a labelled symmetric precubical set $K \to !^S\Sigma$, the following conditions are equivalent:*

1. *The labelled symmetric precubical set $K \to !^S\Sigma$ satisfies the HDA paradigm.*
2. *The map $K \to !^S\Sigma$ satisfies the right lifting property with respect to the set of maps*
$$\left\{\square_S[p] \sqcup_{\partial\square_S[p]} \square_S[p] \to \square_S[p], p \geqslant 2\right\}.$$
3. *The map $K \to !^S\Sigma$ satisfies the right lifting property with respect to the set of maps*
$$\left\{\square_S[p] \sqcup_{\partial\square_S[p]} \square_S[p] \to \square_S[p], p \geqslant 2\right\}$$
and the lift is unique.
4. *The labelled symmetric precubical set $K \to !^S\Sigma$ is orthogonal to the set of maps of labelled symmetric precubical sets*
$$\left\{\square_S[a_1, \dots, a_p] \sqcup_{\partial\square_S[a_1,\dots,a_p]} \square_S[a_1, \dots, a_p] \to \square_S[a_1, \dots, a_p]\right\}$$
for $p \geqslant 2$ and $a_1, \dots, a_p \in \Sigma$.

Proof. The equivalence (1) $\iff$ (2) is due to the "at most" in the definition of the HDA paradigm. The equivalence (3) $\iff$ (4) is due to the definition of a map of labelled symmetric precubical sets. The implication (3) $\Longrightarrow$ (2) is obvious. The implication (2) $\Longrightarrow$ (3) comes from the fact that for every symmetric precubical set K, the set map

$$\square_S^{op}\mathbf{Set}(\square_S[p], K) \to \square_S^{op}\mathbf{Set}(\square_S[p] \sqcup_{\partial\square_S[p]} \square_S[p], K)$$

is one-to-one. $\square$

Corollary 7.4. *The full subcategory, denoted by $\mathbf{HDA}^\Sigma$, of $\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$ containing the objects satisfying the HDA paradigm is a full reflective locally presentable category of the category $\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$ of labelled symmetric precubical sets. In other terms, the inclusion functor $i_\Sigma : \mathbf{HDA}^\Sigma \subset \square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$ has a left adjoint $\mathrm{Sh}_\Sigma : \square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma \to \mathbf{HDA}^\Sigma$.*

When Σ is the singleton $\{\tau\}$, the category $\mathbf{HDA}^\Sigma$ will be simply denoted by $\mathbf{HDA}$.

Proof. This is a corollary of Proposition 7.3 and [AR94, Theorem 1.39]. $\square$

In fact the category $\mathbf{HDA}^\Sigma$ is locally finitely presentable; indeed, the labelled n-cubes for $n \geqslant 0$ are in $\mathbf{HDA}^\Sigma$ by Proposition 7.2, and one can prove that they form a dense set of generators.

Notation 7.5. When Σ is the singleton $\{\tau\}$, let $i := i_\Sigma$ and $\mathrm{Sh} := \mathrm{Sh}_\Sigma$.

One has

$$i_\Sigma(K \to !^S\Sigma) \cong (i(K) \to i(!^S\Sigma)) = (K \to !^S\Sigma),$$
$$\mathrm{Sh}_\Sigma(K \to !^S\Sigma) \cong (\mathrm{Sh}(K) \to \mathrm{Sh}(!^S\Sigma) \cong !^S\Sigma),$$

since the symmetric precubical set $!^S\Sigma$ already belongs to **HDA** by Proposition 6.9.

8. Cubes as labelled symmetric precubical sets and as higher dimensional transition systems

Let us denote by $\mathrm{CUBE}(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)$ the full subcategory of that of labelled symmetric precubical sets containing the labelled cubes $\square_S[a_1,\dots,a_n]$ with $n \geqslant 0$ and $a_1,\dots,a_n \in \Sigma$. Let us denote by $\mathrm{CUBE}(\mathbf{WHDTS})$ the full subcategory of that of weak higher dimensional transition systems containing the labelled cubes $C_n[a_1,\dots,a_n]$ with $n \geqslant 0$ and with $a_1,\dots,a_n \in \Sigma$. This section is devoted to proving that these two small categories are isomorphic (cf. Theorem 8.5). Note that $\mathrm{CUBE}(\mathbf{WHDTS}) \subset \mathbf{HDTS}$ by Proposition 5.2.

Lemma 8.1. *Let $f : \square_S[m] \to \square_S[n]$ be a map of symmetric precubical sets. Then there exists a unique set map $\widehat{f} : \{1,\dots,n\} \to \{1,\dots,m\} \cup \{-\infty,+\infty\}$ such that $f(\epsilon_1,\dots,\epsilon_m) = (\epsilon_{\widehat{f}(1)},\dots,\epsilon_{\widehat{f}(n)})$ for every $(\epsilon_1,\dots,\epsilon_m) \in [m]$ with the conventions $\epsilon_{-\infty} = 0$ and $\epsilon_{+\infty} = 1$. Moreover, the restriction $\overline{f} : \widehat{f}^{-1}(\{1,\dots,m\}) \to \{1,\dots,m\}$ is a bijection.*

By convention, and for the sequel, the set map $\widehat{f}$ will be defined from $\{1,\dots,n\} \cup \{-\infty,+\infty\}$ to $\{1,\dots,m\} \cup \{-\infty,+\infty\}$ by setting $\widehat{f}(-\infty) = -\infty$ and $\widehat{f}(+\infty) = +\infty$.

Proof. If $\widehat{f_1}$ and $\widehat{f_2}$ are two solutions, then one has

$$(\epsilon_{\widehat{f_1}(1)},\dots,\epsilon_{\widehat{f_1}(n)}) = (\epsilon_{\widehat{f_2}(1)},\dots,\epsilon_{\widehat{f_2}(n)})$$

for every $(\epsilon_1,\dots,\epsilon_m) \in [m]$. Let $i \in \{1,\dots,n\}$. If $\widehat{f_1}(i) = -\infty$, then $\epsilon_{\widehat{f_1}(i)} = 0 = \epsilon_{\widehat{f_2}(i)}$ for every $(\epsilon_1,\dots,\epsilon_m) \in [m]$. So in this case, $\widehat{f_1}(i) = \widehat{f_2}(i)$. For the same reason, if $\widehat{f_1}(i) = +\infty$, then $\widehat{f_1}(i) = \widehat{f_2}(i)$. If $\widehat{f_1}(i) \in \{1,\dots,m\}$, then $\epsilon_{\widehat{f_1}(i)} = \epsilon_{\widehat{f_2}(i)}$ for every $(\epsilon_1,\dots,\epsilon_m) \in [m]$. So $\widehat{f_1}(i) = \widehat{f_2}(i)$ again. Thus, one obtains $\widehat{f_1} = \widehat{f_2}$. Hence there is at most one such $\widehat{f}$. Because of the algebraic relations permuting the symmetry and face maps recalled in Section 6, the set map $f_0 : (\square_S[m])_0 = [m] \to (\square_S[n])_0 = [n]$ factors as a composite $[m] \to [m] \to [n]$ where the left-hand map is a composite of symmetry maps and where the right-hand map is a composite of face maps (see also [GM03]). So there exists a permutation σ of $\{1,\dots,m\}$ such that

$$f(\epsilon_1,\dots,\epsilon_m) = \delta_{i_1}^{\alpha_1}\dots\delta_{i_{n-m}}^{\alpha_{n-m}}(\epsilon_{\sigma(1)},\dots,\epsilon_{\sigma(m)})$$

for every $(\epsilon_1, \dots, \epsilon_m) \in [m]$. Because of the cocubical relations satisfied by the face maps, one can suppose that $i_1 > i_2 > \dots > i_{n-m}$. Let $j_1 < \dots < j_m$ such that

$$\{j_1, \dots, j_m\} \cup \{i_1, \dots, i_{n-m}\} = \{1, \dots, n\}.$$

So $f(\epsilon_1, \dots, \epsilon_m) = (\epsilon'_1, \dots, \epsilon'_n)$ with $\epsilon'_{i_k} = \alpha_k$ for all $k \in \{1, \dots, n-m\}$ and $\epsilon'_{j_k} = \epsilon_{\sigma(k)}$. Therefore, the set map $\widehat{f} : \{1, \dots, n\} \to \{1, \dots, m\} \cup \{-\infty, +\infty\}$ defined by $\widehat{f}(i_k) = -\infty$ if $\alpha_k = 0$, $\widehat{f}(i_k) = +\infty$ if $\alpha_k = 1$ and $\widehat{f}(j_k) = \sigma(k)$ is a solution. $\square$

Lemma 8.2. *Let*

$$f : \square_S[a_1, \dots, a_m] \to \square_S[b_1, \dots, b_n],$$
$$g : \square_S[b_1, \dots, b_n] \to \square_S[c_1, \dots, c_p],$$

be two maps of labelled symmetric precubical sets. Then one has $\widehat{g \circ f} = \widehat{f} \circ \widehat{g}$ *with the notations of Lemma 8.1.*

Proof. The set map $\widehat{f} : \{1, \dots, n\} \cup \{-\infty, +\infty\} \to \{1, \dots, m\} \cup \{-\infty, +\infty\}$ is the unique set map such that $f_0(\epsilon_1, \dots, \epsilon_m) = (\epsilon_{\widehat{f}(1)}, \dots, \epsilon_{\widehat{f}(n)})$ for every $(\epsilon_1, \dots, \epsilon_m) \in [m]$ with the same notations as above and with $\widehat{f}(-\infty) = -\infty$ and $\widehat{f}(+\infty) = +\infty$. Therefore, one obtains the equality $g_0(f_0(\epsilon_1, \dots, \epsilon_m)) = g_0(\epsilon_{\widehat{f}(1)}, \dots, \epsilon_{\widehat{f}(n)})$ for every $(\epsilon_1, \dots, \epsilon_m) \in [m]$. The set map $\widehat{g} : \{1, \dots, p\} \cup \{-\infty, +\infty\} \to \{1, \dots, n\} \cup \{-\infty, +\infty\}$ is the unique set map such that $g_0(\epsilon'_1, \dots, \epsilon'_n) = (\epsilon'_{\widehat{g}(1)}, \dots, \epsilon'_{\widehat{g}(p)})$ for every $(\epsilon'_1, \dots, \epsilon'_n) \in [n]$ with the same notations as above and with $\widehat{g}(-\infty) = -\infty$ and $\widehat{g}(+\infty) = +\infty$. Let $\epsilon'_i = \epsilon_{\widehat{f}(i)}$ for $1 \leqslant i \leqslant m$. If $\widehat{g}(i) \in \{-\infty, +\infty\}$, then $\epsilon'_{\widehat{g}(i)} = \epsilon_{\widehat{f}(\widehat{g}(i))}$ since $\widehat{f}(-\infty) = -\infty$ and $\widehat{f}(+\infty) = +\infty$. If $\widehat{g}(i) \notin \{-\infty, +\infty\}$, then $\epsilon'_{\widehat{g}(i)} = \epsilon_{\widehat{f}(\widehat{g}(i))}$ by definition of the family ϵ'. So one obtains

$$g_0(f_0(\epsilon_1, \dots, \epsilon_m)) = g_0(\epsilon_{\widehat{f}(1)}, \dots, \epsilon_{\widehat{f}(n)}) = (\epsilon_{\widehat{f}(\widehat{g}(1))}, \dots, \epsilon_{\widehat{f}(\widehat{g}(p))})$$

for every $(\epsilon_1, \dots, \epsilon_m) \in [m]$. Thus by Lemma 8.1, one obtains $\widehat{g \circ f} = \widehat{f} \circ \widehat{g}$. $\square$

Let $m, n \geqslant 0$ and $a_1, \dots, a_m, b_1, \dots, b_n \in \Sigma$. A map of labelled symmetric precubical sets $f : \square_S[a_1, \dots, a_m] \to \square_S[b_1, \dots, b_n]$ gives rise to a set map

$$\boxed{f_0 : [m] = \{0,1\}^m = \square_S[a_1, \dots, a_m]_0 \to [n] = \{0,1\}^n = \square_S[b_1, \dots, b_n]_0}$$

from the set of states of $C_m[a_1, \dots, a_m]$ to the set of states of $C_n[b_1, \dots, b_n]$ which belongs to $\square_S([m], [n]) = \square_S^{\mathrm{op}}\mathbf{Set}(\square_S[m], \square_S[n])$. By Lemma 8.1, there exists a unique set map $\widehat{f} : \{1, \dots, n\} \to \{1, \dots, m\} \cup \{-\infty, +\infty\}$ such that

$$\boxed{f_0(\epsilon_1, \dots, \epsilon_m) = (\epsilon_{\widehat{f}(1)}, \dots, \epsilon_{\widehat{f}(n)})}$$

for every $(\epsilon_1, \dots, \epsilon_m) \in [m]$ with the conventions $\epsilon_{-\infty} = 0$ and $\epsilon_{+\infty} = 1$. Moreover, the restriction $\overline{f} : \widehat{f}^{-1}(\{1, \dots, m\}) \to \{1, \dots, m\}$ is a bijection. Since $f : \square_S[a_1, \dots, a_m] \to \square_S[b_1, \dots, b_n]$ is compatible with the labelling, one necessarily has $a_i = b_{\overline{f}^{-1}(i)}$ for every $i \in \{1, \dots, m\}$. One deduces a set map

$$\widetilde{f} : \{(a_1, 1), \dots, (a_m, m)\} \to \{(b_1, 1), \dots, (b_n, n)\}$$

from the set of actions of $C_m[a_1, \dots, a_m]$ to the set of actions of $C_n[b_1, \dots, b_n]$ by setting

$$\boxed{\widetilde{f}(a_i, i) = (b_{\overline{f}^{-1}(i)}, \overline{f}^{-1}(i)) = (a_i, \overline{f}^{-1}(i))}.$$

Lemma 8.3. *The two set maps f_0 and $\widetilde{f}$ above defined by starting from a map of labelled symmetric precubical sets $f : \square_S[a_1, \dots, a_m] \to \square_S[b_1, \dots, b_n]$ yield a map of weak higher dimensional transition systems*

$$\boxed{\mathbb{T}(f) : C_m[a_1, \dots, a_m] \to C_n[b_1, \dots, b_n]}.$$

Proof. Let

$$((\epsilon_1, \dots, \epsilon_m), (a_{i_1}, i_1), \dots, (a_{i_r}, i_r), (\epsilon'_1, \dots, \epsilon'_m))$$

be a transition of $C_m[a_1, \dots, a_m]$. One has for every $i \in \{1, \dots, n\}$:

- $\epsilon_{\widehat{f}(i)} \leqslant \epsilon'_{\widehat{f}(i)}$ for every $i \in \{1, \dots, n\}$, by definition of a transition of $C_m[a_1, \dots, a_m]$.
- $\epsilon_{\widehat{f}(i)} = \epsilon'_{\widehat{f}(i)}$ if $i \in \widehat{f}^{-1}(\{-\infty, +\infty\})$.
- $\epsilon_{\widehat{f}(i)} \neq \epsilon'_{\widehat{f}(i)}$ for $i \in \widehat{f}^{-1}(\{1, \dots, m\})$ if and only if $\widehat{f}(i) = \overline{f}(i) \in \{i_1, \dots, i_r\}$, by definition of a transition of $C_m[a_1, \dots, a_m]$ again.

So one has $\epsilon_{\widehat{f}(i)} \neq \epsilon'_{\widehat{f}(i)}$ if and only if $i = \overline{f}^{-1}(i_k)$ for some $k \in \{1, \dots, r\}$. Thus, the $(d+2)$-tuple

$$((\epsilon_{\widehat{f}(1)}, \dots, \epsilon_{\widehat{f}(n)}), (a_{i_1}, \overline{f}^{-1}(i_1)), \dots, (a_{i_r}, \overline{f}^{-1}(i_r)), (\epsilon'_{\widehat{f}(1)}, \dots, \epsilon'_{\widehat{f}(n)}))$$

is a transition of the higher dimensional transition system $C_n[b_1, \dots, b_n]$. □

Proposition 8.4. *Let $\mathbb{T}(\square_S[a_1, \dots, a_n]) := C_n[a_1, \dots, a_n]$. Together with the mapping $f \mapsto \mathbb{T}(f)$ defined in Lemma 8.3, one obtains a well-defined functor from* $\mathrm{CUBE}(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)$ *to* $\mathrm{CUBE}(\mathbf{WHDTS})$.

Proof. The set map $\widehat{\mathrm{Id}_{[m]}}$ is the inclusion

$$\{1, \dots, m\} \subset \{1, \dots, m\} \cup \{-\infty, +\infty\}.$$

So $\mathbb{T}(\mathrm{Id}_{\square_S[a_1, \dots, a_n]}) = \mathrm{Id}_{C_n[a_1, \dots, a_n]}$. Let $f : \square_S[a_1, \dots, a_m] \to \square_S[b_1, \dots, b_n]$ and $g : \square_S[b_1, \dots, b_n] \to \square_S[c_1, \dots, c_p]$ be two maps of labelled symmetric precubical sets. The functoriality of the mapping $K \mapsto K_{\leqslant 0}$ yields the equality $(g \circ f)_0 = g_0 \circ f_0$. One has $\widetilde{g}(\widetilde{f}(a_i, i)) = \widetilde{g}(a_i, \overline{f}^{-1}(i)) = (a_i, \overline{g}^{-1}(\overline{f}^{-1}(i)))$.

The integer $N = \overline{g}^{-1}(\overline{f}^{-1}(i)) \in \{1,..,p\}$ satisfies $i = \widehat{f}(\widehat{g}(N)) = \widehat{g\circ f}(N)$ by Lemma 8.2. So by Lemma 8.1, one has $\overline{g\circ f}^{-1}(i) = N$. Thus, one obtains

$$\widetilde{g}(\widetilde{f}(a_i,i)) = (a_i, \overline{g\circ f}^{-1}(i)) = \widetilde{g\circ f}(a_i,i).$$

Hence the functoriality. $\square$

Theorem 8.5. *The functor* $\mathbb{T} : \mathrm{CUBE}(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma) \to \mathrm{CUBE}(\mathbf{WHDTS})$ *constructed in Proposition 8.4 is an isomorphism of categories.*

Proof. Let us construct a functor

$$\mathbb{T}^{-1} : \mathrm{CUBE}(\mathbf{WHDTS}) \to \mathrm{CUBE}(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)$$

such that $\mathbb{T}\circ\mathbb{T}^{-1} = \mathrm{Id}_{\mathrm{CUBE}(\mathbf{WHDTS})}$ and $\mathbb{T}^{-1}\circ\mathbb{T} = \mathrm{Id}_{\mathrm{CUBE}(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)}$.

Let $\mathbb{T}^{-1}(C_n[a_1,\dots,a_n]) := \square_S[a_1,\dots,a_n]$ for every $n \geqslant 0$ and every $a_1,\dots,a_n \in \Sigma$. Let $f : C_m[a_1,\dots,a_m] \to C_n[b_1,\dots,b_n]$ be a map of weak higher dimensional transition systems. By definition, it gives rise to a set map $f_0 : [m] \to [n]$ between the set of states and to a set map $\widetilde{f} : \{(a_1,1),\dots,(a_m,m)\} \to \{(b_1,1),\dots,(b_n,n)\}$ between the set of actions. Since the map $f : C_m[a_1,\dots,a_m] \to C_n[b_1,\dots,b_n]$ is compatible with the labelling maps of the source and target higher dimensional transition systems, one necessarily has $\widetilde{f}(a_i,i) = (a_i,\underline{f}(i))$ where $\underline{f} : \{1,\dots,m\} \to \{1,\dots,n\}$ denotes a set map. Since the $(m+2)$-tuple

$$(f_0(0,\dots,0),(a_1,\underline{f}(1)),\dots,(a_m,\underline{f}(m)),f_0(1,\dots,1))$$

is a transition of $C_n[b_1,\dots,b_n]$, the map $\underline{f} : \{1,\dots,m\} \to \{1,\dots,n\}$ is one-to-one. Let $\overline{f} : \underline{f}(\{1,\dots,n\}) \to \{1,\dots,m\}$ be the inverse map. Let $(\epsilon_1,\dots,\epsilon_m) < (\epsilon'_1,\dots,\epsilon'_m)$ be two adjacent elements of $[m]$, more precisely, $\epsilon_i = \epsilon'_i$ for all $i \in \{1,\dots,m\}\backslash\{j\}$ and $0 = \epsilon_j < \epsilon'_j = 1$. Then the triple

$$((\epsilon_1,\dots,\epsilon_m),(a_j,j),(\epsilon'_1,\dots,\epsilon'_m))$$

is a 1-transition of $C_m[a_1,\dots,a_m]$. So the triple

$$(f_0(\epsilon_1,\dots,\epsilon_m),(a_j,\underline{f}(j)),f_0(\epsilon'_1,\dots,\epsilon'_m))$$

is a 1-transition of $C_n[b_1,\dots,b_n]$. Thus, the n-tuples $f_0(\epsilon_1,\dots,\epsilon_m)$ and $f_0(\epsilon'_1,\dots,\epsilon'_m)$ are adjacent in $[n]$, and the only difference is the $\underline{f}(j)$-th coordinate. Thus, the mapping $f \mapsto f_0$ yields a set map

$$(-)_0 : \mathbf{WHDTS}(C_m[a_1,\dots,a_m],C_n[b_1,\dots,b_n]) \to \widehat{\square}([m],[n]).$$

The map $f_0 : [m] \to [n]$ factors uniquely as a composite

$$[m] \xrightarrow{\psi} [m] \xrightarrow{\phi} [n]$$

with $\phi \in \square$ since the image $f_0([m])$ is an m-subcube of $[n]$ (see [Gau10, Proposition 3.1 and Proposition 3.11]). Let $\phi = \delta_{i_1}^{\alpha_1}\dots\delta_{i_{n-m}}^{\alpha_{n-m}}$ with $i_1 > i_2 > \dots > i_{n-m}$. Let $\widehat{f} : \{1,\dots,n\}\cup\{-\infty,+\infty\} \to \{1,\dots,m\}\cup\{-\infty,+\infty\}$ be the set map defined by the four mutually exclusive cases:

- $\widehat{f}(-\infty) = -\infty$ and $\widehat{f}(+\infty) = +\infty$,
- $\widehat{f}(k) = \overline{f}(k)$ if $k \in \underline{f}(\{1, \dots, n\})$,
- $\widehat{f}(i_k) = -\infty$ if $\alpha_k = 0$,
- $\widehat{f}(i_k) = +\infty$ if $\alpha_k = 1$.

Since $\widehat{f}(\underline{f}(i)) = i$ one has for every m-tuple $(\epsilon_1, \dots, \epsilon_m)$ of $[m]$ the equality

$$f_0(\epsilon_1, \dots, \epsilon_m) = (\epsilon_{\widehat{f}(1)}, \dots, \epsilon_{\widehat{f}(n)}).$$

So $\psi \in \widehat{\square}$ is one-to-one, and therefore equal to a composite of σ_i maps by Proposition 6.4. Thus, one obtains $f_0 \in \square_S([m], [n])$. The Yoneda bijection

$$\square_S([m], [n]) \cong \square_S^{\mathrm{op}}\mathbf{Set}(\square_S[m], \square_S[n])$$

takes f_0 to a map of symmetric precubical sets $\mathbb{T}^{-1}(f) : \square_S[m] \to \square_S[n]$ preserving the labelling. So $\mathbb{T}^{-1}(f)$ yields a map of labelled symmetric precubical sets, still denoted by $\mathbb{T}^{-1}(f)$, from $\square_S[a_1, \dots, a_m]$ to $\square_S[b_1, \dots, b_n]$ and it is clear that $\mathbb{T}(\mathbb{T}^{-1}(f)) = f$ by construction of $\mathbb{T}$. The equality $\mathbb{T}^{-1}(\mathbb{T}(f)) = f$ is due to the uniqueness of $\widehat{f}$ in Lemma 8.1. $\square$

9. Labelled symmetric precubical sets as weak higher dimensional transition systems

For the sequel, the category of small categories is denoted by $\mathbf{Cat}$. Let $H : I \longrightarrow \mathbf{Cat}$ be a functor from a small category I to $\mathbf{Cat}$. The *Grothendieck construction* $I \int H$ is the category defined as follows [Tho79]: the objects are the pairs (i, a) where i is an object of I and a is an object of $H(i)$; a morphism $(i, a) \to (j, b)$ consists in a map $\phi : i \to j$ and in a map $h : H(\phi)(a) \to b$.

Lemma 9.1 (cf. [Gau10, Lemma 9.3] and [Gau08, Lemma A.1])**.** *Let I be a small category, and $i \mapsto K^i$ be a functor from I to the category of labelled symmetric precubical sets. Let $K = \varinjlim_i K^i$. Let $H : I \to \mathbf{Cat}$ be the functor defined by $H(i) = \square_S{\downarrow}K^i$. Then the functor $\iota : I \int H \to \square_S{\downarrow}K$ defined by $\iota(i, \square_S[m] \to K^i) = (\square_S[m] \to K)$ is final in the sense of [ML98]; that is to say the comma category $k{\downarrow}\iota$ is nonempty and connected for all objects k of $\square_S{\downarrow}K$.*

Theorem 9.2. *There exists a unique colimit-preserving functor*

$$\mathbb{T} : \square_S{}^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma \to \mathbf{WHDTS}$$

extending the functor $\mathbb{T}$ previously constructed on the full subcategory of labelled cubes. Moreover, this functor is a left adjoint.

Proof. Let K be a labelled symmetric precubical set. One necessarily has

$$\mathbb{T}(K) \cong \varinjlim_{\square_S[a_1, \dots, a_n] \to K} C_n[a_1, \dots, a_n]$$

hence the uniqueness. Let $K = \varinjlim K^i$ be a colimit of labelled symmetric precubical sets, and denote by I the base category. By definition, one has the isomorphism

$$\varinjlim \mathbb{T}(K^i) \cong \varinjlim_{i} \varinjlim_{\square_S[a_1,\dots,a_n]\to K^i} C_n[a_1,\dots,a_n].$$

Consider the functor $H : I \longrightarrow \mathbf{Cat}$ defined by $H(i) = \square_S{\downarrow}K^i$. Consider the functor $F_i : H(i) \longrightarrow \mathbf{WHDTS}$ defined by $F_i(\square_S[a_1,\dots,a_n] \to K^i) = C_n[a_1,\dots,a_n]$. Consider the functor $F : I \int H \longrightarrow \mathbf{WHDTS}$ defined by

$$F(i, \square_S[a_1,\dots,a_n] \to K^i) = C_n[a_1,\dots,a_n].$$

Then the composite $H(i) \subset I \int H \to \mathbf{WHDTS}$ is exactly F_i. Therefore one has the isomorphism

$$\varinjlim_{i} \varinjlim_{\square_S[a_1,\dots,a_n]\to K^i} C_n[a_1,\dots,a_n] \cong \varinjlim_{(i,\square_S[a_1,\dots,a_n]\to K^i)} C_n[a_1,\dots,a_n]$$

by [CS02, Proposition 40.2]. The functor $\iota : I \int H \to \square_S{\downarrow}K$ defined by $\iota(i, \square_S[m] \to K^i) = (\square_S[m] \to K)$ is final in the sense of [ML98] by Lemma 9.1. Therefore by [ML98, p. 213, Theorem 1] or [Hir03, Theorem 14.2.5], one has the isomorphism

$$\varinjlim_{(i,\square_S[a_1,\dots,a_n]\to K^i)} C_n[a_1,\dots,a_n] \cong \varinjlim_{\square_S[a_1,\dots,a_n]\to K} C_n[a_1,\dots,a_n] \cong \mathbb{T}(K).$$

Hence the functor $\mathbb{T}$ is colimit-preserving, hence the existence.

Since the category $\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma$ is locally presentable, it is co-wellpowered by [AR94, Theorem 1.58], and also cocomplete. The set of labelled n-cubes

$$\{\square_S[a_1,\dots,a_n], a_1,\dots,a_n \in \Sigma\}$$

is a set of generators. So by $\mathrm{SAFT}^{\mathrm{op}}$ [ML98, Corollary p126], it is a left adjoint. $\square$

Proposition 9.3. *Let $n \geqslant 2$ and $a_1,\dots,a_n \in \Sigma$. The map of labelled symmetric precubical sets*

$$\square_S[a_1,\dots,a_n] \sqcup_{\partial\square_S[a_1,\dots,a_n]} \square_S[a_1,\dots,a_n] \to \square_S[a_1,\dots,a_n]$$

induces an isomorphism of weak higher dimensional transition systems

$$\mathbb{T}(\square_S[a_1,\dots,a_n] \sqcup_{\partial\square_S[a_1,\dots,a_n]} \square_S[a_1,\dots,a_n]) \cong \mathbb{T}(\square_S[a_1,\dots,a_n]).$$

Proof. Since $\mathbb{T}$ is colimit-preserving, one has the pushout diagram of weak higher dimensional transition systems

$$\begin{tikzcd}
\mathbb{T}(\partial\square_S[a_1,\dots,a_n]) \arrow[r] \arrow[d] & \mathbb{T}(\square_S[a_1,\dots,a_n]) \arrow[d] \\
\mathbb{T}(\square_S[a_1,\dots,a_n]) \arrow[r] & \mathbb{T}(\square_S[a_1,\dots,a_n] \sqcup_{\partial\square_S[a_1,\dots,a_n]} \square_S[a_1,\dots,a_n]).
\end{tikzcd}$$

Since **WHDTS** is topological over $\mathbf{Set}^{\{s\}\cup\Sigma}$ by Theorem 3.4, the weak higher dimensional transition system

$$\mathbb{T}(\square_S[a_1,\dots,a_n] \sqcup_{\partial\square_S[a_1,\dots,a_n]} \square_S[a_1,\dots,a_n])$$

is obtained by taking the colimits of the three sets of states, of the three sets of actions and of the three labelling maps, and by endowing the result with the final structure of weak higher dimensional transition system. By Proposition 3.5, this final structure is the closure under the Coherence axiom of the union of the transitions of $\mathbb{T}(\partial\square_S[a_1,\dots,a_n])$ and of the two copies of $\mathbb{T}(\square_S[a_1,\dots,a_n])$. Since the set of transitions of $\mathbb{T}(\partial\square_S[a_1,\dots,a_n])$ is included in the set of transitions of $\mathbb{T}(\square_S[a_1,\dots,a_n])$, the right-hand vertical and bottom horizontal maps are isomorphisms. Since the composite

$$\begin{aligned}\mathbb{T}(\square_S[a_1,\dots,a_n]) \to \mathbb{T}(\square_S[a_1,\dots,a_n] \sqcup_{\partial\square_S[a_1,\dots,a_n]} \square_S[a_1,\dots,a_n])\\ \to \mathbb{T}(\square_S[a_1,\dots,a_n])\end{aligned}$$

is the identity of $\mathbb{T}(\square_S[a_1,\dots,a_n])$, the proof is complete. $\square$

Theorem 9.4. *Let K be a labelled symmetric precubical set. The canonical map $K \to \mathrm{Sh}_\Sigma(K)$ induces an isomorphism of weak higher dimensional transition systems $\mathbb{T}(K) \cong \mathbb{T}(\mathrm{Sh}_\Sigma(K))$.*

Proof. By Proposition 7.3, a labelled symmetric precubical set K belongs to $\mathbf{HDA}^\Sigma$ if and only if the map $K \to !^S\Sigma$ satisfies the right lifting property with respect to the set of maps

$$\{\square_S[n] \sqcup_{\partial\square_S[n]} \square_S[n] \longrightarrow \square_S[n], n \geqslant 2\}.$$

So the labelled symmetric precubical set $\mathrm{Sh}_\Sigma(K)$ can be obtained by a small object argument by factoring the map $K \to !^S\Sigma$ as a composite $K \to \mathrm{Sh}(K) \to !^S\Sigma$ where $K \to \mathrm{Sh}(K)$ is a relative $\{\square_S[n] \sqcup_{\partial\square_S[n]} \square_S[n] \longrightarrow \square_S[n], n \geqslant 2\}$-cell complex and where the map $\mathrm{Sh}(K) \to !^S\Sigma$ satisfies the right lifting property with respect to the same set of morphisms. The small object argument is possible by [Bek00, Proposition 1.3] since the category of symmetric precubical sets is locally finitely presentable. Thanks to Proposition 9.3, the proof is complete. $\square$

Theorem 9.5. *The functor $\mathbb{T} : \square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma \to \mathbf{WHDTS}$ factors uniquely (up to isomorphism of functors) as a composite*

$$\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma \xrightarrow{\mathrm{Sh}_\Sigma} \mathbf{HDA}^\Sigma \xrightarrow{\overline{\mathbb{T}}} \mathbf{WHDTS}.$$

Moreover, the functor $\overline{\mathbb{T}}$ is a left adjoint.

Proof. Let $\overline{\mathbb{T}}_1$ and $\overline{\mathbb{T}}_2$ be two solutions. Then there is the isomorphism of functors $\overline{\mathbb{T}}_1 \circ \mathrm{Sh}_\Sigma \cong \overline{\mathbb{T}}_2 \circ \mathrm{Sh}_\Sigma$. So there are the isomorphisms of functors $\overline{\mathbb{T}}_1 \cong \overline{\mathbb{T}}_1 \circ \mathrm{Sh}_\Sigma \circ i_\Sigma \cong \overline{\mathbb{T}}_2 \circ \mathrm{Sh}_\Sigma \circ i_\Sigma \cong \overline{\mathbb{T}}_2$. Let $\overline{\mathbb{T}} := \mathbb{T} \circ i_\Sigma$. Then there is the isomorphism of functors $\overline{\mathbb{T}} \circ \mathrm{Sh}_\Sigma = \mathbb{T} \circ i_\Sigma \circ \mathrm{Sh}_\Sigma \cong \mathbb{T}$ thanks to Theorem 9.4.

Hence the existence. Let $K = \varinjlim K_i$ be a colimit in $\mathbf{HDA}^{\Sigma}$. Then one has the sequence of natural isomorphisms

$$\begin{aligned}
\overline{\mathbb{T}}(\varinjlim K_i) &\cong \overline{\mathbb{T}}(\varinjlim \mathrm{Sh}_\Sigma(i_\Sigma(K_i))) && \text{since } K_i \cong \mathrm{Sh}_\Sigma(i_\Sigma(K_i))\\
&\cong \overline{\mathbb{T}}(\mathrm{Sh}_\Sigma(\varinjlim i_\Sigma(K_i))) && \text{since } \mathrm{Sh}_\Sigma \text{ is a left adjoint}\\
&\cong \mathbb{T}(i_\Sigma(\mathrm{Sh}_\Sigma(\varinjlim i_\Sigma(K_i)))) && \text{by definition of } \overline{\mathbb{T}}\\
&\cong \mathbb{T}(\varinjlim i_\Sigma(K_i)) && \text{by Theorem 9.4}\\
&\cong \varinjlim \mathbb{T}(i_\Sigma(K_i)) && \text{since } \mathbb{T} \text{ is colimit-preserving}\\
&\cong \varinjlim \overline{\mathbb{T}}(K_i) && \text{by definition of } \overline{\mathbb{T}}.
\end{aligned}$$

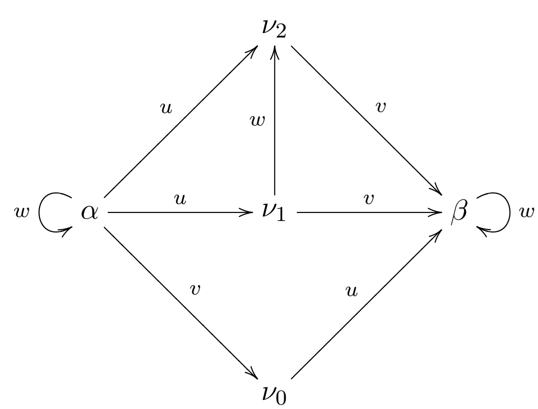
So the functor $\overline{\mathbb{T}}$ is colimit-preserving. Since the category $\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$ is locally presentable, the functor $\overline{\mathbb{T}}$ is a left adjoint for the same reason as in the proof of Theorem 9.2. $\square$

Definition 9.6. A labelled symmetric precubical set K is strong if the weak higher dimensional transition system $\mathbb{T}(K)$ satisfies the Unique intermediate state axiom.

Note that a labelled symmetric precubical set K is strong if and only if $\mathrm{Sh}_\Sigma(K)$ is strong, by Theorem 9.4.

Proposition 9.7. *There exists a labelled symmetric precubical set satisfying the HDA paradigm K which is not strong.*

Sketch of proof. Consider the following 1-dimensional (symmetric) precubical set:



And let us add three squares corresponding to the concurrent execution of u and w (square $(\alpha, \alpha, \nu_1, \nu_2)$), of v and w (square $(\beta, \beta, \nu_1, \nu_2)$), and finally of u and v (square $(\alpha, \nu_0, \beta, \nu_1)$). One obtains a 2-dimensional labelled symmetric precubical set K. The weak higher dimensional transition system $\mathbb{T}(K)$ contains the 2-transition (α, u, v, β). And there exist two distinct states ν_1 and ν_2 such that (α, u, ν_1), (α, u, ν_2), (ν_1, v, β) and (ν_2, v, β) are 1-transitions of the weak higher dimensional transition system $\mathbb{T}(K)$. $\square$

Note that every weak higher dimensional transition system of the form $\mathbb{T}(K)$ where K is a labelled symmetric precubical set satisfies a weak version of the Unique intermediate state axiom (called the Intermediate state axiom):

Proposition 9.8. *Let K be a labelled symmetric precubical set. For every $n \geqslant 2$, every p with $1 \leqslant p < n$ and every transition $(\alpha, u_1, \dots, u_n, \beta)$ of $\mathbb{T}(K)$, there exists a (not necessarily unique) state ν such that both*

$$(\alpha, u_1, \dots, u_p, \nu) \quad \textit{and} \quad (\nu, u_{p+1}, \dots, u_n, \beta)$$

are transitions.

Proof. It suffices to prove that for every pushout diagram of labelled symmetric precubical sets of the form

$$\begin{tikzcd} \partial\square_S[a_1, \dots, a_n] \arrow[r] \arrow[d] & K \arrow[d] \\ \square_S[a_1, \dots, a_n] \arrow[r] & L \end{tikzcd}$$

with $n \geqslant 2$, if $\mathbb{T}(K)$ satisfies the Intermediate state axiom, then $\mathbb{T}(L)$ does too. Since $\mathbb{T}$ is colimit-preserving by Theorem 9.2, one obtains the pushout diagram of weak higher dimensional transition systems

$$\begin{tikzcd} \mathbb{T}(\partial\square_S[a_1, \dots, a_n]) \arrow[r, "f"] \arrow[d] & \mathbb{T}(K) \arrow[d] \\ \mathbb{T}(\square_S[a_1, \dots, a_n]) \arrow[r] & \mathbb{T}(L). \end{tikzcd}$$

It then suffices to observe that for every $1 \leqslant p < n$, there exists a state ν_p of $\mathbb{T}(K)$ such that the tuples

$$(f_0(0_n), \widetilde{f}(a_1, 1), \dots, \widetilde{f}(a_p, p), \nu_p),$$
$$(\nu_p, \widetilde{f}(a_{p+1}, p+1), \dots, \widetilde{f}(a_n, n), f_0(1_n))$$

are transitions of $\mathbb{T}(L)$: take $\nu_p = f_0(\nu'_p)$ where ν'_p is the unique state of $\mathbb{T}(\square_S[a_1, \dots, a_n])$ such that the tuples

$$(0_n, (a_1, 1), \dots, (a_p, p), \nu'_p),$$
$$(\nu'_p, (a_{p+1}, p+1), \dots, (a_n, n), 1_n)$$

are transitions of $\mathbb{T}(\square_S[a_1, \dots, a_n])$ (cf. Proposition 5.2). $\square$

One will see that for every concurrent process P of every process algebra of any synchronization algebra, the interpretation $\square_S[\![P]\!]$ of P as labelled symmetric precubical set is always strong. In fact, it is even always a higher dimensional transition system since CSA1 is also satisfied.

10. Categorical property of the realization

Theorem 10.1. *Let K and L be two labelled symmetric precubical sets with $L \in \mathbf{HDA}^{\Sigma}$. Then the set map*

$$(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)(K,L) \xrightarrow{f \mapsto \mathbb{T}(f)} \mathbf{WHDTS}(\mathbb{T}(K),\mathbb{T}(L))$$

is one-to-one.

Proof. Let K and L be two labelled symmetric precubical sets with $L \in \mathbf{HDA}^{\Sigma}$. Let us consider the commutative diagram of sets

$$\begin{tikzcd}
(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)(K,L) \arrow[r, "\mathbb{T}"] \arrow[d, "(-)_{\leqslant 1}"'] & \mathbf{WHDTS}(\mathbb{T}(K),\mathbb{T}(L)) \arrow[d] \\
\mathbf{Set}(K_{\leqslant 1}, L_{\leqslant 1}) \arrow[r, equal] & \mathbf{Set}(\mathbb{T}(K)_{\leqslant 1}, \mathbb{T}(L)_{\leqslant 1})
\end{tikzcd}$$

where the left-hand vertical map is the restriction to dimension 1 and where the right-hand vertical map is the restriction of a map of weak higher dimensional transition systems to the underlying map between the 1-dimensional parts, i.e., by keeping only the 1-dimensional transitions. The right-hand vertical map is one-to-one by definition of a map of weak higher dimensional transition systems. Let $f, g : K \rightrightarrows L$ be two maps of labelled symmetric precubical sets with $f_{\leqslant 1} = g_{\leqslant 1}$. let us prove by induction on $n \geqslant 1$ that $f_{\leqslant n} = g_{\leqslant n}$. The assertion is true for $n = 1$ by hypothesis. Let us suppose that it is true for some $n \geqslant 1$. Let $x : \square_S[n+1] \to K$ be a $(n+1)$-cube of K. Let $\partial x : \partial\square_S[n+1] \subset \square_S[n+1] \to K$. Consider the diagram of labelled symmetric precubical sets

$$\begin{tikzcd}
\square_S[n+1] \sqcup_{\partial\square_S[n+1]} \square_S[n+1] \arrow[r] \arrow[d] \arrow[rr, bend left, "f(x)\sqcup_{f(\partial x)} g(x)"] & K_{\leqslant n+1} \arrow[r] & L \arrow[d] \\
\square_S[n+1] \arrow[rru, dashed, "k"] \arrow[rr] & & !^S\Sigma.
\end{tikzcd}$$

Since $L \in \mathbf{HDA}^{\Sigma}$, there exists exactly one lift k. Thus, $f(x) = g(x)$ and the induction is complete. $\square$

Corollary 10.2. *The functor* $\overline{\mathbb{T}} : \mathbf{HDA}^{\Sigma} \to \mathbf{WHDTS}$ *is faithful.*

Proposition 10.3. *The functor* $\overline{\mathbb{T}} : \mathbf{HDA}^{\Sigma} \to \mathbf{WHDTS}$ *is not full.*

Sketch of proof. Let us consider the higher dimensional transition system $C_2[u,v]$ with set of states $\{\alpha,\beta,\nu_0,\nu_2\}$. And the inclusion of this higher dimensional transition system to the weak higher dimensional transition system $X = \mathbb{T}(K)$ given in the proof of Proposition 9.7:

$$\begin{tikzcd}
& \nu_2 \arrow[dr, "v"] & \\
\alpha \arrow[loop left, "w"] \arrow[ur, "u"] \arrow[r, "u"] \arrow[dr, "v"'] & \nu_1 \arrow[u, "w"] \arrow[r, "v"] & \beta \arrow[loop right, "w"] \\
& \nu_0 \arrow[ur, "u"'] &
\end{tikzcd}$$

Then this inclusion cannot come from a map of labelled symmetric precubical sets since there are no squares in K with the vertices α,β,ν_0,ν_2. $\square$

Theorem 10.4. *Let K and L be two labelled symmetric precubical sets such that L satisfies the HDA paradigm and such that $\mathbb{T}(L)$ satisfies the Unique intermediate state axiom. Then the set map*

$$(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)(K,L) \xrightarrow{f\mapsto\mathbb{T}(f)} \mathbf{WHDTS}(\mathbb{T}(K),\mathbb{T}(L))$$

is bijective.

Proof. First of all, let us consider the local case, i.e., when

$$K = \square_S[a_1,\dots,a_m]$$

is a labelled m-cube. It suffices to prove that the map

$$(\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)(K,L) \xrightarrow{f\mapsto\mathbb{T}(f)} \mathbf{WHDTS}(\mathbb{T}(K),\mathbb{T}(L))$$

is onto since we already know by Theorem 10.1 that it is one-to-one because L satisfies the HDA paradigm. Since $\mathbb{T}$ is colimit-preserving, one has the isomorphism

$$\mathbb{T}(L) \cong \varinjlim_{\square_S[b_1,\dots,b_n]\to L} C_n[b_1,\dots,b_n].$$

Let $f \in \mathbf{WHDTS}(C_m[a_1,\dots,a_m],\mathbb{T}(L))$. The $(m+2)$-tuple

$$(f_0(0,\dots,0),\widetilde{f}(a_1,1),\dots,\widetilde{f}(a_m,m),f_0(1,\dots,1))$$

is an m-transition of $\mathbb{T}(L)$. By Theorem 4.7, since each cube $C_n[b_1,\dots,b_n]$ as well as $\mathbb{T}(L)$ satisfy the Unique intermediate state axiom, there exists a labelled cube $g : \square_S[b_1,\dots,b_n] \to L$ of L such that the $(m+2)$-tuple

$$(f_0(0,\dots,0), \widetilde{f}(a_1,1),\dots,\widetilde{f}(a_m,m), f_0(1,\dots,1))$$

comes from an m-transition of $C_n[b_1,\dots,b_n]$. In other terms the composite

$$C_m[a_1,\dots,a_m]^{\mathrm{ext}} \subset C_m[a_1,\dots,a_m] \xrightarrow{f} \mathbb{T}(L)$$

factors as a composite

$$C_m[a_1,\dots,a_m]^{\mathrm{ext}} \longrightarrow C_n[b_1,\dots,b_n] \xrightarrow{\mathbb{T}(g)} \mathbb{T}(L).$$

Since $C_n[b_1,\dots,b_n]$ is a higher dimensional transition system by Proposition 5.2, the latter map factors as a composite

$$C_m[a_1,\dots,a_m]^{\mathrm{ext}} \subset C_m[a_1,\dots,a_m] \xrightarrow{H} C_n[b_1,\dots,b_n] \xrightarrow{\mathbb{T}(g)} \mathbb{T}(L)$$

by Theorem 5.6. Since $\mathbb{T}(L)$ satisfies the Unique intermediate state axiom, one obtains that the map $f \in \mathbf{WHDTS}(C_m[a_1,\dots,a_m], \mathbb{T}(L))$ is equal to the composite

$$C_m[a_1,\dots,a_m] \xrightarrow{H} C_n[b_1,\dots,b_n] \xrightarrow{\mathbb{T}(g)} \mathbb{T}(L)$$

thanks to Theorem 5.6. By Theorem 8.5, the left-hand morphism H is of the form $\mathbb{T}(h)$ where $h : \square_S[a_1,\dots,a_m] \to \square_S[b_1,\dots,b_n]$ is a map of labelled symmetric precubical sets. Hence $f = \mathbb{T}(gh)$.

Let us treat now the passage from the local to the global case. Since the functor $\mathbb{T}$ is colimit-preserving by Theorem 9.5, one has the isomorphism of weak higher dimensional transition systems

$$\mathbb{T}(K) \cong \varinjlim_{\square_S[a_1,\dots,a_m]\to K} C_m[a_1,\dots,a_m].$$

The set map

$$\varprojlim_{\square_S[a_1,\dots,a_m]\to K} (\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma)(\square_S[a_1,\dots,a_m], L) \longrightarrow \varprojlim_{\square_S[a_1,\dots,a_m]\to K} \mathbf{WHDTS}(C_m[a_1,\dots,a_m], \mathbb{T}(L))$$

is bijective since it is an inverse limit of bijections. This completes the proof. $\square$

Note that it is also possible to prove that the set map of Theorem 10.4 is onto without using Theorem 10.1. Indeed, the category of cubes of a labelled symmetric precubical set is a dualizable generalized Reedy category in the sense of [BM08]. So one obtains the same result by applying [BM08, Corollary 1.7] to the category of diagrams from the category of cubes to the opposite $\mathbf{Set}^{\mathrm{op}}$ of the category of sets and by endowing $\mathbf{Set}$ with the unique fibrantly generated model structure such that the fibrations are the onto maps [Gau05, Theorem 4.6].

Corollary 10.5. *Let K and L be two strong labelled symmetric precubical sets. Let us suppose that the two weak higher dimensional transition systems $\mathbb{T}(K)$ and $\mathbb{T}(L)$ are isomorphic. Then there is an isomorphism of labelled symmetric precubical sets $\mathrm{Sh}_\Sigma(K) \cong \mathrm{Sh}_\Sigma(L)$.*

Proof. By Theorem 10.4 and Theorem 9.4, the isomorphism $\mathbb{T}(\mathrm{Sh}_\Sigma(K)) \cong \mathbb{T}(K) \cong \mathbb{T}(L) \cong \mathbb{T}(\mathrm{Sh}_\Sigma(L))$ is of the form $\mathbb{T}(f)$ for some map $f : \mathrm{Sh}_\Sigma(K) \to \mathrm{Sh}_\Sigma(L)$ of labelled symmetric precubical sets. And symmetrically, there exists a map $g : \mathrm{Sh}_\Sigma(L) \to \mathrm{Sh}_\Sigma(K)$ such that $\mathbb{T}(g) = \mathbb{T}(f)^{-1}$. By Corollary 10.2, one has $f \circ g = \mathrm{Id}_{\mathrm{Sh}_\Sigma(L)}$ and $g \circ f = \mathrm{Id}_{\mathrm{Sh}_\Sigma(K)}$. Hence the result. $\square$

Corollary 10.6. *Let K and L be two strong labelled symmetric precubical sets such that the weak higher dimensional transition systems $\mathbb{T}(K)$ and $\mathbb{T}(L)$ are isomorphic. Then the two weak higher dimensional transition systems $\mathbb{T}(K)$ and $\mathbb{T}(L)$ have the same set of actions.*

11. Higher dimensional transition systems are labelled symmetric precubical sets

We want to compare now the two settings of higher dimensional transition systems and labelled symmetric precubical sets. Let us start with some definitions and notations.

- $\mathbf{HDA}^\Sigma_{\mathrm{hdts}}$ denotes the full subcategory of $\mathbf{HDA}^\Sigma$ of labelled symmetric precubical sets K such that $\overline{\mathbb{T}}(K)$ is a higher dimensional transition system, i.e., such that the weak higher dimensional transition system $\overline{\mathbb{T}}(K)$ satisfies CSA1 and the Unique Intermediate axiom.
- $\overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}})$ is the full subcategory of $\mathbf{HDTS}$ of higher dimensional transition systems of the form $\overline{\mathbb{T}}(K)$ with $K \in \mathbf{HDA}^\Sigma_{\mathrm{hdts}}$; this subcategory is isomorphism-closed.
- An action u of a weak higher dimensional transition system is *used* if there exists a transition (α, u, β).
- The *cubification* of $X \in \mathbf{WHDTS}$ is the weak higher dimensional transition system

$$\underline{\mathrm{Cub}}(X) := \varinjlim_{C_n[a_1,\dots,a_n] \to X} C_n[a_1, \dots, a_n],$$

the colimit being calculated in $\mathbf{WHDTS}$. Note that the natural map

$$p_X : \underline{\mathrm{Cub}}(X) \to X$$

induces a bijection between the set of states for any weak higher dimensional transition system X.

Proposition 11.1. *The cubification functor satisfies the following properties:*

(1) *It induces a functor*

$$\underline{\mathrm{Cub}} : \mathbf{HDTS} \to \overline{\mathbb{T}}(\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}).$$

(2) *The natural map* $p_X : \underline{\mathrm{Cub}}(X) \to X$ *is an isomorphism for every* $X \in \overline{\mathbb{T}}(\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}})$.

(3) *For every higher dimensional transition system* Y*, one has a natural isomorphism* $\underline{\mathrm{Cub}}(\underline{\mathrm{Cub}}(Y)) \cong \underline{\mathrm{Cub}}(Y)$.

Proof. One has, the colimit being taken in **WHDTS**,

$$\underline{\mathrm{Cub}}(X) = \overline{\mathbb{T}}\left(\varinjlim_{C_n[a_1,\dots,a_n]\to X} \square_S[a_1,\dots,a_n]\right)$$

by Theorem 8.5 and Theorem 9.5. By Proposition 9.8, the weak higher dimensional transition system satisfies the Intermediate State axiom. The canonical map $p_X : \underline{\mathrm{Cub}}(X) \to X$ is a bijection on states. Therefore if X satisfies the Unique Intermediate State axiom, then so does $\underline{\mathrm{Cub}}(X)$. By Theorem 4.7, the set of transitions of $\underline{\mathrm{Cub}}(X)$ is the union of the transitions of the cubes $C_n[a_1,\dots,a_n]$. So there is a bijection between the 1-transitions of $\underline{\mathrm{Cub}}(X)$ and the map of the form $C_1[x] \to X$. Let (α,u,β) and (α,v,β) be two transitions of $\underline{\mathrm{Cub}}(X)$ with $\mu(u)=\mu(v)\in\Sigma$, μ being the labelling map of $\underline{\mathrm{Cub}}(X)$. Since X satisfies CSA1, one has $\widetilde{p_X}(u) = \widetilde{p_X}(v)$. We obtain $u = v$ and $\underline{\mathrm{Cub}}(X)$ satisfies CSA1. Hence the first assertion.

Let $K \in \mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}$. Then one has

$$\begin{aligned}\underline{\mathrm{Cub}}(\overline{\mathbb{T}}(K)) &= \varinjlim_{C_n[a_1,\dots,a_n]\to\overline{\mathbb{T}}(K)} C_n[a_1,\dots,a_n]\\ &\cong \overline{\mathbb{T}}\left(\varinjlim_{\square_S[a_1,\dots,a_n]\to K} \square_S[a_1,\dots,a_n]\right) \cong \overline{\mathbb{T}}(K)\end{aligned}$$

by Theorem 8.5, Theorem 10.4 and Theorem 9.5. Hence the second assertion.

For every higher dimensional transition system Y, there exists

$$K \in \mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}$$

such that $\underline{\mathrm{Cub}}(Y) = \overline{\mathbb{T}}(K)$. Hence the third assertion. □

Proposition 11.2. *The restriction functor* $\overline{\mathbb{T}} : \mathbf{HDA}^{\Sigma} \to \mathbf{WHDTS}$ *induces an equivalence of categories*

$$\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}} \simeq \overline{\mathbb{T}}(\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}) \simeq \mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$$

where $\mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$ *is the categorical localization of* **HDTS** *by the maps* f *such that* $\underline{\mathrm{Cub}}(f)$ *is an isomorphism.*

Proof. The restriction functor $\overline{\mathbb{T}} : \mathbf{HDA}^\Sigma \to \mathbf{WHDTS}$ induces an equivalence of categories $\mathbf{HDA}^\Sigma_{\mathrm{hdts}} \simeq \overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}})$: indeed, it is faithful by Corollary 10.2, full by Theorem 10.4 and Proposition 4.6 and essentially surjective by construction. It remains to prove that the pair of functors

$$i : \overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}}) \leftrightarrows \mathbf{HDTS} : \underline{\mathrm{Cub}}$$

where $i : \overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}}) \subset \mathbf{HDTS}$ is the inclusion functor induces an equivalence of categories

$$\overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}}) \simeq \mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}].$$

That $\underline{\mathrm{Cub}} : \mathbf{HDTS} \to \overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}})$ factors uniquely as a composite

$$\mathbf{HDTS} \to \mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}] \to \overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}})$$

comes from the universal property of the localization. For every $X \in \overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}})$, there is a natural isomorphism $p_X : \underline{\mathrm{Cub}}(X) \cong X$ by Proposition 11.1 (2). For every $Y \in \mathbf{HDTS}$, the map $p_Y : \underline{\mathrm{Cub}}(Y) \to Y$ is an isomorphism of $\mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$ since $\underline{\mathrm{Cub}}(p_Y)$ is an isomorphism by Proposition 11.1 (3). Hence the desired categorical equivalence. $\square$

Proposition 11.3. *The category $\overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}})$ is a coreflective locally finitely presentable subcategory of* $\mathbf{HDTS}$*.*

Proof. By Proposition 11.1 (2), one has the commutative diagram of higher dimensional transition systems

$$\begin{tikzcd}
\underline{\mathrm{Cub}}(X) \arrow[r, "\underline{\mathrm{Cub}}(f)"] & \underline{\mathrm{Cub}}(Y) \arrow[d, "p_Y"] \\
X \arrow[u, "p_X^{-1}"] \arrow[r, "f"] & Y
\end{tikzcd}$$

for every map $f : X \to Y$ where X is an object of $\overline{\mathbb{T}}(\mathbf{HDA}^\Sigma_{\mathrm{hdts}})$ and Y a higher dimensional transition system. So the set map $h \mapsto p_Y \circ h$ from $\mathbf{HDTS}(X, \underline{\mathrm{Cub}}(Y))$ to $\mathbf{HDTS}(X, Y)$ is onto. Let $f, g : X \rightrightarrows \underline{\mathrm{Cub}}(Y)$ be two maps such that $p_Y \circ f = p_Y \circ g$. Since the set map $(p_Y)_0$ from the set of states of $\underline{\mathrm{Cub}}(Y)$ to the one of Y is bijective, one has $f_0 = g_0$, i.e., f and g coincide on the set of states. Let u be an action of X. Let (α, u, β) be a transition of X: all actions of X are used since $X = \mathbb{T}(K)$ for some K. Then $(f_0(\alpha), \widetilde{f}(u), f_0(\beta))$ and $(g_0(\alpha), \widetilde{g}(u), g_0(\beta))$ are two transitions of $\underline{\mathrm{Cub}}(Y)$. Since $f_0 = g_0$ and since $\underline{\mathrm{Cub}}(Y)$ satisfies CSA1 by Proposition 11.1 (1), one obtains $\widetilde{f}(u) = \widetilde{g}(u)$. So $f = g$ and the map $f \mapsto p_Y \circ f$ from $\mathbf{HDTS}(X, \underline{\mathrm{Cub}}(Y))$ to $\mathbf{HDTS}(X, Y)$ is one-to-one. [1]

[1] The set map $\widetilde{p_Y}$ from the set of actions of $\underline{\mathrm{Cub}}(Y)$ to that of Y is not necessarily one-to-one. See Equation (1).

Therefore, the cubification functor $\underline{\mathrm{Cub}} : \mathbf{HDTS} \to \overline{\mathbb{T}}(\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}})$ is right adjoint to the inclusion $i : \overline{\mathbb{T}}(\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}) \subset \mathbf{HDTS}$. So the category $\overline{\mathbb{T}}(\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}})$ is cocomplete, as a coreflective isomorphism-closed subcategory of the cocomplete category $\mathbf{HDTS}$. Since $\underline{\mathrm{Cub}}(\overline{\mathbb{T}}(K)) = \overline{\mathbb{T}}(K)$, the cubes $C_n[a_1, \dots, a_n]$ with $n \geqslant 0$ and $a_1, \dots, a_n \in \Sigma$ form a dense (and hence strong) generator of $\overline{\mathbb{T}}(\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}})$. So the category $\overline{\mathbb{T}}(\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}})$ is locally finitely presentable by [AR94, Theorem 1.20]. $\square$

Proposition 11.4. *A labelled symmetric precubical set K is in $\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}$ if and only if K is orthogonal to the set of maps*

$$\{\square_S[a_1, \dots, a_p] \sqcup_{\partial\square_S[a_1,\dots,a_p]} \square_S[a_1, \dots, a_p] \to \square_S[a_1, \dots, a_p], p \geqslant 1 \\ \textit{and } a_1, \dots, a_p \in \Sigma\}$$

and the weak higher dimensional transition system $\mathbb{T}(K)$ satisfies the Unique intermediate state axiom.

Proof. This is a consequence of Proposition 4.6. $\square$

Proposition 11.5. *The inclusion functor $\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}} \subset \square^{op}_S\mathbf{Set}{\downarrow}!^S\Sigma$ is limit-preserving and finitely accessible.*

Proof. Limit-preserving. Let I be a small category. Let $\underline{K} : I \to \mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}$ be a diagram of objects of $\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}$. Then the labelled symmetric precubical set $\varprojlim \underline{K}$ (limit taken in the category of labelled symmetric precubical sets) is orthogonal to the set of maps

$$\{\square_S[a_1, \dots, a_p] \sqcup_{\partial\square_S[a_1,\dots,a_p]} \square_S[a_1, \dots, a_p] \to \square_S[a_1, \dots, a_p], p \geqslant 1 \\ \text{and } a_1, \dots, a_p \in \Sigma\}$$

by [AR94, Theorem 1.39]. It remains to prove that the weak higher dimensional transition system $\mathbb{T}(\varprojlim \underline{K})$ satisfies the Unique intermediate state axiom by Proposition 11.4. Consider the canonical map of weak higher dimensional transition systems $\mathbb{T}(\varprojlim \underline{K}) \to \varprojlim(\mathbb{T} \circ \underline{K})$. The right-hand limit is taken in $\mathbf{HDTS}$ or $\mathbf{WHDTS}$ since the inclusion functor $\mathbf{HDTS} \subset \mathbf{WHDTS}$ is a right adjoint by Corollary 5.8. Since the category $\mathbf{WHDTS}$ is topological, the set of states of $\varprojlim(\mathbb{T} \circ \underline{K})$ is equal to the inverse limit of the sets of states of the $\mathbb{T}(\underline{K}(i))$, i.e., the inverse limit of the sets of 0-cubes of $\underline{K}(i)$ by definition of the functor $\mathbb{T}$. So the canonical map $\mathbb{T}(\varprojlim \underline{K}) \to \varprojlim(\mathbb{T} \circ \underline{K})$ induces a bijection between the set of states. Consequently, $\mathbb{T}(\varprojlim \underline{K})$ satisfies the Unique Intermediate State axiom since two intermediate states for the same transition would be mapped to the same state in $\varprojlim(\mathbb{T} \circ \underline{K})$. Hence, the inclusion functor $\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}} \subset \square^{op}_S\mathbf{Set}{\downarrow}!^S\Sigma$ is limit-preserving.

<u>Finitely accessible</u>. Let us now suppose that $\underline{K}$ is directed. Then the colimit $\varinjlim \underline{K}$ taken in $\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$ is orthogonal to the set of maps

$$\big\{\square_S[a_1,\dots,a_p] \sqcup_{\partial\square_S[a_1,\dots,a_p]} \square_S[a_1,\dots,a_p] \to \square_S[a_1,\dots,a_p], p \geqslant 1 \\ \text{and } a_1,\dots,a_p \in \Sigma\big\}.$$

since the inclusion functor is accessible by [AR94, Theorem 1.39] and since every labelled cube $\square_S[a_1,\dots,a_p]$ and its boundary $\partial\square_S[a_1,\dots,a_p]$ are finitely presentable. Moreover, one has $\mathbb{T}(\varinjlim \underline{K}) = \varinjlim(\mathbb{T}\circ \underline{K})$ by Theorem 9.2. So the weak higher dimensional transition system $\mathbb{T}(\varinjlim \underline{K})$ is a higher dimensional transition system since the inclusion functor $\mathbf{HDTS} \subset \mathbf{WHDTS}$ is finitely accessible as explained at the very end of Section 5. So the inclusion functor $\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}} \subset \square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$ is finitely accessible. $\square$

Theorem 11.6. *The categorical localization* $\mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$ *of* $\mathbf{HDTS}$ *by the maps* f *such that* $\underline{\mathrm{Cub}}(f)$ *is an isomorphism is equivalent to a full reflective locally finitely presentable subcategory of the category of labelled symmetric precubical sets.*

Proof. The theorem is a consequence of Proposition 11.2, Proposition 11.3, Proposition 11.5 and [AR94, Theorem 1.66]. $\square$

Let us explain what the localization $\mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$ consists of. The first effect of the cubification functor is to removed all unused actions. Let $x \in \Sigma$. Let $\underline{x} = (\varnothing, \{x\} \subset \Sigma, \varnothing)$ be a higher dimensional transition system with no states and no transitions, and a unique action x; then $\underline{\mathrm{Cub}}(\underline{x}) = \varnothing$. The second effect of the cubification functor is to use different actions for two 1-transitions which are not related by higher dimensional cubes. For example, one has the isomorphism

$$C_1[x] \sqcup C_1[x] \cong \underline{\mathrm{Cub}}\left(\varinjlim\left(C_1[x] \leftarrow \underline{x} \to C_1[x]\right)\right). \tag{1}$$

So, in $\mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$, two higher dimensional transition systems are isomorphic if they have the same cubes modulo their unused actions. Given a higher dimensional transition system X all of whose actions are used, one can show that the canonical map $\underline{\mathrm{Cub}}(X) \to X$ is bijective on states, surjective on actions, and surjective on transitions. Using Theorem 4.7, this proves that the set of transitions of a higher dimensional transition system is always the union of the set of transitions of its cubes.

12. Geometric realization of a weak higher dimensional transition system

The category **Top** of *compactly generated topological spaces* (i.e., of weak Hausdorff k-spaces) is complete, cocomplete and cartesian closed (more details for these kinds of topological spaces are in [Bro06], [May99], the appendix of [Lew78] and also in the preliminaries of [Gau03]). For the sequel, all topological spaces will be supposed to be compactly generated. A *compact space* is always Hausdorff.

Definition 12.1 ([Gau03])**.** A (time) flow X is a small topological category without identity maps. The set of objects is denoted by X^0. The topological space of morphisms from α to β is denoted by $\mathbb{P}_{\alpha,\beta}X$. The elements of X^0 are also called the states of X. The elements of $\mathbb{P}_{\alpha,\beta}X$ are called the (non-constant) execution paths from α to β. A flow X is loopless if for every $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.

Notation 12.2. Let

$$\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}_{\alpha,\beta}X.$$

The topological space $\mathbb{P}X$ is called the *path space* of X. The source map (resp. the target map) $\mathbb{P}X \to X^0$ is denoted by s (resp. t).

Definition 12.3. Let X be a flow, and let $\alpha \in X^0$ be a state of X. The state α is *initial* if $\alpha \notin t(\mathbb{P}X)$, and the state α is *final* if $\alpha \notin s(\mathbb{P}X)$.

Definition 12.4. A morphism of flows $f : X \to Y$ consists of a set map $f^0 : X^0 \to Y^0$ and a continuous map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ compatible with the structure. The corresponding category is denoted by **Flow**.

The strictly associative composition law

$$\begin{cases} \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X \\ (x,y) \mapsto x * y \end{cases}$$

models the composition of non-constant execution paths. The composition law $*$ is extended in the usual way to states, that is to constant execution paths, by $x * t(x) = x$ and $s(x) * x = x$ for every non-constant execution path x.

Here are two fundamental examples of flows:

(1) Let S be a set. The flow associated with S, also denoted by S, has S as its set of states and the empty space as its path space. This construction induces a functor **Set** $\to$ **Flow** from the category of sets to that of flows. The flow associated with a set is loopless.
(2) Let $(P, \leqslant)$ be a poset. The flow associated with $(P, \leqslant)$, also denoted by P, is defined as follows: the set of states of P is the underlying set of P; the space of morphisms from α to β is empty if $\alpha \geqslant \beta$ and is equal to $\{(\alpha, \beta)\}$ if $\alpha < \beta$, and the composition law is defined by $(\alpha, \beta) * (\beta, \gamma) = (\alpha, \gamma)$. This construction induces a functor **PoSet** $\to$ **Flow** from the category of posets together with the strictly increasing maps to the category of flows. The flow associated with a poset is loopless.

The model structure of **Flow** is characterized as follows [Gau03]:

$$\begin{tikzcd} (\widehat{0},\widehat{0}) \arrow[r, "(\widehat{0},*)"] \arrow[d, "(*,\widehat{0})"'] \arrow[dr, "(*,*)"] & (\widehat{0},\widehat{1}) \arrow[d, "(*,\widehat{1})"] \\ (\widehat{1},\widehat{0}) \arrow[r, "(\widehat{1},*)"] & (\widehat{1},\widehat{1}) \end{tikzcd}$$

FIGURE 3. The flow $\{\widehat{0}<\widehat{1}\}^2$ ($(*,*) = (\widehat{0},*)*(*,\widehat{1}) = (*,\widehat{0})*(\widehat{1},*)$)

- The weak equivalences are the *weak S-homotopy equivalences*, i.e., the morphisms of flows $f : X \longrightarrow Y$ such that $f^0 : X^0 \longrightarrow Y^0$ is a bijection of sets and such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence.
- The fibrations are the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration[2].

This model structure is cofibrantly generated. The cofibrant replacement functor is denoted by $(-)^{\mathrm{cof}}$.

A state of the flow associated with the poset $\{\widehat{0}<\widehat{1}\}^n$ (i.e., the product of n copies of $\{\widehat{0}<\widehat{1}\}$) is denoted by an n-tuple of elements of $\{\widehat{0},\widehat{1}\}$. By convention, $\{\widehat{0}<\widehat{1}\}^0 = \{()\}$. The unique morphism/execution path from $(x_1,\dots,x_n)$ to $(y_1,\dots,y_n)$ is denoted by an n-tuple $(z_1,\dots,z_n)$ of $\{\widehat{0},\widehat{1},*\}$ with $z_i = x_i$ if $x_i = y_i$ and $z_i = *$ if $x_i < y_i$. For example in the flow $\{\widehat{0}<\widehat{1}\}^2$ (cf. Figure 3), one has the algebraic relation $(*,*) = (\widehat{0},*)*(*,\widehat{1}) = (*,\widehat{0})*(\widehat{1},*)$.

Let $\square \to \mathbf{PoSet} \subset \mathbf{Flow}$ be the functor defined on objects by the mapping $[n] \mapsto \{\widehat{0}<\widehat{1}\}^n$ and on morphisms by the mapping

$$\delta_i^\alpha \mapsto ((\epsilon_1,\dots,\epsilon_{n-1}) \mapsto (\epsilon_1,\dots,\epsilon_{i-1},\alpha,\epsilon_i,\dots,\epsilon_{n-1})),$$

where the ϵ_i's are elements of $\{\widehat{0},\widehat{1},*\}$.

Let $\square_S \to \mathbf{PoSet} \subset \mathbf{Flow}$ be the functor defined on objects by the mapping $[n] \mapsto \{\widehat{0}<\widehat{1}\}^n$ and on morphisms as follows. Let $f : [m] \to [n]$ be a map of $\square_S$ with $m, n \geqslant 0$. Let $(\epsilon_1,\dots,\epsilon_m) \in \{\widehat{0},\widehat{1},*\}^m$ be a r-cube. Since f is adjacency-preserving, the two elements $f(s(\epsilon_1,\dots,\epsilon_m))$ and $f(t(\epsilon_1,\dots,\epsilon_m))$ are respectively the initial and final states of a unique r-dimensional subcube denoted by $f(\epsilon_1,\dots,\epsilon_m)$ of $[n]$ with $f(\epsilon_1,\dots,\epsilon_m) \in \{\widehat{0},\widehat{1},*\}^n$. Note that the composite functor $\square \subset \square_S \to \mathbf{PoSet} \subset \mathbf{Flow}$ is the functor defined above.

[2]that is, a continuous map having the RLP with respect to the inclusion $\mathbf{D}^n \times 0 \subset \mathbf{D}^n \times [0,1]$ for any $n \geqslant 0$ where $\mathbf{D}^n$ is the n-dimensional disk.

Definition 12.5 ([Gau08] [Gau10])**.** Let K be a labelled symmetric precubical set. The geometric realization of K is the flow

$$|K|_{\mathrm{flow}} := \varinjlim_{\square_S[n]\to K} [n]^{\mathrm{cof}}.$$

Because cubes in labelled symmetric precubical sets and in weak higher dimensional transition systems can be identified (Theorem 8.5), there is a well-defined functor from weak higher dimensional transition systems to flows as follows.

Definition 12.6. Let X be a weak higher dimensional transition system. The geometric realization of X is the flow

$$|X| := \varinjlim_{C_n[a_1,\dots,a_n]\to X} [n]^{\mathrm{cof}}.$$

Theorem 12.7. *Let K be a strong labelled symmetric precubical set satisfying the HDA paradigm, i.e., such that $\mathbb{T}(K)$ satisfies the Unique intermediate state axiom. Then there is a natural isomorphism of flows $|K|_{\mathrm{flow}} \cong |\mathbb{T}(K)|$.*

Proof. Since K is strong and since it satisfies the HDA paradigm, the set map

$$\mathbf{HDA}^{\Sigma}(\square_S[a_1,\dots,a_n], K) \to \mathbf{WHDTS}(C_n[a_1,\dots,a_n], \mathbb{T}(K))$$

is bijective by Theorem 10.4. So the two colimits

$$\varinjlim_{\square_S[a_1,\dots,a_n]\to K} [n]^{\mathrm{cof}}$$

and

$$\varinjlim_{C_n[a_1,\dots,a_n]\to \mathbb{T}(K)} [n]^{\mathrm{cof}}$$

are calculated for the same diagram of flows. $\square$

The isomorphism $|K|_{\mathrm{flow}} \cong |\mathbb{T}(K)|$ is false in general. Consider the non-strong labelled symmetric precubical set K of Proposition 9.7. There exists a map $C_2[u,v] \to \mathbb{T}(K)$ which does not come from a square of K. So the geometric realization $|\mathbb{T}(K)|$ contains a homotopy which is not in $|K|_{\mathrm{flow}}$.

13. Process algebras and strong labelled symmetric precubical sets

First we recall the semantics of process algebra given in [Gau08] and [Gau10]. The *CCS process names* are generated by the following syntax:

$$P ::= nil \mid a.P \mid (\nu a)P \mid P + P \mid P||P \mid \mathrm{rec}(x)P(x)$$

where $P(x)$ means a process name with one free variable x. The variable x must be *guarded*, that is it must lie in a prefix term $a.x$ for some $a \in \Sigma$. The set of process names is denoted by $\mathbf{Proc}_{\Sigma}$.

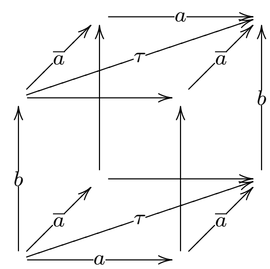


FIGURE 4. Representation of $\square_S[a,b]_{\leqslant 1} \times_\Sigma \square_S[\overline{a}]$

The set $\Sigma\backslash\{\tau\}$, which may be empty, is supposed to be equipped with an involution $a \mapsto \overline{a}$. In Milner's calculus of communicating systems (CCS) [Mil89], which is the only case treated here, one has $a \neq \overline{a}$. We do not use this hypothesis. The involution on $\Sigma\backslash\{\tau\}$ is used only in Definition 13.1 of the fibered product of two 1-dimensional labelled symmetric precubical sets over Σ.

Definition 13.1. Let K and L be two 1-dimensional labelled symmetric precubical sets. The fibered product of K and L over Σ is the 1-dimensional labelled symmetric precubical set $K \times_\Sigma L$ defined as follows:

- $(K \times_\Sigma L)_0 = K_0 \times L_0$,
- $(K \times_\Sigma L)_1 = (K_1 \times L_0) \sqcup (K_0 \times L_1) \sqcup \{(x,y) \in K_1 \times L_1, \overline{\ell(x)} = \ell(y)\}$,
- $\partial_1^\alpha(x,y) = (\partial_1^\alpha(x), y)$ for every $(x,y) \in K_1 \times L_0$,
- $\partial_1^\alpha(x,y) = (x, \partial_1^\alpha(y))$ for every $(x,y) \in K_0 \times L_1$,
- $\partial_1^\alpha(x,y) = (\partial_1^\alpha(x), \partial_1^\alpha(y))$ for every $(x,y) \in K_1 \times L_1$,
- $\ell(x,y) = \ell(x)$ for every $(x,y) \in K_1 \times L_0$,
- $\ell(x,y) = \ell(y)$ for every $(x,y) \in K_0 \times L_1$,
- $\ell(x,y) = \tau$ for every $(x,y) \in K_1 \times L_1$ with $\overline{\ell(x)} = \ell(y)$.

The 1-cubes (x,y) of $(K \times_\Sigma L)_1 \cap (K_1 \times L_1)$ are called synchronizations of x and y.

Definition 13.2. A labelled symmetric precubical set $\ell : K \to !^S\Sigma$ decorated by process names is a labelled precubical set together with a set map $d : K_0 \to \mathbf{Proc}_\Sigma$ called the decoration.

Let $(\square_S)_n \subset \square_S$ be the full subcategory of $\square_S$ containing the $[p]$ only for $p \leqslant n$. By [Gau10, Proposition 5.4], the truncation functor

$$\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma \to (\square_S)_n^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma$$

has a right adjoint $\mathrm{cosk}_n^{\square_S,\Sigma} : (\square_S)_n^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma \to \square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma$.

Definition 13.3. Let K be a 1-dimensional labelled symmetric precubical set with $K_0 = [p]$ for some $p \geqslant 0$. The labelled symmetric directed coskeleton of K is the labelled symmetric precubical set $\overrightarrow{\mathrm{cosk}}_S^{\Sigma}(K)$ defined as the subobject of $\mathrm{cosk}_1^{\square_S,\Sigma}(K)$ such that:

- $\overrightarrow{\mathrm{cosk}}_S^{\Sigma}(K)_{\leqslant 1} = \mathrm{cosk}_1^{\square_S,\Sigma}(K)_{\leqslant 1}$.
- For every $n \geqslant 2$, $x \in \mathrm{cosk}_1^{\square_S,\Sigma}(K)_n$ is an n-cube of $\overrightarrow{\mathrm{cosk}}_S^{\Sigma}(K)$ if and only if the set map $x_0 : [n] \to [p]$ is non-twisted, i.e., $x_0 : [n] \to [p]$ is a composite[3]
$$x_0 : [n] \xrightarrow{\phi} [q] \xrightarrow{\psi} [p],$$
where ψ is a morphism of the small category $\square$ and where ϕ is of the form
$$(\epsilon_1, \ldots, \epsilon_n) \mapsto (\epsilon_{i_1}, \ldots, \epsilon_{i_q})$$
such that $\{1, \ldots, n\} \subset \{i_1, \ldots, i_q\}$.

Let us recall that for every $m, n \geqslant 0$ and $a_1, \ldots, a_m, b_1, \ldots, b_n \in \Sigma$, the labelled symmetric precubical set $\overrightarrow{\mathrm{cosk}}_S^{\Sigma}(\square_S[a_1, \ldots, a_m]_{\leqslant 1} \times_\Sigma \square_S[b_1, \ldots, b_n]_{\leqslant 1})$ satisfies the HDA paradigm. In particular, one has the isomorphism of labelled symmetric precubical sets

$$\boxed{\square_S[a_1, \ldots, a_m] \cong \overrightarrow{\mathrm{cosk}}_S^{\Sigma}(\square_S[a_1, \ldots, a_m]_{\leqslant 1})}.$$

Definition 13.4. Let K and L be two labelled symmetric precubical sets. The tensor product with synchronization (or synchronized tensor product) of K and L is

$$K \otimes_\Sigma L := \varinjlim_{\square_S[a_1,\ldots,a_m]\to K} \varinjlim_{\square_S[b_1,\ldots,b_n]\to L} \overrightarrow{\mathrm{cosk}}_S^{\Sigma}(\square_S[a_1, \ldots, a_m]_{\leqslant 1} \times_\Sigma \square_S[b_1, \ldots, b_n]_{\leqslant 1}).$$

Let us define by induction on the syntax of the CCS process name P the decorated labelled symmetric precubical set $\square_S[\![P]\!]$ (see [Gau08] for further explanations). The labelled symmetric precubical set $\square_S[\![P]\!]$ has always a unique initial state canonically decorated by the process name P and its other states will be decorated as well in an inductive way. Therefore for every process name P, $\square_S[\![P]\!]$ is an object of the double comma category $\{i\}{\downarrow}\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma$. One has $\square_S[\![nil]\!] := \square_S[0]$, $\square_S[\![\mu.nil]\!] := \mu.nil \xrightarrow{(\mu)} nil$, $\square_S[\![P+Q]\!] := \square_S[\![P]\!] \oplus \square_S[\![Q]\!]$ with the binary coproduct taken in

[3] The factorization is necessarily unique.

$\{i\}{\downarrow}\square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma$, the pushout diagram of symmetric precubical sets

$$\begin{tikzcd}
\square_S[0] = \{0\} \arrow[r, "0\mapsto nil"] \arrow[d, "0\mapsto P"] & \square_S[\![\mu.nil]\!] \arrow[d] \\
\square_S[\![P]\!] \arrow[r] & \square_S[\![\mu.P]\!],
\end{tikzcd}$$

the pullback diagram of symmetric precubical sets

$$\begin{tikzcd}
\square_S[\![(\nu a)P]\!] \arrow[r] \arrow[d] & \square_S[\![P]\!] \arrow[d] \\
!^S(\Sigma\backslash\{a,\overline{a}\}) \arrow[r] & !^S\Sigma,
\end{tikzcd}$$

the formula giving the interpretation of the parallel composition with synchronization

$$\square_S[\![P||Q]\!] := \square_S[\![P]\!] \otimes_\Sigma \square_S[\![Q]\!]$$

and finally $\square_S[\![\mathrm{rec}(x)P(x)]\!]$ defined as the least fixed point of $P(-)$. The condition imposed on $P(x)$ implies that for all process names Q_1 and Q_2 with $\square_S[\![Q_1]\!] \subset \square_S[\![Q_2]\!]$, one has $\square_S[\![P(Q_1)]\!] \subset \square_S[\![P(Q_2)]\!]$. So by starting from the inclusion of labelled symmetric precubical sets $\square_S[\![nil]\!] \subset \square_S[\![P(nil)]\!]$ given by the unique initial state of $\square_S[\![P(nil)]\!]$, the labelled symmetric precubical set

$$\square_S[\![\mathrm{rec}(x)P(x)]\!] := \varinjlim_n \square_S[\![P^n(nil)]\!] \cong \bigcup_{n\geqslant 0} \square_S[\![P^n(nil)]\!]$$

will be equal to the least fixed point of $P(-)$.

Proposition 13.5. *Let $m,n \geqslant 0$ and $a_1,\dots,a_m,b_1,\dots,b_n \in \Sigma$. The weak higher dimensional transition system*

$$\mathbb{T}\Big(\overrightarrow{\mathrm{cosk}}_S^\Sigma(\square_S[a_1,\dots,a_m]_{\leqslant 1} \times_\Sigma \square_S[b_1,\dots,b_n]_{\leqslant 1})\Big)$$

is a higher dimensional transition system.

Proof. The proof is similar to the proof of Proposition 5.2. $\square$

Theorem 13.6. *For every CCS process name P, the labelled symmetric precubical set $\square_S[\![P]\!]$ belongs to $\mathbf{HDA}^\Sigma$ and the weak higher dimensional transition system $\mathbb{T}(\square_S[\![P]\!])$ satisfies CSA1 and the Unique intermediate state axiom, i.e., $\mathbb{T}(\square_S[\![P]\!]) \in \mathbf{HDTS}$.*

Sketch of proof. That $\square_S[\![P]\!]$ belongs to $\mathbf{HDA}^\Sigma$ is proved by induction on the syntax of P, as in [Gau08, Theorem 5.2]. If $\square_S[\![P]\!]$ and $\square_S[\![Q]\!]$ belong to $\mathbf{HDA}^\Sigma$, then $\square_S[\![P+Q]\!]$ belongs to $\mathbf{HDA}^\Sigma$ since every map $\partial\square_S[a_1,\dots,a_p] \to \square_S[\![P+Q]\!]$ with $p \geqslant 2$ factors as a composite

$$\partial\square_S[a_1,\dots,a_p] \to \square_S[\![P]\!] \to \square_S[\![P+Q]\!]$$

or as a composite $\partial\square_S[a_1,\dots,a_p] \to \square_S[\![Q]\!] \to \square_S[\![P+Q]\!]$. If $\square_S[\![P]\!]$ belongs to $\mathbf{HDA}^\Sigma$, then $\square_S[\![(\nu aP)]\!]$ belongs to $\mathbf{HDA}^\Sigma$ since $\square_S[\![(\nu a)P]\!] \subset \square_S[\![P]\!]$. If for every $n \geqslant 0$, the labelled symmetric precubical set $\square_S[\![P^n(nil)]\!]$ belongs to $\mathbf{HDA}^\Sigma$, then $\square_S[\![\mathrm{rec}(x)P(x)]\!]$ belongs to $\mathbf{HDA}^\Sigma$ since the inclusion functor $\mathbf{HDA}^\Sigma \subset \square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$ is accessible by Corollary 7.4. Finally, let P and Q be two process names such that both $\square_S[\![P]\!]$ and $\square_S[\![Q]\!]$ belong to $\mathbf{HDA}^\Sigma$. For a given map $\partial\square_S[a_1,\dots,a_p] \to \square_S[\![P||Q]\!]$ with $p \geqslant 2$, the category $\partial\square_S[a_1,\dots,a_p]{\downarrow}(\square_S \times \square_S){\downarrow}\square_S[\![P||Q]\!]$ has an initial object [4] otherwise $\square_S[\![P]\!]$ or $\square_S[\![Q]\!]$ would not satisfy the HDA paradigm. Hence the labelled symmetric precubical set $\square_S[\![P||Q]\!]$ satisfies the HDA paradigm too, since the labelled symmetric precubical set

$$\overrightarrow{\mathrm{cosk}}^\Sigma_S(\square_S[a_1,\dots,a_m]_{\leqslant 1} \times_\Sigma \square_S[b_1,\dots,b_n]_{\leqslant 1})$$

does for every $m, n \geqslant 0$ and for every $a_1,\dots,a_m,b_1,\dots,b_n \in \Sigma$.

It is clear that CSA1 is always satisfied by $\square_S[\![P]\!]$. That $\square_S[\![P]\!]$ is a strong labelled symmetric precubical set, i.e., that $\mathbb{T}(\square_S[\![P]\!])$ satisfies the Unique intermediate state axiom, is proved by induction on the syntax of P as follows. It is obvious that if $\square_S[\![P]\!]$ and $\square_S[\![Q]\!]$ are strong, then $\square_S[\![P+Q]\!]$ is strong too. It is also obvious that $\square_S[\![(\nu a)P]\!]$ is strong since the weak higher dimensional transition system $\mathbb{T}(\square_S[\![(\nu a)P]\!])$ is included in the higher dimensional transition system $\mathbb{T}(\square_S[\![P]\!])$. If for every $n \geqslant 0$, the labelled symmetric precubical set $\square_S[\![P^n(nil)]\!]$ is strong, then $\square_S[\![\mathrm{rec}(x)P(x)]\!]$ is strong too by Theorem 11.6. It remains to prove that if the two weak higher dimensional transition systems $\mathbb{T}(\square_S[\![P]\!])$ and $\mathbb{T}(\square_S[\![Q]\!])$ satisfy the Unique intermediate axiom, then the weak higher dimensional transition systems $\mathbb{T}(\square_S[\![P]\!] \otimes_\Sigma \square_S[\![Q]\!])$ does as well. Since $\mathbb{T}$ is colimit-preserving by Theorem 9.2, the weak higher dimensional transition system

$$\mathbb{T}(\square_S[\![P]\!] \otimes_\Sigma \square_S[\![Q]\!])$$

is isomorphic to

$$\varinjlim_{\square_S[a_1,\dots,a_m]\to K}\ \varinjlim_{\square_S[b_1,\dots,b_n]\to L} \mathbb{T}\Big(\overrightarrow{\mathrm{cosk}}^\Sigma_S(\square_S[a_1,\dots,a_m]_{\leqslant 1} \times_\Sigma \square_S[b_1,\dots,b_n]_{\leqslant 1})\Big).$$

By Theorem 4.7 and Proposition 13.5, an n-transition of the higher dimensional transition system $\mathbb{T}\Big(\overrightarrow{\mathrm{cosk}}^\Sigma_S(\square_S[a_1,\dots,a_m]_{\leqslant 1} \times_\Sigma \square_S[b_1,\dots,b_n]_{\leqslant 1})\Big)$ is of the form

$$((\alpha,\beta),(u_1,v_1),\dots,(u_n,v_n),(\gamma,\delta))$$

with three mutually exclusive cases for the (u_i,v_i): (1) Both u_i and v_i are actions of respectively $\mathbb{T}(\square_S[\![P]\!])$ and $\mathbb{T}(\square_S[\![Q]\!])$; in this case $u_i = \overline{v_i}$ and $\mu(u_i,v_i) = \tau$; this case corresponds to a synchronization. (2) u_i is an action of $\mathbb{T}(\square_S[\![P]\!])$ and v_i is a state of $\mathbb{T}(\square_S[\![Q]\!])$. (3) u_i is a state of $\mathbb{T}(\square_S[\![P]\!])$ and

[4] There is an error in [Gau08, Theorem 5.2], which says that this small category is directed. It should say that this category always has an initial object.

$$\begin{tikzcd}
\mathbf{Proc}_\Sigma \arrow[r, "\square_S[\![-]\!]"] & \mathbf{HDA}^\Sigma_{\mathrm{hdts}} \arrow[r, "\cong"] \arrow[d, "\subset"'] & \mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}] \arrow[d, "\subset"'] \arrow[dr, "|-|_{\mathrm{flow}}"] & \\
& \square_S^{\mathrm{op}}\mathbf{Set}{\downarrow}!^S\Sigma \arrow[r, "\mathbb{T}"] & \mathbf{WHDTS} \arrow[r, "|-|"] & \mathbf{Flow}
\end{tikzcd}$$

FIGURE 5. Recapitulation: one has the inclusions of full subcategories $\mathbf{HDTS}[\mathrm{Opt}^{-1}] \subset \mathbf{HDTS} \subset \mathbf{WHDTS}$

v_i is an action of $\mathbb{T}(\square_S[\![Q]\!])$. For such an n-transition, the tuples obtained from $(\alpha, u_1, \dots, u_n, \gamma)$ and $(\beta, v_1, \dots, v_n, \delta)$ by removing the u_i and v_i which are states are transitions of respectively $\mathbb{T}(\square_S[\![P]\!])$ and $\mathbb{T}(\square_S[\![Q]\!])$. So the union of the transitions of the

$$\mathbb{T}\Big(\overrightarrow{\mathrm{cosk}}^{\Sigma}_S(\square_S[a_1, \dots, a_m]_{\leqslant 1} \times_\Sigma \square_S[b_1, \dots, b_n]_{\leqslant 1})\Big)$$

satisfies the Unique intermediate state axiom since $\mathbb{T}(\square_S[\![P]\!])$ and $\mathbb{T}(\square_S[\![Q]\!])$ do. So by Theorem 4.7 again, this union is the final structure, that is the colimit. Hence, the weak higher dimensional transition system

$$\mathbb{T}(\square_S[\![P]\!] \otimes_\Sigma \square_S[\![Q]\!])$$

satisfies the Unique intermediate state axiom. $\square$

Corollary 13.7. *The mapping taking each CCS process name P to the flow*

$$|\square_S[\![P]\!]|_{\mathrm{flow}}$$

factors through the category of higher dimensional transition systems.

14. Concluding remarks and perspectives

The commutative diagram of Figure 5 summarizes the two main results of this paper. In $\mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$, two higher dimensional transition systems are isomorphic if they have the same cubes modulo their unused actions. This category is equivalent to a full coreflective subcategory of the category **HDTS** of higher dimensional transition systems, and the latter is a reflective full subcategory of that of weak higher dimensional transition systems **WHDTS**. The category $\mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$ is also equivalent to $\mathbf{HDA}^\Sigma_{\mathrm{hdts}}$ which is a full reflective subcategory of that of labelled symmetric precubical sets, and even a full reflective subcategory of those satisfying the HDA paradigm ($\mathbf{HDA}^\Sigma$). The inclusion $\mathbf{HDA}^\Sigma_{\mathrm{hdts}} \subset \mathbf{HDA}^\Sigma$ is strict since the non-strong labelled symmetric precubical set K used for proving Proposition 9.7 satisfies the HDA paradigm.

All these constructions illustrate the expressiveness of the category of flows and of the other topological models of concurrency. Indeed, using the geometric realization functors from **HDTS** to **Flow**, one can associate a flow with any transition system with independence, with any Petri net, with any domain of configurations of prime event structures [CS96], and of course with any process algebra as already explained in [Gau08].

It would be interesting to find a geometric sufficient condition for a labelled symmetric precubical set K to be strong, for example by proving that $\mathbf{HDA}^{\Sigma}_{\mathrm{hdts}}$ is a small-orthogonality class. It would be also interesting to find the analogue of the notion of weak higher dimensional transition system for the labelled symmetric transverse precubical sets (the presheaves over $\widehat{\square}$) introduced in [Gau10]. Weak higher dimensional transition systems are transition systems indexed by finite multisets of actions. The analogous notion for labelled symmetric transverse precubical sets should be a notion of transition system indexed by partially ordered finite multisets of actions. By restricting to transitions labelled by finite multisets endowed with a discrete ordering, one should get back a weak higher dimensional transition system. Understanding the link between labelled transverse symmetric precubical sets and higher dimensional transition systems is necessary since the space of morphisms of flows from $|\square_S[m]|_{\mathrm{flow}}$ to itself for $m \geqslant 0$ is homotopy equivalent to $\widehat{\square}([m],[m])$, not to $\square_S([m],[m])$, and the inclusion $\square_S([m],[m]) \subset \widehat{\square}([m],[m])$ is strict for $m \geqslant 2$. In particular, it contains the set map $(\epsilon_1,\dots,\epsilon_m) \mapsto (\max(\epsilon_1,\epsilon_2),\min(\epsilon_1,\epsilon_2),\epsilon_3,\dots,\epsilon_m)$. These questions will be hopefully the subject of future works.

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Laboratoire PPS (CNRS UMR 7126), Université Paris 7–Denis Diderot, Site Chevaleret, Case 7014, 75205 PARIS Cedex 13, France
http://www.pps.jussieu.fr/~gaucher/

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