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Dynatomic cycles for morphisms of projective varieties

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Abstract. We prove the effectivity of the zero-cycles of formal periodic points, dynatomic cycles, for morphisms of projective varieties. We then analyze the degrees of the dynatomic cycles and multiplicities of formal periodic points and apply these results to the existence of periodic points with arbitrarily large minimal periods.

Contents

1. Introduction

Consider an analytic function $f : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$[z_1, \ldots, z_N] \mapsto [f_1(z_1, \ldots, z_N), \ldots, f_N(z_1, \ldots, z_N)].$$

We can iterate the function f and denote the n^{th} *iterate* as $f^n = f(f^{n-1})$ to create a (discrete) dynamical system. The *periodic points* of f are the points $P \in \mathbb{C}^N$ such that $f^n(P) = P$ for some integer n. We call n the *period* of

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P and the least such n the *minimal* period of P. Denote the coordinate functions of the n^{th} iterate as $f^n = [f_1^n, \ldots, f_N^n]$. Then, the set of periodic points of period n, but not necessarily minimal period n, for f is the set of solutions to the system of equations

$$f_i^n(z_1, \ldots, z_N) = z_i \quad \text{for } 1 \leq i \leq N.$$

To find the points of minimal period n, we could attempt to remove the points of period strictly less than n from this set. In the case of $f(z) \in \mathbb{C}[z]$, we can do this through division. Consider the zeros of

$$\prod_{d \mid n} (f^d(z) - z)^{\mu\left(\frac{n}{d}\right)} \tag{1.1}$$

where μ is the Möbius function defined as $\mu(1) = 1$ and

$$\mu(n) = \begin{cases} (-1)^{\omega} & n \text{ is square-free with } \omega \text{ distinct prime factors} \\ 0 & n \text{ is not square-free.} \end{cases}$$

For example, for $n = 6$ we consider

$$\prod_{d \mid 6} (f^d(z) - z)^{\mu\left(\frac{6}{d}\right)} = \frac{(f^6(z) - z)(f(z) - z)}{(f^3(z) - z)(f^2(z) - z)}.$$

Two fundamental questions come to mind. Are the zeros of the resulting function exactly the set of periodic points of minimal period n? Does the resulting function have poles as well as zeros? In the case $f(z) = z^2 - \frac{3}{4}$ for $n = 2$ we compute

$$\frac{f^2(z) - z}{f(z) - z} = (2z + 1)^2 \quad \text{and} \quad f\left(-\frac{1}{2}\right) = -\frac{1}{2}.$$

Thus, the answer to the first question is no, not all zeros are periodic points of minimal period n. Additionally, this phenomenon of higher multiplicity through collision of periodic cycles allows for the possible existence of poles. In the single variable polynomial case, both of these questions were answered by Morton [17, Theorem 2.4 and Theorem 2.5]. He showed that the points of minimal period n are among the zeros of the resulting function, any zero with minimal period strictly less than n must have multiplicity (as a zero) greater than one, and the resulting function is always a polynomial. Morton and Silverman conjectured the nonexistence of poles in a much more general setting [19, Conjecture 1.1]. This article will address the minimal period of zeros and the existence of poles in the more general setting.

We now state the more general problem. Let K be an algebraically closed field and X/K a projective variety. Let $\phi : X/K \to X/K$ be a morphism defined over K, a function locally representable as a system of homogeneous polynomials with no common zeros. We can iterate the morphism ϕ, denoted ϕ^n, and consider the periodic points for ϕ. As the proofs will require tools from both dynamical systems and algebraic geometry in which the word cycle has two different meanings, we adopt the terminology: *periodic cycle* to be the points in the orbit of a periodic point and *algebraic zero-cycle* as a formal sum of points with integer multiplicities (only finitely many nonzero). If all of the multiplicities of an algebraic zero-cycle are nonnegative, we call it *effective*. For example, for $\phi \in K[z]$, if the algebraic zero-cycle of periodic points of period n is effective, then the function $\phi^n(z) - z$ has no poles. To generalize construction (1.1) we follow [19] and consider the graph of ϕ^n in the product variety $X \times X$ defined as

$$\Gamma_n = \{(x, \phi^n(x)) : x \in X\}$$

and the diagonal in $X \times X$ defined as

$$\Delta = \{(x, x) : x \in X\}.$$

Their intersection is precisely the periodic points of period n, and we can determine the multiplicity of points as the multiplicity of the intersection (see for example [22] for some basic intersection theory). Denote the intersection multiplicity of Γ_n and Δ at a point $(P, P) \in X \times X$ to be $a_P(n)$ and the algebraic zero-cycle of periodic points of period n as

$$\Phi_n(\phi) = \sum_{P \in X} a_P(n)(P).$$

Following construction (1.1), define

$$a^*_P(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) a_P(d)$$

and

$$\Phi^*_n(\phi) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \Phi_d(\phi) = \sum_{P \in X} a^*_P(n)(P).$$

Definition 1.1. We call $\Phi^*_n(\phi)$ the n^{th} *dynatomic cycle*[1] and $a^*_P(n)$ the *multiplicity* of P in $\Phi^*_n(\phi)$. If $a^*_P(n) > 0$, then we call P a periodic point of *formal* period n.

Remark. In the one variable polynomial case, Φ^*_n is called a *dynatomic polynomial* and has been studied extensively such as in [14, 17, 23]. For a more complete background and additional references in this area see [24].

To state our results precisely we need one more definition.

Definition 1.2. For $n \geq 1$, we say that ϕ^n is *nondegenerate* if Δ and Γ_n intersect properly; in other words, if $\Delta \cap \Gamma_n$ is a finite set of points.

Remark. If ϕ^n is nondegenerate, then ϕ^d is nondegenerate for all $d \mid n$ since $\Delta \cap \Gamma_d \subseteq \Delta \cap \Gamma_n$. Conversely, ϕ may be nondegenerate with ϕ^n degenerate, such as when ϕ is a nontrivial automorphism of a curve with finite order.

[1] This term is inspired by "cyclotomic" much like "Tribonacci" was inspired by "Fibonacci".

We now describe the results and organization of this article. In Section 2 we prove that $\Phi_n^*(\phi)$ is effective for morphisms of nonsingular, irreducible, projective varieties with ϕ^n nondegenerate and describe the possible values of n for which a periodic point P of ϕ has nonzero multiplicity in $\Phi_n^*(\phi)$ resolving the conjecture of Morton and Silverman [19, Conjecture 1.1] in the affirmative.

Theorem 1.3. *Let $X \subset \mathbb{P}^N_K$ be a nonsingular, irreducible, projective variety of dimension b defined over an algebraically closed field K and let $\phi : X \to X$ be a morphism defined over K. Let P be a point in $X(K)$. Define integers*

$p =$ the characteristic of K.

$m =$ the minimal period of P for ϕ (set $m = \infty$ if $P \notin \operatorname{Per}(\phi)$).

If m is finite, let $d\phi_P^m$ be the map induced by ϕ^m on the cotangent space of X at P, and let $\lambda_1, \ldots, \lambda_l$ be the distinct eigenvalues of $d\phi_P^m$. Define

$r_i =$ the multiplicative period of λ_i in K^* (set $r_i = \infty$ if λ_i is not a root of unity).

Then:

1. *For all $n \geq 1$ such that ϕ^n is nondegenerate, $a_P^*(n) \geq 0$.*
2. *Let $n \geq 1$. If ϕ^n is nondegenerate and $a_P^*(n) \geq 1$, then $m \neq \infty$ and n has one of the following forms:*
   (a) *$n = m$.*
   (b) *$n = m \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})$ for some $1 \leq k \leq l$.*
   (c) *$n = m \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k}) p^e$ for some $1 \leq k \leq l$ and some $e \geq 1$.*

As in the one-dimensional case, the proof is carried out by carefully examining when the multiplicity of a fixed point P in $\Phi_n(\phi)$ is greater than the multiplicity of P in $\Phi_1(\phi)$. However, several new ideas and a lot of additional work are needed in the higher dimensional case. Some of the difficulties encountered are taking into account the higher Tor modules in the intersection theory (Definition 2.1), using the theory of standard bases to obtain information about the multiplicity of a point in $\Phi_n(\phi)$ (Proposition 2.18), and iterating local power series representations of the morphism.

From this detailed analysis of the multiplicities, in Section 3 we show that periodic points of formal period n with multiplicity one and $\operatorname{char} K \nmid n$ have minimal period n. In other words, $a_P^*(n) = 1$ for $\operatorname{char} K \nmid n$ implies that P is a periodic point of minimal period n, generalizing [17, Theorem 2.5].

In Section 4.1 we state some basic properties of $\Phi_n(\phi)$ and $\Phi_n^*(\phi)$ including the fact that all periodic points of minimal period n are points of formal period n (Proposition 4.1(2)). In Section 4.2 we note the similarity to periodic Lefschetz numbers, and in Section 4.3 we state results similar to those of [8, 13, 25] on the existence of periodic points. In particular, if P is a periodic point, then the sequence $\{a_P(n)\}$ for $\operatorname{char} K \nmid n$ is bounded (Theorem 4.11), and if $\deg(\Phi_n)$ is unbounded for $\operatorname{char} K \nmid n$, then there are periodic points with arbitrarily large minimal periods and infinitely many periodic points (Corollary 4.12). In Section 4.4 these results are applied to dynamical systems on Wehler K3 surfaces studied in [5, 23] and in Section 4.5 to dynamical systems arising from morphisms of projective space.

The cycles $\Phi_n(\phi)$ and $\Phi_n^*(\phi)$ occur with great frequency in the literature under a variety of notations and with a number of results stemming from the fact they are effective, see for example [14, 15, 16, 17, 18, 19, 20, 26]. In particular, [17, 26] contain Galois theoretic results in the single-variable polynomial case where $\Phi_n^*(\phi)$ has no points of multiplicity greater than one; many of the arguments of these two articles carry through to the higher dimensional case given that $\Phi_n^*(\phi)$ is effective (see [10, Chapter 4]).

For an introduction to the algebraic geometry needed such as varieties, morphisms, local power series representations, and basic intersection theory see [22], for a reference for the homological and local algebra needed see [12, 21], and, finally, for background and more discussion of the algebraic dynamics see [24]. Much of this work is from the author's doctoral thesis [10, Chapter 3].

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2. Effectivity of $\Phi_n^*(\phi)$

Recall that we have defined K to be an algebraically closed field, X/K a projective variety of dimension b, and $\phi : X \to X$ a morphism defined over K. Let $P \in X(K)$ and let R_P be the local ring (see for example [22, II.1]) of $X \times X$ at (P, P) and let $I_\Delta, I_{\Gamma_n} \subset R_P$ be the ideals of Δ (the diagonal) and Γ_n (the graph of ϕ^n), respectively. The following steps outline the proof of the effectivity of $\Phi_n^*(\phi)$.

(1) Define the intersection multiplicity and show that $\Phi_n^*(\phi)$ is an algebraic zero-cycle.

(2) Show that the naive intersection theory is, in fact, correct (Theorem 2.4). Specifically, show that

$$\operatorname{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n}) = 0 \quad \text{for all } i > 0$$

using properties of Cohen–Macaulay modules from [21].

(3) Determine conditions on n for when $a_P(n) > a_P(1)$ (Proposition 2.18). In particular, we show that if $a_p(n) > a_p(1)$, there must be a least monomial (Definition 2.13) H with $H \in \operatorname{supp}(\phi)$ and $H \notin \operatorname{supp}(\phi^n)$. Therefore, given a least monomial $H \in \operatorname{supp}(\phi)$ we determine in general the coefficient of $H \in \operatorname{supp}(\phi^n)$. The conditions on n are then determined by when such a coefficient is 0 and, hence, $H \notin \operatorname{supp}(\phi^n)$.

(4) Show that $a^*_P(n) \geq 0$ for all P and n (Theorem 1.3). Specially, using the conditions on n from Proposition 2.18 for $a_p(n) > a_p(1)$, we check several cases to determine that $a^*_p(n) \geq 0$.

In what follows, the concept of dimension will be used in several different contexts. Recall that for a ring R, we call a sequence $P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq \cdots \subsetneq P_n$ of prime ideals of R a *chain of length* n. The *Krull dimension* of R is given by the supremum of the length of chains of prime ideals in R [21, Section III.A.1]. We will denote

- $\dim R$ for the Krull dimension of a ring R,
- $\dim M$ for the Krull dimension of $R/\operatorname{Ann}(M)$ where $\operatorname{Ann}(M)$ is the annihilator of the R-module M, and
- $\dim_K V$ for the dimension of the finite dimensional K-vector space V.

2.1. Intersection multiplicity. For two irreducible curves C and D on a nonsingular projective surface X the intersection multiplicity of a point P on X is defined as $\dim(\mathcal{O}_{P,X}/(f,g))$ where $\mathcal{O}_{P,X}$ is the local ring of X at P and (f,g) is the ideal generated by the local equations f and g for C and D at P. This returns a nonnegative integer which can be shown to be independent of the choice of local equations f and g. Following this model, we could try and define the intersection multiplicity of I_{Γ_n} and I_Δ at a point (P,P) as $\dim_K(R_P/(I_\Delta + I_{\Gamma_n})) = \operatorname{length}(R_P/I_\Delta \otimes R_P/I_{\Gamma_n})$. However, this is too simplistic in general and does not give the correct value, see for example [9, Example A.1.1.1]. We will use Serre's definition of intersection multiplicity [9, Appendix A] using Tor-modules. We first recall the definition of Tor-modules, see [12] for a more extensive treatment.

Definition 2.1. Let M, N be R-modules. Let

$$\cdots \to E_i \to E_{i-1} \to \cdots \to E_0 \to M \to 0$$

be a free or projective resolution of M, in other words, an exact sequence where each E_i is a free or projective R-module. Then we define $\operatorname{Tor}_i(M,N)$ to be the i^{th} homology (in other words, $\ker(d_i)/\operatorname{im}(d_{i+1})$) of

$$\cdots \xrightarrow{d_{i+1}} E_i \otimes N \xrightarrow{d_i} E_{i-1} \otimes N \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} E_0 \otimes N \xrightarrow{d_0} 0.$$

In particular,

$$\operatorname{Tor}_0(R_P/I_\Delta, R_P/I_{\Gamma_n})) \cong R_P/I_\Delta \otimes R_P/I_{\Gamma_n} \cong R_P/(I_\Delta + I_{\Gamma_n})$$

recovering our "naive" intersection multiplicity as

$$\dim_K(\operatorname{Tor}_0(R_P/I_\Delta, R_P/I_{\Gamma_n})).$$

Now we can state Serre's definition of intersection multiplicity [9, Appendix A],

$$a_P(n) = i(\Delta, \Gamma_n; P) = \sum_{i=0}^{b-1} (-1)^i \dim_K(\operatorname{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n})).$$

In what follows, we will actually work over the completion $\widehat{R_P}$ of R_P so that we may consider our problem over a local power series ring.

Since ϕ^n is nondegenerate, Δ and Γ_n intersect properly. We also know $X \times X$ has dimension $2b$, Δ has dimension b, and Γ_n has dimension b. Consequently, $\Phi_n(\phi)$ is an algebraic zero-cycle. Thus, $\Phi_n^*(\phi)$ is also an algebraic zero-cycle.

In local coordinates, we have

$$\widehat{R_P} \cong K[[x_1, \ldots, x_b, y_1, \ldots, y_b]].$$

Definition 2.2. Let $\phi(\mathbf{x}) = [\phi_1(\mathbf{x}), \ldots, \phi_b(\mathbf{x})]$, where $\mathbf{x} = (x_1, \ldots, x_b)$. Then denote

$$\phi^n(\mathbf{x}) = [\phi_1^n(\mathbf{x}), \ldots, \phi_b^n(\mathbf{x})]$$

as the coordinates of the n^{th} iterate of ϕ.

Then we have

$$I_\Delta = (x_1 - y_1, \ldots, x_b - y_b) \quad \text{and} \quad I_{\Gamma_n} = (\phi_1^n(\mathbf{x}) - y_1, \ldots, \phi_b^n(\mathbf{x}) - y_b).$$

We will use the nondegeneracy of ϕ^n and the following theorem to show that $\mathrm{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n}) = 0$ for all $i > 0$.

Theorem 2.3 ([21, Corollary to Theorem V.B.4])**.** *Let $(R, \mathfrak{m})$ be a regular local ring of dimension b, and let M and N be two nonzero finitely generated R-modules such that $M \otimes N$ is of finite length. Then $\mathrm{Tor}_i(M,N) = 0$ for all $i > 0$ if and only if M and N are Cohen–Macaulay modules and $\dim M + \dim N = b$.*

Theorem 2.4. *Let X be a nonsingular, irreducible, projective variety defined over a field K and $\phi : X \to X$ a morphism defined over K such that ϕ^n is nondegenerate. Let $P \in X(K)$. Then, $\mathrm{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n}) = 0$ for all $i > 0$.*

Proof. Let $b = \dim X$, then we have $\dim X \times X = 2b$ and $\dim \Delta = \dim \Gamma_n = b$. The ideals I_Δ and I_{Γ_n} are each generated by b elements and Δ and Γ_n intersect properly. Therefore,

$$\dim_K(R_P/(I_\Delta + I_{\Gamma_n})) = \mathrm{length}(R_P/I_\Delta \otimes R_P/I_{\Gamma_n}) < \infty.$$

By [21, Proposition III.B.6] the union of the generators of I_Δ and the generators of I_{Γ_n} are a system of parameters for R_P. Because R_P is Cohen–Macaulay by [21, Corollary 3 to Theorem IV.D.9] we can apply [21, Corollary to Theorem IV.B.2] to I_Δ and its generators to conclude that R_P/I_Δ is Cohen–Macaulay of dimension b and, similarly with I_{Γ_n}, to conclude that R_P/I_{Γ_n} is Cohen–Macaulay of dimension b.

We have fulfilled the hypotheses of Theorem 2.3; consequently, we have that

$$\mathrm{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n}) = 0 \quad \text{for all } i > 0. \qquad \square$$

2.2. $\mathbf{Tor_0}$ module. If P is not a periodic point, then $a_P(n) = 0$ for all n, so we will assume that P is a periodic point. If $a_P(1) = 0$, then P has some minimal period $m > 1$. If $m \nmid n$ then $a_P(n) = 0$, so we may replace ϕ by ϕ^m and assume that P is a fixed point for ϕ and, hence, $a_P(1) > 0$. For P, a fixed point of ϕ, we can iterate a local representation of ϕ as a family of power series.

From Theorem 2.4 we know the naive intersection multiplicity

$$\begin{aligned} a_P(n) &= \dim_K(\mathrm{Tor}_0(R_P/I_\Delta, R_P/I_{\Gamma_n})) \\ &= \dim_K(R_P/(I_\Delta + I_{\Gamma_n})) \end{aligned}$$

is, in fact, correct in our situation. To prove the effectivity of $\Phi_n^*(\phi)$, we will use conditions on n for $a_P(n) > a_P(1)$. To determine these conditions, we will consider local power series representations of ϕ and the theory of standard bases. For information on standard bases, see [4, Chapter 4]. Below, we recall the needed terminology.

Definition 2.5. Recall that a formal power series $f \in K[[X_1, \ldots, X_h]]$, may be written as

$$f = \sum_{v \in \mathbb{N}^h} f_v X^v.$$

An *admissible monomial ordering* is an ordering on the set T of terms in $X_1, \ldots, X_h$ such that $1 < t$ for all $t \in T$ and if $t_1 < t_2$ for $t_1, t_2 \in T$, then $st_1 < st_2$ for all $s \in T$, see for example [3].

The monomial *support of f* is defined as

$$\mathrm{supp}(f) = \{f_v X^v : f_v \neq 0\}.$$

If $f \neq 0$, then $\mathrm{supp}(f)$ has a least element under any admissible monomial ordering. We call this least element the *leading monomial of f*, denoted by $LM(f)$. We denote $v(f)$ the exponent of the leading monomial. Then

$$LM(f) = f_{v(f)} X^{v(f)}$$

and we call $X^{v(f)}$ the *leading term of f* and denote it by $LT(f)$.

Let I be an ideal in $K[[X_1, \ldots, X_h]]$. We define the *leading term ideal of I* as

$$\begin{aligned} &LT(I) = \\ &\qquad \text{the polynomial ideal generated by } \{X^v : \exists f \in I \text{ with } LT(f) = X^v\}. \end{aligned}$$

Definition 2.6. A nonzero element $f \in K[[X_1, \ldots, X_h]]$ is called *self-reduced* with respect to an admissible monomial ordering if

$$LT(f) \nmid F \text{ for all } F \in \mathrm{supp}(f) - LT(f).$$

For the most part, we will not be concerned with the particular admissible ordering that is used, so in what follows we fix an admissible monomial ordering. When necessary, we will specify a particular ordering. Finally, we recall three facts that we will need (see [4, Chapter 4.4]).

Theorem 2.7. *The following are equivalent.*

(1) *There exists a standard basis for I.*
(2) *Every $f \in K[[X_1, \ldots, X_h]]$ has a unique standard remainder modulo I.*
(3) *Every $f \in K[[X_1, \ldots, X_h]]$ has a standard remainder modulo I.*

Theorem 2.8. *Every ideal $I \subset K[[X_1, \ldots, X_h]]$ has a universal standard basis.*

Theorem 2.9. *Let $I \subset K[[X_1, \ldots, X_h]]$ be an ideal with*

$$\dim K[[X_1, \ldots, X_h]]/I = 0.$$

Then $K[[X_1, \ldots, X_h]]/I$ is isomorphic as a K-vector space to

$$\operatorname{Span}(X^v \mid X^v \notin LT(I)).$$

For notational convenience, define $I_n = I_\Delta + I_{\Gamma_n}$.

Corollary 2.10. *Consider the ideal $I_n \subset \widehat{R_P}$. Then*

$$a_P(n) = \dim_K(\widehat{R_P}/I_n) = \dim_K(\operatorname{Span}(X^v \mid X^v \notin LT(I_n))).$$

Proof. Apply Theorem 2.9 to $\widehat{R_P}$ and I_n. □

Lemma 2.11. *Assume ϕ^n is nondegenerate. Then $a_P(n) \geq a_P(1)$ for all $n \in \mathbb{N}$.*

Proof. It is clear that

$$\Gamma_1 \cap \Delta \subseteq \Gamma_n \cap \Delta$$

and we have a local representation of $\phi = [\phi_1, \ldots, \phi_b]$ at the fixed point P. Iterating this representation involves taking combinations of the ϕ_i and hence are all elements of the original ideal I_{Γ_1}. Hence, we have

$$I_{\Gamma_n} + I_\Delta = I_n \subseteq I_1 = I_{\Gamma_1} + I_\Delta.$$

Therefore,

$$LT(I_n) \subseteq LT(I_1)$$

which implies $a_P(n) \geq a_P(1)$ by Corollary 2.10. □

We next show that we may reduce to the case where the generators of the ideal are self-reduced.

Remark. By [3, Corollary 2.2] applied to $LT(I) = LT((f_1, \ldots, f_m))$, we know each there exist units $u_i \in K[[X_1, \ldots, X_h]]$ such that each $u_i f_i$ is self-reduced.

Lemma 2.12. *Let $I \subset K[[X_1, \ldots, X_h]]$ be an ideal generated by $\{f_1, \ldots, f_m\}$ with $\dim K[[X_1, \ldots, X_h]]/I = 0$. Let $u_i \in K[[X_1, \ldots, X_h]]$ be a unit such that $u_i f_i$ is self-reduced for each $1 \leq i \leq h$ and define $uI = (u_1 f_1, \ldots, u_m f_m)$. Then*

$$\dim_K(\operatorname{Span}(X^v \mid X^v \notin LT(I))) = \dim_K(\operatorname{Span}(X^v \mid X^v \notin LT(uI))).$$

Proof. Since each u_i is a unit, we have $v(LT(u_i)) = 0$ and $LT(u_if_i) = LT(f_i)$ (and similarly for any combinations of the f_i). Hence we have

$$LT(I) = LT((f_1, \dots, f_m)) = LT((u_1f_1, \dots, u_mf_m)). \qquad \square$$

Definition 2.13. Let $G, H \in \operatorname{supp}(\phi_i)$. A monomial G is said to *contribute to H through iteration* if for some n when we substitute $\phi_j^{n-1}(x_1, \dots, x_b)$ for x_j for all $1 \leq j \leq b$ for each monomial in $\operatorname{supp}(\phi_i)$ and formally expand the terms to obtain ϕ_i^n as a power series in $x_1, \dots, x_b$, one of these terms is the monomial H.

Let $H \in \operatorname{supp}(\phi_i)$ for some $1 \leq i \leq b$. We say that H is a *least monomial for ϕ_i* if the only monomials contributing to H through iteration are x_i and H.

Example 2.14. We have

$$\begin{aligned}\phi_1(x,y,z) &= x + x^4 + \boxed{x^2z^2} + xy\\ \phi_2(x,y,z) &= y + y^4 + xz^2\\ \phi_3(x,y,z) &= z + z^4.\end{aligned}$$

To iterate the system

$$\phi(x,y,z) = (\phi_1(x,y,z), \phi_2(x,y,z), \phi_3(x,y,z))$$

we take

$$\phi^2(x,y,z) = \phi(\phi_1(x,y,z), \phi_2(x,y,z), \phi_3(x,y,z)).$$

Note that we have $x^2z^2 \in \operatorname{supp}(\phi_1^2)$ with coefficient 3.

(1) One x^2z^2 occurs from $x^2z^2 \in \operatorname{supp}(\phi_1)$ when ϕ_1 is substituted for $x \in \operatorname{supp}(\phi_1)$, and we say x contributes to x^2z^2 through iteration.
(2) A second x^2z^2 occurs from $x \in \operatorname{supp}(\phi_1)$ and $z \in \operatorname{supp}(\phi_3)$ when ϕ_1 and ϕ_3 are substituted for x and z in $x^2z^2 \in \operatorname{supp}(\phi_1)$, and we say x^2z^2 contributes to x^2z^2 through iteration.
(3) The third x^2z^2 occurs from $x \in \operatorname{supp}(\phi_1)$ and $xz^2 \in \operatorname{supp}(\phi_2)$ when ϕ_1 and ϕ_2 are substituted for x and y in $xy \in \operatorname{supp}(\phi_1)$, and we say xy contributes to x^2z^2 through iteration.

It is (3) that causes x^2z^2 to not be a least monomial.

The justification for this definition is Lemma 2.17, but first we examine the coefficients of least monomials under iteration.

Denote $d\phi_P$ as the map induced by ϕ on the cotangent space of X at P. Recall that we are assuming that P is a fixed point of ϕ and that K is algebraically closed. Therefore, $d\phi_P$ is a $b \times b$ matrix and can always be put in Jordan canonical form, with Jordan blocks $J_1, \dots, J_k$ of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0\\ 0 & \lambda_i & 1 & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & \lambda_i \end{pmatrix}.$$

Lemma 2.15. *Let $F = \prod_{i=1}^{b} x_i^{e_i}$ with $e_i \geq 0$ and assume $F \in \operatorname{supp}(\phi_{i_t})$ is a least monomial with $1 \leq i_t \leq b$. Assume that ϕ_{i_t} is in a Jordan block of $d\phi_P$ of size $\mathfrak{v} \geq 1$ with eigenvalue λ. Label the rows $1, \ldots, \mathfrak{v}$ corresponding to $\phi_{i_1}, \ldots, \phi_{i_{\mathfrak{v}}}$. Label the coefficient of F in $\phi_{i_t}^n$ as c_n with $c_1 \neq 0$. Let $\alpha = \sum_{j=1}^{\mathfrak{v}} e_{i_j}$. Then:*

(1) *If $\deg F = 1$, then*
$$c_n = \lambda^n.$$
(2) *If $\deg F > 1$, then*
$$c_n = \left(\sum_{j=0}^{n-1} \lambda^{j(\alpha-1)+n-1} \prod_{\substack{x_i | F \\ i \notin \{i_1, \ldots, i_{\mathfrak{v}}\}}} \lambda_i^{e_i j} \right) c_1.$$

Proof. We will prove both statements by induction.

(1) For the base case of $n = 1$, we expect to have
$$c_1 = \lambda$$
which corresponds to the linear term of a Jordan block and so verifies the statement for $n = 1$. We will assume now that the formula for c_n holds and consider c_{n+1}.

The contribution to F in ϕ_{i_t} through iteration is given by
$$\lambda x_{i_t}.$$
Hence,
$$c_{n+1} = \lambda(\lambda^n) = \lambda^{n+1},$$
confirming the formula.

(2) For the base case of $n = 1$, we have $j = 0$. We verify
$$c_1 = \left(\lambda^0 \prod_{\substack{x_i | F \\ i \notin \{i_1, \ldots, i_{\mathfrak{v}}\}}} \lambda_i^0 \right) c_1 = c_1.$$
We now assume that the formula holds for c_n and consider c_{n+1}.

The contribution to F in ϕ_{i_t} through iteration is given by
$$\lambda x_{i_t} + c_1 F,$$
and, hence,
$$c_{n+1} = \lambda \left(\sum_{j=0}^{n-1} \lambda^{j(\alpha-1)+n-1} \prod_{\substack{x_i | F \\ i \notin \{i_1, \ldots, i_{\mathfrak{v}}\}}} \lambda_i^{e_i j} \right) c_1 + c_1 (\lambda^n)^{\alpha} \prod_{\substack{x_i | F \\ i \notin \{i_1, \ldots, i_{\mathfrak{v}}\}}} (\lambda_i^n)^{e_i}, \tag{2.1}$$

where the first term comes from substituting $c_{n-1}F$ for x_{i_t} into λx_{i_t} and the second from substituting $\lambda^n x_{i_1}, \ldots, \lambda^n x_{i_{\mathfrak{v}}}$ for each $x_i \mid F$ into F. Combining the sum and the term of (2.1) we have

$$c_{n+1} = \left(\sum_{j=0}^{n} \lambda^{j(\alpha-1)+n} \prod_{\substack{x_i \mid F \\ i \notin \{i_1,\ldots,i_{\mathfrak{v}}\}}} \lambda_i^{e_i j} \right) c_1 \tag{2.2}$$

confirming the formula. □

Remark. If $F \in \operatorname{supp}(\phi_i)$ with $\lambda_i = 0$ then we know that F does not effect $LT(I_1)$ since x_i either divides $LT(f)$ or is relatively prime to $LT(f)$ for all $f \in I_1$. In the former, case we take the normal form of f with respect to the known leading terms. In the latter case, we see that every term in the local analogue of the S-polynomials is divisible by the known leading terms and hence is already in the leading term ideal.

If $F \in \operatorname{supp}(\phi_j)$ with $x_i \mid F$ and $\lambda_i = 0$ then we know that F does not effect $LT(I_1)$ since $x_i \in LT(I_n)$ for all n. So we exclude from consideration the Jordan block(s) with eigenvalue 0 and monomials divisible by x_i with $\lambda_i = 0$.

Lemma 2.16. *Let $F = \prod_{i=1}^{b} x_i^{e_i}$ with $e_i \geq 0$, $\deg F > 1$ and*

$$F \in \operatorname{supp}(\phi_{i_t} - x_{i_t})$$

with $1 \leq i_t \leq b$. Assume that ϕ_{i_t} is in a Jordan block of $d\phi_P$ of size $\mathfrak{v} \geq 1$ with eigenvalue $\lambda \neq 0$. Label the rows $1, \ldots, \mathfrak{v}$ corresponding to $\phi_{i_1}, \ldots, \phi_{i_{\mathfrak{v}}}$. Let $\alpha = \sum_{j=1}^{\mathfrak{v}} e_{i_j}$. The following are conditions for the coefficient of F in $(\phi_{i_t}^n - x_{i_t})$ to be divisible by p.

1. *If $\lambda = 1$ and $\lambda_i = 1$ for all i such that $x_i \mid F$ and $i \notin \{i_1, \ldots, i_{\mathfrak{v}}\}$, then $p \mid n$.*
2. *Assume $\lambda \neq 1$ and $\lambda_i = 1$ for all i such that $x_i \mid F$ and $i \notin \{i_1, \ldots, i_{\mathfrak{v}}\}$.*
   (a) *If $\alpha = 0$, then λ is an r^{th} root of unity for some $r \mid n$.*
   (b) *If $\alpha > 0$, then $\lambda^{\alpha-1}$ is an r^{th} root of unity for some $r \mid n$.*
3. *If $\lambda = 1$ and $\lambda_i \neq 1$ for at least one i such that $x_i \mid F$, then*
$$\prod_{\substack{x_i \mid F \\ i \notin \{i_1,\ldots,i_{\mathfrak{v}}\}}} \lambda_i^{e_i}$$
is an r^{th} root of unity for some $r \mid n$.
4. *If $\lambda \neq 1$ and $\lambda_i \neq 1$ for at least one i such that $x_i \mid F$ and $i \notin \{i_1, \ldots, i_{\mathfrak{v}}\}$, then*
$$\lambda^{\alpha-1} \prod_{\substack{x_i \mid F \\ i \notin \{i_1,\ldots,i_{\mathfrak{v}}\}}} \lambda_i^{e_i}$$

is an r^{th} root of unity with $r \mid n$.

Proof. We will use the description of the coefficients of F under iteration from Lemma 2.15. Label the coefficient of F in $\phi_{i_t}^n$ as c_n with $c_1 \neq 0$.

If $\deg F = 1$, then $F = x_{i_t}$ and $c_1 = \lambda$ and

$$c_n = \lambda^n c_1.$$

Since $p \nmid \lambda$ this coefficient is never divisible by p. So we restrict to the case $\deg F > 1$.

(1) We want c_n to be divisible by p and c_n is given by

$$c_n = \left(\sum_{j=0}^{n-1} 1 \right) c_1 = n c_1$$

with $p \nmid c_1$. Hence, we must have $p \mid n$.

(2a) We want c_n to be divisible by p and c_n is given by

$$c_n = \left(\sum_{j=0}^{n-1} \lambda^j \right) c_1$$

with $p \nmid c_1$. Hence, we must have

$$\lambda^n \equiv 1 \mod p$$

and so λ is an r^{th} root of unity modulo p for some $r \mid n$.

(2b) We want c_n to be divisible by p and c_n is given by

$$c_n = \left(\sum_{j=0}^{n-1} \lambda^{j(\alpha-1)+n-1} \right) c_1$$

with $p \nmid c_1$. Hence, $\lambda^{\alpha-1}$ must be an r^{th} root of unity modulo p for some $r \mid n$.

(3) We want c_n to be divisible by p and c_n is given by

$$c_n = \left(\sum_{j=0}^{n-1} \prod_{\substack{x_i \mid F \\ i \notin \{i_1,\ldots,i_{\mathfrak{v}}\}}} \lambda_i^{e_i j} \right) c_1$$

with $p \nmid c_1$. Hence,

$$\prod_{\substack{x_i \mid F \\ i \notin \{i_1,\ldots,i_{\mathfrak{v}}\}}} \lambda_i^{e_i}$$

must be an r^{th} root of unity modulo p for some $r \mid n$.

(4) We want c_n to be divisible by p and c_n is given by

$$c_n = \left(\sum_{j=0}^{n-1} \lambda^{j(\alpha-1)+n-1} \prod_{\substack{x_i \mid F \\ i \notin \{i_1,\ldots,i_{\mathfrak{v}}\}}} \lambda_i^{e_i j} \right) c_1.$$

Hence,

$$\lambda^{\alpha-1} \prod_{\substack{x_i \mid F \\ i \notin \{i_1,\ldots,i_{\mathfrak{v}}\}}} \lambda_i^{e_i}$$

must be an r^{th} root of unity modulo p for some $r \mid n$. □

We have now established necessary conditions on n for a least monomial in $\operatorname{supp}(\phi_i - x_i)$ to not be in $\operatorname{supp}(\phi_i^n - x_i)$.

Lemma 2.17. *Assume that ϕ^n is nondegenerate. If $LT(I_1) \neq LT(I_n)$, then for some i with $1 \leq i \leq b$ there is a least monomial in* $\operatorname{supp}(\phi_i - x_i)$ *which is not in* $\operatorname{supp}(\phi_i^n - x_i)$.

Proof. Assume first that $\operatorname{supp}(\phi_i - x_i) \subseteq \operatorname{supp}(\phi_i^n - x_i)$ for each $1 \leq i \leq b$ and that $LT(I_1) \neq LT(I_n)$. In particular, there exists at least one monomial $H \in LT(I_1)$ such that $H \notin LT(I_n)$. We will establish a contradiction by showing that $H \in LT(I_{mn})$ for some $m \in \mathbb{N}$.

Since $\operatorname{supp}(\phi_i - x_i) \subseteq \operatorname{supp}(\phi_i^n - x_i)$ we have $p \nmid n$ and $r_i \nmid n$ for each i such that $r_i \neq 1$. To show that $H \in LT(I_{mn})$ for some m, we will modify the combination of the $(\phi_i - x_i)$ which has leading term H to produce a combination of the $(\phi_i^{mn} - x_i)$ with leading term H; this can be thought of as modifying the finite sequence of the local analogue of S-polynomials for I_1 which results in a leading term of H. Consider the combination of the $(\phi_i - x_i)$ that results in a leading term of H. This is some polynomial combination of the coefficients of monomials in $\operatorname{supp}(\phi_i - x_i)$ for $1 \leq i \leq b$. For m large enough the monomials up to some finite degree in $\operatorname{supp}(\phi_i^{mn})$ are in $\operatorname{supp}(\phi_i^{(m+1)n})$ for each $1 \leq i \leq b$. The combination of the $(\phi_i - x_i)$ which had leading term H, may or may not result in an element with leading term H as a combination of the $(\phi_i^{mn} - x_i)$. If not and the coefficient of H is nonzero in the resulting element, then some monomial G of degree at most $\deg(H)$ is the leading term. However, since $LT(I_{mn}) \subset LT(I_1)$ and $\deg(G) \leq \deg(H)$, G must also be the leading term of some other combination of the $(\phi_i - x_i)$, consider this combination for the $(\phi_i^{mn} - x_i)$. The resulting element may or may not have G as leading term. If it does not, there is some monomial G' as the leading term with $\deg(G') \leq \deg(G)$. The monomial G' is an element of $LT(I_{mn}) \subset LT(I_1)$ and we repeat the process. Since the degrees of these leading terms are bounded above, continuing to examine the leading monomial for each new combination we will eventually arrive at a combination of the $(\phi_i^{mn} - x_i)$ with a leading term already encountered. We can then back substitute to either get H as the leading term of a combination of the $(\phi_i^{mn} - x_i)$ or the coefficient of H in the combination is 0. In particular, we have a polynomial combination in the coefficients of monomials in $\operatorname{supp}(\phi_i^{mn} - x_i)$ that determines the coefficient of H as the leading term of the S-polynomial. The polynomial combination is not identically 0 since $H \in LT(I_1)$, so we have some m so that $H \in LT(I_{mn})$ which contradicts the fact that $H \notin LT(I_n)$ and $LT(I_{mn}) \subset LT(I_n)$.

Now assume that each monomial $H \in \operatorname{supp}(\phi_i - x_i)$ for some $1 \leq i \leq b$ which satisfies $H \notin \operatorname{supp}(\phi_i^n - x_i)$ is not a least monomial for ϕ_i. The coefficient of H in $\operatorname{supp}(\phi_i^{mn} - x_i)$ is a polynomial F_H in the coefficients of monomials in $\operatorname{supp}(\phi_j^{mn-1} - x_j)$ for all $1 \leq j \leq b$. For m large enough, the monomials up to some finite degree in $\operatorname{supp}(\phi_i^{mn})$ are in $\operatorname{supp}(\phi_i^{(m+1)n})$ for each $1 \leq i \leq b$ and, hence, F_H is the same for every m large enough. The polynomial F_H is not identically 0 since $H \in \operatorname{supp}(\phi_i - x_i)$. This is true for each such H, and there are only finitely many H that can affect $LT(I_n)$. Hence, there is some m, such that $\operatorname{supp}(\phi_i) \subseteq \operatorname{supp}(\phi_i^{mn})$ for each $1 \leq i \leq b$, which was treated in the previous case. □

In particular, Lemma 2.17 says that if we have $1 \leq a_P(1) < a_P(n)$, then for some i we must have a least monomial in $\operatorname{supp}(\phi_i - x_i)$ not in $\operatorname{supp}(\phi_i^n - x_i)$. However, this condition is not sufficient for $a_P(n) \neq a_P(1)$. Regardless, the necessary conditions on n from Lemma 2.16 will be enough to show that $\Phi_n^*(\phi)$ is an effective algebraic zero-cycle for all $n \geq 1$.

The next proposition gathers our knowledge of $a_P(n)$.

Proposition 2.18. *Let $X \subset \mathbb{P}_K^N$ be a nonsingular, irreducible, projective variety of dimension b defined over K. Let $\phi : X \to X$ be a morphism defined over K and $P \in X(K)$ be a fixed point of ϕ. Denote $d\phi_P$ as the map induced by ϕ on the cotangent space of X at P. Let $\lambda_1, \ldots, \lambda_l$ be the distinct eigenvalues of $d\phi_P$ with minimal multiplicative orders $r_1, \ldots, r_l$ (set $r_i = \infty$ if λ_i is not a root of unity). Then for all $n \geq 1$ such that ϕ^n is nondegenerate:*

1. $a_P(n) \geq a_P(1)$.
2. $a_P(n) = 1 \Leftrightarrow \lambda_i^n \neq 1$ *for all* $1 \leq i \leq l$.
3. *If* $a_P(n) > a_P(1)$, *then at least one of the following is true.*
   - (a) $r_i \mid n$ *for at least one* i *for* $1 \leq i \leq l$ *with* $r_i \neq 1$.
   - (b) $\operatorname{char}(K) \mid n$, *for* $\operatorname{char}(K) \neq 0$.

Proof. (1) Lemma 2.11.

(2) It is clear that $a_P(n) = 1$ if and only if I_n generates the maximal ideal of $\widehat{R_P}$. This is true if and only if

$$\{x_1 - y_1, \ldots, x_b - y_b, \phi_1^n(\mathbf{x}) - y_1, \ldots, \phi_b^n(\mathbf{x}) - y_b\}$$

is a regular local system of parameters. Zariski and Samuel [28, Corollary 2 page 137] state that this occurs if and only if the power series

$$\{x_1 - y_1, \ldots, x_b - y_b, \phi_1^n(\mathbf{x}) - y_1, \ldots, \phi_b^n(\mathbf{x}) - y_b\}$$

contain independent linear terms. This is true if and only if $\lambda_i^n \neq 1$ for all $1 \leq i \leq l$.

(3) We know from Corollary 2.10 that $a_P(n) > a_P(1)$ if and only if certain monomials F has zero coefficients after iteration. Any such monomial must be a least monomial by Lemma 2.12 and Lemma 2.17. Lemma 2.16 gives necessary conditions on n for when a least monomial has zero coefficient after iteration. Note that cases (2b), (3), and (4) of Lemma 2.16 are cases where $a_P(n) = a_P(1)$ since $\lambda_i \neq 1$ for some $x_i \mid F$. Hence, the absence of this monomial has no effect on the leading term ideal. So we are concerned only with the conditions (1) and (2a) of Lemma 2.16. □

2.3. Proof of effectivity. We will consider several different maps over the course of the proof, so to avoid confusion we include the map in the notation as $a_P(\phi, n)$ and $a_P^*(\phi, n)$. We will also use properties of the Möbius function throughout the rest of the article so we recall them here and will refer to them as (M1), (M2), and (M3) in what follows. The Möbius function satisfies the following properties [11, Chapter 2 §2]:

(M1) For $\gcd(m,n) = 1$ we have $\mu(mn) = \mu(m)\mu(n)$.

(M2) $\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1. \end{cases}$

(M3) Möbius inversion: if $F(n) = \sum_{d|n} f(d)$ for all $n \geq 1$, then

$$f(n) = \sum_{d|n} \mu(n/d) F(d)$$

for all $n \geq 1$.

Lemma 2.19. *Let p be a prime in $\mathbb{Z}$ and let $n = Mp^e$ in $\mathbb{Z}^+$ with $e \geq 1$ and $p \nmid M$.*

(1) *If $e = 1$, then*

$$a_P^*(\phi, n) = a_P^*(\phi^p, M) - a_P^*(\phi, M).$$

(2) *If $e \geq 2$, then*

$$a_P^*(\phi, n) = a_P^*(\phi^{p^{e-1}}, Mp).$$

(3) *Let $n = qM$ where $\gcd(q, M) = 1$. Then*

$$a_P^*(\phi, n) = \sum_{d|q} \mu\left(\frac{q}{d}\right) a_P^*(\phi^d, M).$$

Proof. Computing, we get

$$a_P^*(\phi, n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(\phi, d)$$

$$= \sum_{pd|n} \mu\left(\frac{n}{pd}\right) a_P(\phi, pd) + \sum_{d|M} \mu\left(\frac{n}{d}\right) a_P(\phi, d)$$

$$= \sum_{d|Mp^{e-1}} \mu\left(\frac{Mp^{e-1}}{d}\right) a_P(\phi^p, d) + \sum_{d|M} \mu\left(\frac{Mp^e}{d}\right) a_P(\phi, d)$$

$$= a^*_P(\phi^p, Mp^{e-1}) + \sum_{d|M} \mu\left(\frac{Mp^e}{d}\right) a_P(\phi, d).$$

So we have

$$a^*_P(\phi, n) = a^*_P(\phi^p, Mp^{e-1}) + \sum_{d|M} \mu\left(\frac{Mp^e}{d}\right) a_P(\phi, d). \tag{2.3}$$

(1) Considering (2.3) with $e = 1$, we have

$$a^*_P(\phi, n) = a^*_P(\phi^p, M) + \sum_{d|M} \mu\left(\frac{Mp}{d}\right) a_P(\phi, d)$$

$$= a^*_P(\phi^p, M) + \sum_{d|M} \mu(p)\mu\left(\frac{M}{d}\right) a_P(\phi, d)$$

$$= a^*_P(\phi^p, M) - a^*_P(\phi, M),$$

where the middle equality comes from the fact that μ is multiplicative and $(p, M) = 1$, property (M1).

(2) Considering (2.3) with $e > 1$, we have

$$a^*_P(\phi, n) = a^*_P(\phi^p, Mp^{e-1}) + \sum_{d|M} \mu\left(\frac{Mp^e}{d}\right) a_P(\phi, d)$$

$$= a^*_P(\phi^p, Mp^{e-1}) + 0,$$

where the second equality comes from the fact that $\frac{Mp^e}{d}$ is not square-free for all $d \mid M$. Replacing ϕ by ϕ^p and n by n/p, we may repeat the argument to conclude that

$$a^*_P(\phi, n) = a^*_P(\phi^{p^{e-1}}, Mp).$$

(3) Using the multiplicativity of the Möbius function for relatively prime numbers (M1), we get

$$a^*_P(\phi, n) = \sum_{d|qM} \mu\left(\frac{qM}{d}\right) a_P(\phi, d)$$

$$= \sum_{d_1|q} \sum_{d_2|M} \mu\left(\frac{qM}{d_1 d_2}\right) a_P(\phi, d_1 d_2)$$

$$= \sum_{d_1|q} \sum_{d_2|M} \mu\left(\frac{q}{d_1}\right) \mu\left(\frac{M}{d_2}\right) a_P(\phi, d_1 d_2)$$

$$= \sum_{d_1|q} \mu\left(\frac{q}{d_1}\right) \sum_{d_2|M} \mu\left(\frac{M}{d_2}\right) a_P(\phi^{d_1}, d_2)$$
$$= \sum_{d_1|q} \mu\left(\frac{q}{d_1}\right) a_P^*(\phi^{d_1}, M). \qquad \square$$

The next lemma provides a formula for $a_P^*(n)$ when $n = \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})$ for some subset $\{r_{i_1}, \ldots, r_{i_k}\}$ of $\{r_1, \ldots, r_l\}$. We clearly need that each r_{i_t} is finite, in other words, that λ_{i_t} has finite order, and we will also assume that each $r_{i_t} \neq 1$.

Lemma 2.20. *Let P be a fixed point of ϕ. Let r_i be the minimal order of λ_i in K^* for all $1 \leq i \leq l$ (set $r_i = \infty$ if λ_i is not a root of unity). If $n = \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})$ for some subset of nontrivial finite orders $\{r_{i_1}, \ldots, r_{i_k}\} \subseteq \{r_1, \ldots, r_l\}$ with* $\operatorname{char} K \nmid n$ *and square-free with no other r_i dividing n, then we have*

$$\begin{aligned}
a_P^*(\phi, n) = \\
&\sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(1) \\
&+ \sum_{t=1}^{k} \sum_{d|\frac{n}{r_{i_t}}} \mu\left(\frac{n}{d r_{i_t}}\right) c_{i_t} \\
&+ \sum_{t1=1}^{k} \sum_{\substack{t2=1 \\ i_{t2} \neq i_{t1}}}^{k} \sum_{d|\frac{n}{\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}})}} \mu\left(\frac{n}{d \operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}})}\right) c_{i_{t1}, i_{t2}} \\
&\vdots \\
&+ \sum_{t=1}^{k} \sum_{d|\frac{n}{\operatorname{lcm}(r_{i_1}, \ldots, \widehat{r_{i_t}}, \ldots, r_{i_k})}} \mu\left(\frac{n}{d \operatorname{lcm}(r_{i_1}, \ldots, \widehat{r_{i_t}}, \ldots, r_{i_k})}\right) c_{i_1, \ldots, \widehat{c_{i_t}}, \ldots, i_k} \\
&+ \sum_{d|1} c_{i_1, \cdots, i_k}
\end{aligned}$$

for some nonnegative constants c_α.

Proof. Recall that $LT(I_n) \subseteq LT(I_1)$. In particular, we know that

$$\operatorname{Span}(X^v \mid X^v \notin LT(I_1)) \subseteq \operatorname{Span}(X^v \mid X^v \notin LT(I_n)).$$

From Proposition 2.18, we know that $a_P(r_{i_t}) > a_P(1)$ for each r_{i_t} since $r_{i_t} \neq 1$. Similarly for $i_{t1} \neq i_{t2}$, by replacing ϕ with $\phi^{r_{i_{t1}}}$, we have

$$\begin{aligned}
a_P(\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}})) &> a_P(r_{i_{t1}}) \text{ if } r_{i_{t2}} \nmid r_{i_{t1}} \\
a_P(\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}})) &= a_P(r_{i_{t1}}) \text{ if } r_{i_{t2}} \mid r_{i_{t1}}
\end{aligned}$$

since in the second case $\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}}) = r_{i_{t1}}$. Continuing in the same manner, we have

$$\begin{aligned} &a_P(\operatorname{lcm}(r_{i_1}, \ldots, r_{i_j}, r_{i_\gamma})) > a_P(\operatorname{lcm}(r_{i_1}, \ldots, r_{i_j})) \quad \text{if } r_{i_\gamma} \nmid \operatorname{lcm}(r_{i_1}, \ldots, r_{i_j}) \\ &a_P(\operatorname{lcm}(r_{i_1}, \ldots, r_{i_j}, r_{i_\gamma})) = a_P(\operatorname{lcm}(r_{i_1}, \ldots, r_{i_j})) \quad \text{if } r_{i_\gamma} \mid \operatorname{lcm}(r_{i_1}, \ldots, r_{i_j}, r_{i_\gamma}). \end{aligned}$$

Again in the second case, we have $\operatorname{lcm}(r_{i_1}, \ldots, r_{i_j}, r_{i_\gamma}) = \operatorname{lcm}(r_{i_1}, \ldots, r_{i_j})$, so we have left to consider the first case. In particular, for any β defined as the least common multiple of any j of $\{r_{i_1}, \ldots, r_{i_j}, r_{i_\gamma}\}$, we have

$$\{X^v : X^v \notin LT(I_{\operatorname{lcm}(r_{i_1}, \ldots, r_{i_j}, r_{i_\gamma})})\}$$

containing at least one element not in $\{X^v : X^v \notin LT(I_\beta)\}$. To see this, consider the ordering

$$x_{i_1} < x_{i_2} < \cdots < x_{i_j} < x_\gamma < x_{i_t} < \cdots < x_{i_{b-j-1}}.$$

Then for each β, one of the linear terms $x_{i_1}, \ldots, x_{i_j}, x_{i_\gamma}$ is contained in $LT(I_\beta)$ since it is a leading term of the associated $\phi_i^\beta(x_1, \ldots, x_b) - x_i$. Also, none of the linear terms $x_{i_1}, \ldots, x_{i_j}, x_{i_\gamma}$ are in $LT(I_{\operatorname{lcm}(r_{i_1}, \ldots, r_{i_j}, r_{i_\gamma})})$. Hence, the monomial

$$x_{i_1} x_{i_2} \cdots x_{i_j} x_\gamma$$

is in $\{X^v : X^v \notin LT(I_{\operatorname{lcm}(r_{i_1}, \ldots, r_{i_j}, r_{i_\gamma})})\}$ but not in any $\{X^v : X^v \notin LT(I_\beta)\}$. This argument ensures the nonnegativity of the constants c_α defined below.

We have $a_P(1) \geq 1$ since P is a fixed point and since

$$\{X^v : X^v \notin LT(I_1)\} \subseteq \{X^v : X^v \notin LT(I_\kappa)\}$$

for all $\kappa \geq 1$, we have a contribution of $a_P(1)$ to $a_P(d)$ for all $d \mid n$.

Let $c_{i_t} = a_P(r_{i_t}) - a_P(1) > 0$ for all $1 \leq t \leq k$ since $r_{i_t} \neq 1$ by assumption for all $1 \leq t \leq k$. Since

$$\{X^v : X^v \notin LT(I_{r_{i_t}})\} \subseteq \{X^v : X^v \notin LT(I_\kappa)\}$$

for all κ with $r_{i_t} \mid \kappa$, we have a contribution of c_{i_t} to $a_P(d)$ for all $d \mid \frac{n}{r_{i_t}}$.

Let

$$c_{i_{t1}, i_{t2}} = a_P(\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}})) - \#\{X^v : X^v \notin LT(I_{r_{i_{t1}}})\} \cup \{X^v : X^v \notin LT(I_{r_{i_{t2}}})\}.$$

If $r_{i_{t2}} \mid r_{i_{t1}}$, then $c_{i_{t1}, i_{t2}} = 0$ since $\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}}) = r_{i_{t1}}$. Otherwise, by the argument at the beginning of the proof, there is at least one monomial not in $LT(I_{r_{t1}, r_{t2}})$ that is not in the complement of $LT(I_{r_{t1}})$ or $LT(I_{r_{t2}})$. Hence $c_{i_{t1}, i_{t2}} \geq 0$. Since

$$\{X^v : X^v \notin LT(I_{\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}})})\} \subseteq \{X^v : X^v \notin LT(I_\kappa)\}$$

for all κ with $\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}}) \mid \kappa$, we have a contribution of $c_{i_{t1}, i_{t2}}$ to $a_P(d)$ for all $d \mid \frac{n}{\operatorname{lcm}(r_{i_{t1}}, r_{i_{t2}})}$.

Similarly, for $2 \leq j \leq k$, let β be the least common multiple of j elements of $\{r_{i_{t1}}, \ldots, r_{i_{tj}}, r_{i_\gamma}\}$ and let

$$c_{i_{t1},\ldots,i_{tj},i_\gamma} = a_P(\mathrm{lcm}(r_{i_{t1}}, \ldots, r_{i_{tj}}, r_{i_\gamma})) - \#\left(\bigcup_{\beta}\{X^v : X^v \notin LT(I_\beta)\}\right).$$

If $r_{i_\gamma} \mid \mathrm{lcm}(r_{i_{t1}}, \ldots, r_{i_{tj}})$, then $c_{i_{t1},\ldots,i_{tj},i_\gamma} = 0$ since $\mathrm{lcm}(r_{i_{t1}}, \ldots, r_{i_{tj}}, r_{i_\gamma}) = \mathrm{lcm}(r_{i_{t1}}, \ldots, r_{i_{tj}})$. Otherwise, by the argument at the beginning of the proof, there is at least one monomial not in $LT(I_{r_{i_{t1}},\ldots,r_{i_{tj}},r_{i_\gamma}})$ that is not in the complement of $LT(I_\beta)$ for each β. Hence $c_{i_{t1},\ldots,i_{tj},i_\gamma} \geq 0$. Since

$$\{X^v : X^v \notin LT(I_{\mathrm{lcm}(r_{i_{t1}},\ldots,r_{i_{tj}},r_{i_\gamma})})\} \subseteq \{X^v : X^v \notin LT(I_\kappa)\}$$

for all κ with $\mathrm{lcm}(r_{i_{t1}}, \ldots, r_{i_{tj}}, r_{i_\gamma}) \mid \kappa$, we have a contribution of $c_{i_{t1},\ldots,i_{tj},i_\gamma}$ to $a_P(d)$ for all $d \mid \frac{n}{\mathrm{lcm}(r_{i_{t1}},\ldots,r_{i_{tj}},r_{i_\gamma})}$.

Notice that by construction, none of the monomials in $\{X^v : X^v \notin LT(I_n)\}$ are counted in multiple constants c_α, and all of them have been counted. Hence, the formula holds. $\square$

Remark. Notice that Lemma 2.20 implies that $a_P^*(n) \geq 0$ for all $n = \mathrm{lcm}(r_{i_1}, \ldots, r_{i_k})$ since each line is either 0 or c_α by property (M2) of the Möbius function and the constants c_α are all nonnegative. We may assume n is square-free by Lemma 2.19(2).

We are now ready to prove Theorem 1.3.

Proof. Fix a point $P \in X$ and let $n \geq 1$ be an integer such that ϕ^n is nondegenerate. By definition, we have

$$a_P^*(\phi, n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) a_P(\phi, d).$$

Suppose that $\phi^n(P) \neq P$. Then $\phi^d(P) \neq P$ for all $d \mid n$, so $a_P(\phi, d) = 0$ for all $d \mid n$ since the graph Γ_d of ϕ^d and the diagonal Δ will not intersect at (P, P). Hence, $a_P^*(\phi, n) = 0$, proving the theorem in this situation. We now assume that $\phi^n(P) = P$.

It follows that P is a periodic point for ϕ, so m is finite with $m \mid n$ and $a_P(\phi, d) \geq 1$ if and only if $m \mid d$. Computing $a_P^*(\phi, n)$ in terms of ϕ^m, we see that

$$\begin{aligned} a_P^*(\phi, n) &= \sum_{d \mid n \text{ with } m \mid d} \mu\left(\frac{n}{d}\right) a_P(\phi, d) \\ &= \sum_{d \mid (n/m)} \mu\left(\frac{n}{md}\right) a_P(\phi, md) \\ &= \sum_{d \mid (n/m)} \mu\left(\frac{n/m}{d}\right) a_P(\phi^m, d) \end{aligned}$$

$$= a^*_P(\phi^m, n/m).$$

Therefore, we can replace ϕ by ϕ^m and n by n/m and assume that $m = 1$.

We will consider a number of cases, but first we recall from Proposition 2.18 that $a_P(\phi, 1) = 1$ if and only if $r_i \neq 1$ for all $1 \leq i \leq l$.

Case 1. $n = 1$, in other words $n = m$.

In this case, we have

$$a^*_P(\phi, n) = a_P(\phi, 1).$$

Since P is assumed to be fixed by ϕ,

$$a^*_P(\phi, n) = a_P(\phi, 1) \geq \begin{cases} 1 & \text{always} \\ 2 & \text{if } r_i = 1 \text{ for some } i. \end{cases}$$

Case 2. $n > 1$ and $a_P(\phi, n) = a_P(\phi, 1)$.

Let $d \mid n$; then Proposition 2.18 states that

$$a_P(\phi, 1) = a_P(\phi, n) = a_P(\phi^d, n/d) \geq a_P(\phi^d, 1) = a_P(\phi, d) \geq a_P(\phi, 1).$$

Hence, $a_P(\phi, d) = a_P(\phi, 1)$ for all $d \mid n$. So

$$a^*_P(\phi, n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) a_P(\phi, 1) = 0$$

by property (M2) of the Möbius function, since $n > 1$ by assumption.

Case 3. $a_P(\phi, n) > a_P(\phi, 1)$ and $\operatorname{char} K \nmid n$.

By the assumptions in this case, we know that at least one $r_i \mid n$. Let $n = \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})M$ where M is not divisible by any r_i. Then we have

$$a^*_P(\phi, n) = \sum_{d \mid M} a^*_P(\phi^d, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k}))$$

by Lemma 2.19(3). However, since $r_i \nmid M$ for all $1 \leq i \leq l$, for $d \mid M$, we also have $r_i \nmid d$ for all $1 \leq i \leq l$. Additionally, $n \neq 0$ implies $p \nmid n$, so we cannot be in any condition of Proposition 2.18(3). Consequently,

$$\begin{aligned} a^*_P(\phi, n) &= \sum_{d \mid M} \mu\left(\frac{M}{d}\right) a^*_P(\phi^d, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})) \\ &= \sum_{d \mid M} \mu\left(\frac{M}{d}\right) a^*_P(\phi, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})) \\ &= 0 \end{aligned}$$

by property (M2) since $a^*_P(\phi^d, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k}))$ is constant over $d \mid M$. So we can assume that $n = \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})$ and $r_i \nmid n$ for $i \notin \{i_1, \ldots, i_k\}$. If

n is not square-free, then by applying Lemma 2.19 to any prime factor $q_j^{e_j}$ with $e_j > 1$, we get

$$a_P^*(\phi, n) = a_P^*\left(\phi^{q_1^{e_1-1}\cdots q_j^{e_j-1}}, \frac{n}{q_1^{e_1-1}\cdots q_j^{e_j-1}}\right).$$

So we may replace n by $\frac{n}{q_1^{e_1-1}\cdots q_j^{e_j-1}}$ and ϕ by $\phi^{q_1^{e_1-1}\cdots q_j^{e_j-1}}$ and assume that n is square-free. We are now in the case of Lemma 2.20 and use the same notation. Since every inner sum is either 0 or c_α by property (M2) of the Möbius function, we have that $a_P^*(\phi, n) \geq 0$ because every c_α is nonnegative. By assumption, at least one r_i divides n, so we know that c_n will be positive since it will have at least one additional monomial. Additionally, the sum associated to c_n will be c_n by property (M2) since it is summing over the divisors of 1. So we have shown that

$$\begin{cases} a_P^*(\phi, n) \geq 1 & \text{if M=1} \\ a_P^*(\phi, n) = 0 & \text{otherwise.} \end{cases}$$

Case 4. $a_P(\phi, n) > a_P(\phi, 1)$ and $\operatorname{char} K \nmid n$.

We can write $n = \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p^e M$ with $(r_{i_t}, M) = 1 = (p, M)$ by Proposition 2.18. If $M > 1$, then

$$a_P^*(\phi, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p^e) = a_P^*(\phi, d\operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p^e)$$

for all $d \mid M$ since M is not in one of the forms of Proposition 2.18(3). So

$$\begin{aligned} a_P^*(\phi, n) &= \sum_{d \mid M} \mu\left(\frac{n}{d}\right) a_P(\phi^d, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p^e) \\ &= \sum_{d \mid M} \mu\left(\frac{n}{d}\right) a_P(\phi, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p^e) \\ &= 0, \end{aligned}$$

by property (M2) where the first equality is from Lemma 2.19(3). So assume $M = 1$. Computing, we have

$$\begin{aligned} &a_P^*(\phi, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p^e) \\ &= a_P^*(\phi^{p^{e-1}}, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p) \\ &= a_P^*(\phi^{p^{e-1}}, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p) - a_P^*(\phi^{p^{e-1}}, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})) \\ &= a_P^*(\phi^{p^e}, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})) - a_P^*(\phi^{p^{e-1}}, \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})). \end{aligned}$$

Considering the maps ϕ^{p^e} and $\phi^{p^{e-1}}$, we have $\operatorname{char} K \nmid \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})$. As in Case 3, we may assume that $\operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})$ is square-free and use Lemma 2.20 to write $a_P^*(\operatorname{lcm}(r_{i_1}, \ldots, r_{i_k}))$ in terms of the nonnegative constants c_α. Since we are working with constants c_α for different maps, we include the map in the notation as $c_\alpha(\phi^{p^e})$. The constants that contribute to

$a^*_P(\mathrm{lcm}(r_{i_1},\ldots,r_{i_k}))$ are associated to $\alpha = \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})$ since the Möbius sum is not identically 0 in that case by property (M2). So if

$$a_P(\phi, \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})p^e) = a_P(\phi, \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})p^{e-1}),$$

then $c_\alpha(\phi^{p^e}) = c_\alpha(\phi^{p^{e-1}})$. If we get additional least monomials with zero coefficient after iteration, in other words,

$$a_P(\phi, \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})p^e) > a_P(\phi, \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})p^{e-1}),$$

then $c_\alpha(\phi^{p^e}) > c_\alpha(\phi^{p^{e-1}})$. Hence,

$$\begin{cases} a^*_P(\phi,n) = 0 & \text{if } a_P(\phi, \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})p^e) = a_P(\phi, \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})p^{e-1}) \\ a^*_P(\phi,n) > 0 & \text{if } a_P(\phi, \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})p^e) > a_P(\phi, \mathrm{lcm}(r_{i_1},\ldots,r_{i_k})p^{e-1}). \end{cases}$$

Hence, $a^*_P(\phi,n) \geq 0$ always; and if $a^*_P(\phi,n) > 0$, then n is in one of the stated forms. □

Remark. If $\mathrm{char}\, K = 0$, then in Theorem 1.3 we have $a^*_P(n) \geq 1$ *if and only if* $n = m$ or $n = m\,\mathrm{lcm}(r_{i_1},\ldots,r_{i_k})$ since we know precisely the conditions for $a_P(n) > a_P(1)$.

Note that Morton and Silverman [19, Corollary 3.3] show that for $\dim X = b = 1$, if $n_1 \nmid n_2$ and $n_2 \nmid n_1$, then $\Phi^*_{n_1}(\phi)$ and $\Phi^*_{n_2}(\phi)$ have disjoint support. They use this fact to construct units in K called dynatomic units similar to the construction of cyclotomic and elliptical units. In the general case, the nondivisibility condition may not imply disjoint supports because there are more possible forms of n. In particular, $n_1 = mr_1$ and $n_2 = mr_2$ could satisfy the divisibility condition, but $\Phi^*_{n_1}(\phi)$ and $\Phi^*_{n_2}(\phi)$ do not have disjoint support.

3. Periodic points of formal period n with multiplicity one have minimal period n.

In this section we use the detailed description of the multiplicities from Section 2 to show that periodic points of formal period n with $\mathrm{char}\, K \nmid n$ and multiplicity one have minimal period n, generalizing [17, Theorem 2.5].

Theorem 3.1. *If P is a periodic point of minimal period m for ϕ, then $a^*_P(n) \geq 2$ for all integers $n > m$ with* $\mathrm{char}\, K \nmid n$ *and $a^*_P(n) \neq 0$.*

Proof. Let $n > m \geq 1$ be any integer for which $a^*_P(n) \neq 0$. Since

$$a^*_P(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(d)$$

and $a_P(d) \neq 0$ only for $m \mid d$, we must have m divides n. Computing $a^*_P(\phi,n)$ in terms of ϕ^m, we know that

$$a^*_P(\phi,n) = a^*_P(\phi^m, n/m).$$

Hence, we may replace ϕ by ϕ^m and n by n/m and assume that P is a fixed point. From Theorem 1.3 we know that for $a^*_P(n) \neq 0$ and $\operatorname{char} K \nmid n$ we have that n is of the form

$$n = \operatorname{lcm}(r_{i_1}, \dots, r_{i_k}).$$

Case 1. $n = r_i$ for some $1 \leq i \leq b$ (in other words, $k = 1$).

If λ_i is in a Jordan block of $d\phi_P$ of size > 1 then consider as i the first row of the Jordan block. Let β be the size of the Jordan block. In other words $x_i, \dots, x_{i+\beta}$ are the rows of the Jordan block. Let

$$\delta = \begin{cases} i & \beta = 1 \\ i+\beta & \beta > 1. \end{cases}$$

Case 1.1. $\lambda_j \neq 1$ and $\lambda_j^n \neq 1$ for all $j \neq i$.

We have $a_P(1) = 1$ and need to compute $a_P(n)$. We know

$$\begin{cases} x_j \in \operatorname{supp}(\phi_j^n(\mathbf{x}) - x_j) & j \neq i, \beta = 1 \\ x_{j+1} \in \operatorname{supp}(\phi_j^n(\mathbf{x}) - x_j) & i \leq j < i+\beta, \beta > 1 \\ x_j \in \operatorname{supp}(\phi_j^n(\mathbf{x}) - x_j) & j \notin \{i, \dots, i+\beta\}, \beta > 1 \end{cases}$$

and, using Lemma 2.15 for the description of the coefficients of a monomial after iteration, we know that

$$x_i^2 \notin \operatorname{supp}(\phi_\delta^n(\mathbf{x}) - x_\delta). \tag{3.1}$$

With the appropriate choice of admissible monomial ordering, we have x_j for $j \neq i$ is a leading term of one of the $\phi_k^n(\mathbf{x}) - x_k$ for $k \neq \delta$. Since all of these leading terms are relatively prime they are part of the generating set of a standard basis and we need only consider the monomial

$$x_i^e \in \operatorname{supp}(\phi_\delta^n(\mathbf{x}) - x_\delta).$$

From (3.1) we must have $e \geq 3$. So then we have

$$LT(I_n) \subset \{x_1, \dots, x_{i-1}, x_i^3, x_{i+1}, \dots, x_b\}.$$

By Lemma 2.20, we have added at least $\{x_i, x_i^2\}$ to the complement of the leading term ideal and so

$$a^*_P(n) \geq 2.$$

Case 1.2. $\lambda_j^n = 1$ for some $j \neq i$.

We have x_i in $LT(I_d)$ for any $d < n$ but not in $LT(I_n)$ and $x_j \notin LT(I_n)$. So we have added at least

$$\{x_i, x_i x_j\}$$

to the complement of $LT(I_n)$, and by Lemma 2.20 we have

$$a^*_P(n) \geq 2.$$

Case 2. $k > 1$.

We have that $n = \operatorname{lcm}(r_{i_1}, \dots, r_{i_k})$.

Case 2.1. $\lambda_j \neq 1$ for $j \notin \{i_1, \ldots, i_k\}$.

We have $a_P(1) = 1$. From Case 1.1 we know that $a_P(r_i) \geq 3$ for each $i \in \{i_1, \ldots, i_k\}$. Hence, we add at least

$$\{x_{i_1} \cdots x_{i_k}, x_{i_1}^2 \cdots x_{i_k}\}$$

to the complement of the $LT(I_n)$. So by Lemma 2.20 we have

$$a_P^*(n) \geq 2.$$

Case 2.2. $\lambda_j = 1$ for some $j \notin \{i_1, \ldots, i_k\}$.

We know $x_j \notin LT(I_1)$ and hence $x_j \notin LT(I_n)$. Additionally, $x_{i_1} \cdots x_{i_k} \in LT(I_h)$ for $h \mid n$ with $h < n$, but $x_{i_1} \cdots x_{i_k} \notin LT(I_n)$ since r_{i_t} divides n for each $1 \leq t \leq k$. Consequently, we add at least

$$\{x_{i_1} \cdots x_{i_k}, x_{i_1} \cdots x_{i_k} x_j\}$$

to the complement of $LT(I_n)$. So by Lemma 2.20 we have

$$a_P^*(n) \geq 2. \qquad \square$$

Example 3.2. Theorem 3.1 does not hold for $\operatorname{char} K \mid n$. In other words, we may have $a_P^*(n) = 1$, but P is a periodic point of minimal period strictly less than n if $\operatorname{char} K \mid n$. For example, consider $\operatorname{char} K = 3$, $\dim X = 2$, and $\phi : X \to X$ defined near a fixed point P as

$$\phi_1(x_1, x_2) = x_1 + x_1^2 + x_1 x_2$$
$$\phi_2(x_1, x_2) = 2x_2 + x_1^2.$$

Then with the monomial ordering $x_2 < x_1$, the leading term ideal is generated by $\{x_1^2, x_2\}$ and, hence, $a_P(1) = 2$. Iterating, we have

$$\phi_1^3(x_1, x_2) = x_1 + x_1^3 + x_1 x_2 + \text{higher order terms}$$
$$\phi_2^3(x_1, x_2) = 2x_2 + x_1^2 + \text{higher order terms}.$$

Then we have the leading term ideal is generated by $\{x_1^3, x_2\}$ and, hence, $a_P(3) = 3$. Then computing

$$a_P^*(3) = a_P(3) - a_P(1) = 1,$$

but P is a fixed point for ϕ.

4. Properties and consequences

Unless otherwise stated, we assume that X is a nonsingular, irreducible, projective variety of dimension b defined over K and that $\phi : X \to X$ is a morphism defined over K such that ϕ^n is nondegenerate.

4.1. Basic properties.

Proposition 4.1. *Let $m, n \geq 1$ be integers such that ϕ^{mn} is nondegenerate. Then*

(1) *If $a_P(n) > 0$, then $a_P(mn) > 0$ for all m.*
(2) *If P is a periodic point of minimal period n for ϕ, then $a^*_P(n) \neq 0$. In particular, points of minimal period n are points of formal period n.*
(3) *$a_P(n) = \sum_{d|n} a^*_P(d)$.*
(4) *If $a_P(n) > 0$, then for m the minimal period of P for ϕ we have $a^*_P(m) > 0$, for all $d < m$ we have $a^*_P(d) = 0$, and*

$$a^*_P(\phi, n) = a^*_P(\phi^m, n/m).$$

Proof. (1) The multiplicity $a_P(n) > 0$ implies that P is a periodic point of period n. In other words, $\phi^n(P) = P$ and, consequently, $\phi^{nm}(P) = P$. Therefore, P is a periodic point of period mn, so it has nonzero multiplicity in $\Phi_{mn}(\phi)$.

(2) Since $\phi^d(P) \neq P$ for all $d < n$, we have

$$a_P(d) = 0 \quad \text{for all } d < n.$$

So we have that

$$a^*_P(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(d) = a_P(n) \neq 0,$$

where the last inequality comes from the fact that P is a periodic point of period n.

(3) The definition of $\Phi^*_n(\phi)$ is

$$\Phi^*_n(\phi) = \sum_{P \in X} a^*_P(n)(P).$$

We also have

$$a^*_P(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(n).$$

We can apply Möbius inversion (M3) to get

$$a_P(n) = \sum_{d|n} a^*_P(d),$$

which gives the factorization as desired.

(4) The multiplicity $a_P(n) > 0$ implies that $\phi^n(P) = P$ and, hence, that P is a periodic point. Consequently, P has some minimal period $m \leq n$. By (2), m satisfies $a^*_P(m) > 0$. It is the minimal such value because for any $d < m$ we have that P is not a periodic point of period d and, hence,

$a_P(d) = 0$. So we have $a^*_P(d) = 0$ for $d < m$. Finally, computing $a^*_P(\phi, n)$ in terms of ϕ^m we have

$$\begin{aligned} a^*_P(\phi, n) &= \sum_{d|n \text{ with } m|d} \mu\left(\frac{n}{d}\right) a_P(\phi, d) \\ &= \sum_{d|(n/m)} \mu\left(\frac{n}{md}\right) a_P(\phi, md) \\ &= \sum_{d|n/m} \mu\left(\frac{B}{d}\right) a_P(\phi^m, d) \\ &= a^*_P(\phi^m, n/m). \end{aligned}$$

□

In the next proposition, we summarize some of the facts about $a^*_P(n)$ in terms of $\Phi^*_n(\phi)$.

Proposition 4.2. *Let $m, n \geq 1$ be integers with ϕ^{mn} nondegenerate.*

(1) *$a^*_P(\phi, mn) \geq a^*_P(\phi^m, n)$.*
(2) *If $(n, m) = 1$, then $\Phi^*_n(\phi^m) = \sum_{d|m} \Phi^*_{nd}(\phi)$.*
(3) *Let $m = p^e$ for some prime p and $e \geq 2$. Then $\Phi^*_{np^e}(\phi) = \Phi^*_{np}(\phi^{p^{e-1}})$.*
(4) *If $n = p_1^{e_1} \cdots p_r^{e_r}$ for distinct primes $p_1, \ldots, p_r$ with $e_1, \ldots, e_r \geq 2$ and $m = p_1^{e_1-1} \cdots p_r^{e_r-1}$, then $\Phi^*_n(\phi) = \Phi^*_{p_1 \cdots p_r}(\phi^m)$.*

Proof. (1) This is clear from Lemma 2.19.

(2) We need to see that

$$a^*_P(\phi^m, n) = \sum_{d|m} a^*_P(\phi, nd).$$

By the Möbius inversion formula (M3), this is equivalent to

$$a^*_P(\phi, mn) = \sum_{d|m} \mu\left(\frac{m}{d}\right) a^*_P(\phi^d, n).$$

Computing the right-hand side, we have

$$\begin{aligned} \sum_{d|m} \mu\left(\frac{m}{d}\right) a^*_P(\phi^d, n) &= \sum_{d|m} \mu\left(\frac{m}{d}\right) \sum_{d'|n} \mu\left(\frac{n}{d'}\right) a_P(\phi^d, d') \\ &= \sum_{d|m} \sum_{d'|n} \mu\left(\frac{m}{d}\right) \mu\left(\frac{n}{d'}\right) a_P(\phi, dd') \\ &= \sum_{d|m} \sum_{d'|n} \mu\left(\frac{nm}{dd'}\right) a_P(\phi, dd') \\ &= \sum_{d''|nm} \mu\left(\frac{nm}{d''}\right) a_P(\phi, d'') \\ &= a^*_P(\phi, nm). \end{aligned}$$

(3) This is Lemma 2.19(2).

(4) This is Lemma 2.19(2) applied to each p_i. ☐

Proposition 4.3. $\deg(\Phi_n^*(\phi)) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \deg(\Phi_d(\phi))$.

Proof. Computing:

$$\begin{aligned}\deg(\Phi_n^*(\phi)) &= \sum_{P\in X}\sum_{d|n} \mu\left(\frac{n}{d}\right) i(\Gamma_d, \Delta_X; P)\\ &= \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{P\in X} i(\Gamma_d, \Delta_X; P)\\ &= \sum_{d|n} \mu\left(\frac{n}{d}\right) \deg(\Phi_d(\phi)).\end{aligned}$$ ☐

4.2. Similarities to periodic Lefschetz numbers.

Proposition 4.3 looks remarkably similar to the definition of periodic Lefschetz numbers. In this section we describe the connection.

Definition 4.4. Following the notation of [8], let $\phi : M \to M$ be a continuous map on a complex manifold M. Define the *Lefschetz number* of ϕ as

$$L(\phi) = \sum_{x\in \mathrm{Fix}(\phi)} \mathrm{ind}(\phi, x),$$

where $\mathrm{Fix}(\phi) \subset M$ is the set of fixed points and $\mathrm{ind}(\phi, x)$ is the Poincaré index of ϕ at x. So $L(\phi)$ is the sum of the multiplicities of the fixed points of ϕ with either a negative or positive sign depending on whether $\phi - \mathrm{id}$ preserves or reverses orientation at x. We refer the reader to [8] for a more detailed definition of $L(\phi)$.

The *periodic Lefschetz number of period* n is then defined as

$$l(\phi^n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) L(\phi^d)$$

The Lefschetz Fixed Point Theorem states that $L(\phi^n) \neq 0$ implies that ϕ^n has a fixed point, in other words, ϕ has a point of period n, but this does not imply that the point is of minimal period n. The periodic Lefschetz numbers were defined to help address this situation. Several papers, including [8, 13], have studied when $l(\phi^n) \neq 0$ implies that there exists a periodic point of minimal period n. We will address the relationship between $\deg(\Phi_n)$, $\deg(\Phi_n^*)$, $L(\phi^n)$, $l(\phi^n)$, and the existence of period points.

Definition 4.5. A map ϕ is *transversal* if $a_P(1) = 1$ for fixed points P.

Definition 4.6. We define the *degree* of an algebraic zero-cycle to be the sum of multiplicities of the points.

Proposition 4.7. (1) $\deg \Phi_n(\phi) \geq L(\phi^n)$.

(2) *If ϕ^n is transversal, then:*

(a) $a^*_P(n) = 1$ *if and only if* P *is a point of minimal period* n *for* ϕ *and* $a^*_P(n) = 0$ *otherwise.*
(b) $\deg(\Phi_n(\phi))$ *is the number of periodic points of period* n *for* ϕ.
(c) $\deg(\Phi^*_n(\phi))$ *is the number of periodic points of minimal period* n *for* ϕ. *In particular, if* $\deg(\Phi^*_n(\phi)) \neq 0$, *then there exists a periodic point of minimal period* n.

Proof. (1) Since $\deg(\Phi_n(\phi))$ is the sum of the multiplicities of the periodic points of period n for ϕ, in other words, the fixed points of ϕ^n and $L(\phi^n)$ is the sum of the same multiplicities with either a positive or negative sign, we have the desired inequality.

(2a) The map ϕ^n is transversal implies that ϕ^d is transversal for all $d \mid n$ and hence $a_P(d) = 1$ for all periodic points P of period $d \mid n$. Therefore, if the minimal period of P is n, then we have $a^*_P(n) = a_P(n) = 1$ since $a_P(d) = 0$ for $d < n$.

Assume that P is a periodic point of minimal period $m \mid n$ and compute

$$a^*_P(m) = a^*_P(\phi^m, 1) = a_P(\phi^m, 1) = 1.$$

Since $a^*_P(m) = a^*_P(\phi^m, 1)$, we may replace ϕ by ϕ^m and assume that $m = 1$. Now computing $a^*_P(n)$ we have

$$a^*_P(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(d) = \sum_{d|n} \mu\left(\frac{n}{d}\right) 1 = 0$$

by property (M2) of the Möbius function.

Properties (2b) and (2c) follow directly from the definition of transversal and (2a). □

Remark. Proposition 4.7(2) is similar to [8, Theorem A].

4.3. Applications. Before we state the results, we note that in the case of a polarized algebraic dynamical systems Fakhruddin [7] has shown that the periodic points are Zariski dense in X, which implies the existence of infinitely many periodic points. However, polarization is a stronger assumption than what is needed below. Although, both the dynamical systems on Wehler K3 surfaces and from morphisms of $\mathbb{P}^N$ are examples of polarized algebraic dynamical systems. Regardless, we include a discussion of these two situations because we are able to determine properties of $\deg(\Phi_n)$ which is of interest in and of itself; and for Wehler K3 surface we resolve an additional question of Silverman.

Proposition 4.8. *There are only finitely many points of minimal period* n *for any fixed* n *with* ϕ^n *nondegenerate.*

Proof. Fix any integer $n \geq 1$ with ϕ^n nondegenerate. Proposition 4.3 provides a formula for the degree of $\Phi^*_n(\phi)$. Since ϕ^n is assumed to be nondegenerate, Bézout's Theorem states that Γ_d and Δ intersect in a finite number of points for all $d \mid n$; in other words, $\deg(\Phi_d(\phi))$ is finite. Hence,

$\deg(\Phi_n^*(\phi))$ is finite, so there can only be finitely many periodic points of minimal period n. ☐

Theorem 4.9. *There exists $M > 0$ such that for all q prime, $q > M$, $\deg(\Phi_q^*(\phi)) \neq 0$, and ϕ^q nondegenerate, there exists a periodic point with minimal period q for ϕ.*

Proof. We want to show that there exists a P with $a_P^*(q) \neq 0$ that is a periodic point of minimal period q. We know that for q prime we have

$$\deg(\Phi_q^*(\phi)) = \deg(\Phi_q(\phi)) - \deg(\Phi_1(\phi)).$$

There are only finitely many fixed points for ϕ by Proposition 4.8, and for each fixed point only finitely many n relatively prime to $\operatorname{char} K$ such that $a_P(n) > a_P(1)$ by Theorem 1.3. Hence, after excluding those finitely many numbers (including $\operatorname{char} K$), each time $\deg(\Phi_q(\phi)) > \deg(\Phi_1(\phi))$ the additional degree comes from at least one periodic point of minimal period q. ☐

Corollary 4.10. *If there are infinitely many primes $q \neq \operatorname{char} K$ such that $\deg(\Phi_q^*(\phi)) \neq 0$ and ϕ^q is nondegenerate, then there exists $P \in X$ with an arbitrarily large minimal period for ϕ, and ϕ has infinitely many periodic points.*

Proof. By assumption, we have infinitely many primes q with $\deg(\Phi_q^*(\phi)) \neq 0$. Applying Theorem 4.9, we then have infinitely many primes q with a periodic point of minimal period q. ☐

Remark. Corollary 4.10 appears to be similar to applications of periodic Lefschetz numbers such as those in [6, 8].

Theorem 4.11. *If P is a fixed point of ϕ and ϕ^n is nondegenerate for all but finitely many $n \in \mathbb{N}$ with $\operatorname{char} K \nmid n$, then the sequence*

$$\{a_P(n)\}_{\substack{n\in\mathbb{N} \\ \operatorname{char} K \nmid n}}$$

is bounded.

Proof. From Theorem 1.3 we have that for a fixed point P for ϕ, $a_P(n) \neq a_P(1)$ for only finitely many n with $\operatorname{char} K \nmid n$. Hence the sequence must be bounded. ☐

Corollary 4.12. *If $\deg(\Phi_n(\phi))$ is unbounded for $\operatorname{char} K \nmid n$ and ϕ^n nondegenerate, then there are infinitely many periodic points for ϕ and, hence, periodic points with arbitrarily large minimal periods.*

Proof. Consider the prime numbers $q \in \mathbb{Z}$ with $q \neq \operatorname{char} K$. We know that $\deg(\Phi_q(\phi))$ is unbounded, and the only contributions come from fixed points or points of minimal period q. Since the sequence $\{a_P(q)\}$ is bounded for all fixed points P, there must be contributions to $\deg(\Phi_q(\phi))$ from periodic points of minimal period q for infinitely many primes q. ☐

Remark. Theorem 4.11 and Corollary 4.12 are similar to [25].

4.4. Wehler K3 surfaces. A Wehler K3 surface $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth surface given by the intersection of an effective divisor of degree (1,1) and an effective divisor of degree (2,2). Wehler [27, Theorem 2.9] shows that a general such surface has an infinite automorphism group. Briefly, the two projection maps

$$p_x : S \to \mathbb{P}^2_x \quad \text{and} \quad p_y : S \to \mathbb{P}^2_y$$

are in general double covers, allowing us to define two involutions of S, σ_x and σ_y, respectively. To define σ_x, we consider $p_x^{-1}(\mathbf{x}) = \{(\mathbf{x},\mathbf{y}),(\mathbf{x},\mathbf{y}')\}$ and define $\sigma_x((\mathbf{x},\mathbf{y})) = (\mathbf{x},\mathbf{y}')$. To define σ_y, we consider

$$p_y^{-1}(\mathbf{y}) = \{(\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y})\}$$

and define $\sigma_y((\mathbf{x},\mathbf{y})) = (\mathbf{x}',\mathbf{y})$. These two involutions do not commute, generating an infinite automorphism group. The associated dynamical systems are studied in [1, 2, 5, 23].

Theorem 4.13. *Dynamical systems on Wehler K3 surfaces have points with arbitrarily large minimal period and infinitely many periodic points. In particular, there exists a constant M such that for all primes $q > M$ there exists a periodic point of minimal period q.*

Proof. From [23, page 358] we know that the Lefschetz numbers of the maps $\phi^n = (\sigma_x \circ \sigma_y)^n$ are given by

$$L(\phi^n) = (2+\sqrt{3})^{2n} + (2+\sqrt{3})^{-2n} + 22.$$

So we have

$$L(\phi^n) \geq 2^{2n}. \tag{4.1}$$

By Proposition 4.7(1)

$$\deg(\Phi_n(\phi)) \geq L(\phi^n), \tag{4.2}$$

hence, we have that $\deg(\Phi_n(\phi))$ is unbounded as n increases. Applying Corollary 4.12, we have the result.

To show the second portion, recall that

$$\deg(\Phi_q^*(\phi)) = \deg(\Phi_q(\phi)) - \deg(\Phi_1(\phi)).$$

In other words, whenever $\deg(\Phi_q(\phi)) > \deg(\Phi_1(\phi))$ we have $\deg(\Phi_q^*(\phi)) \neq 0$. Combining (4.1) and (4.2), we have that for k larger than some constant C we have $\deg(\Phi_q^*(\phi)) \neq 0$. Applying Theorem 4.9 now gives the desired result. ☐

Definition 4.14. Let S be a Wehler K3 surface and let $\mathcal{A}$ be the subgroup of the automorphism group of S generated by σ_x and σ_y. Let $B_k \subset \mathcal{A}$ be the cyclic subgroup generated by $\phi^k = (\sigma_x \circ \sigma_y)^k$. Let $\mathcal{A}_P = \{\phi \in \mathcal{A} : \phi(P) = P\}$. For any subgroup $B \subset \mathcal{A}$, let $S[B] = \{P \in S(K) : \mathcal{A}_P = B\}$. Recall that we are assuming K is algebraically closed.

The following proposition addresses a remark of Silverman from [23, page 358].

Proposition 4.15. *$\#S[B_q] \to \infty$ as $q \to \infty$ for q prime.*

Proof. From Theorem 4.13 we have that there are periodic points of infinitely large prime minimal period and, in particular, periodic points of prime minimal period for all primes larger than some constant M. Hence, $S[B_q]$ will increase as q increases. $\square$

4.5. Morphisms of projective space. We also apply our results to morphisms of projective space. Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ given by $N+1$ homogeneous forms of degree d with no common zeros. We need to compute the intersection number for Δ and Γ_n, which are contained in $\mathbb{P}^N \times \mathbb{P}^N$.

Let D_1 and D_2 be the pullbacks in $\mathbb{P}^N \times \mathbb{P}^N$ of a hyperplane class D in $\mathbb{P}^N$ by the first and second projections, respectively.

Proposition 4.16. (1) *The class of Δ is given by*

$$\sum_{j=0}^{N} D_1^{N-j} D_2^j.$$

(2) *The class of Γ_n is given by*

$$\sum_{j=0}^{N} d^{N-j} D_1^{N-j} D_2^j.$$

Proof. (1) By the Künneth formula, the diagonal must be a class in

$$H_N(\mathbb{P}^N \times \mathbb{P}^N) = \sum_{j=0}^{N} H_{N-j}(\mathbb{P}^N) \otimes H_j(\mathbb{P}^N).$$

Now, $H_{N-j}(\mathbb{P}^N) \otimes H_j(\mathbb{P}^N)$ is a 1-dimensional space for all $0 \leq j \leq N$, spanned by the Poincaré dual of $D_1^{N-j} D_2^j$. We can write

$$\Delta = \sum_{j=0}^{N} a_j D_1^{N-j} D_2^j.$$

To determine the coefficient a_j, we should intersect Δ with the dual of $D_1^{N-j} D_2^j$. This is $D_1^j D_2^{N-j}$. So let $i_\Delta : \Delta \hookrightarrow \mathbb{P}^N \times \mathbb{P}^N$ and compute

$$(D_1^{N-j} D_2^j) \cdot (\Delta) = i_\Delta^*(D_1^{N-j} D_2^j) \cdot \mathbb{P}^N = D^N \cdot \mathbb{P}^N = 1$$

using the fact that $i_\Delta^*(D_1) = i_\Delta^*(D_2) = D$, a hyperplane class on $\mathbb{P}^N$.

(2) Again, by the Künneth formula, the graph must be a class in

$$H_N(\mathbb{P}^N \times \mathbb{P}^N) = \sum_{j=0}^{N} H_{N-j}(\mathbb{P}^N) \otimes H_j(\mathbb{P}^N),$$

and we can write

$$\Gamma_n = \sum_{j=0}^{N} a_j D_1^{N-j} D_2^j.$$

Let $i_{\Gamma_n} : \Gamma_n \hookrightarrow \mathbb{P}^N \times \mathbb{P}^N$. To determine the coefficients a_j, we compute

$$(D_1^{N-j} D_2^j) \cdot (\Gamma_n) = i_{\Gamma_n}^*(D_1^{N-j} D_2^j) \cdot \mathbb{P}^N = d^{N-j} D^N \cdot \mathbb{P}^N = d^{N-j}$$

using the facts that $i_{\Gamma_n}^*(D_1) = dD$ and $i_{\Gamma_n}^*(D_2) = D$. □

Proposition 4.17. *A morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ given by $N+1$ homogeneous forms of degree d has*

$$\deg(\Phi_n(\phi)) = \sum_{j=0}^{N} (d^n)^j.$$

Proof. We first compute the number of fixed points of ϕ. By Proposition 4.16, we compute the intersection number of Γ_1 and Δ.

$$\begin{aligned}(\Gamma_1) \cdot (\Delta) &= \left(\sum_{j=0}^{N} D_1^{N-j} D_2^j \right) \cdot \left(\sum_{k=0}^{N} d^{N-k} D_1^{N-k} D_2^k \right) \\ &= 0 + \sum_{j=0}^{N} d^{N-j} D_1^N D_2^N = \sum_{j=0}^{N} d^j.\end{aligned}$$

Since each fixed point has multiplicity at least 1, $\sum_{j=0}^{N} d^j$ is the maximum possible number of fixed points.

The points of period n are fixed points of ϕ^n and ϕ^n is given by $N+1$ homogeneous forms of degree d^n. Therefore,

$$\deg(\Phi_n(\phi)) = \sum_{j=0}^{N} (d^n)^j. \qquad \square$$

Theorem 4.18. *A morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ given by $N+1$ homogeneous forms of degree $d > 1$ has periodic points with arbitrarily large minimal periods and infinitely many periodic points. In particular, there exists a constant M such that for all primes $q > M$ there exist periodic points of minimal period q.*

Proof. The degree of $\Phi_n(\phi)$ is clearly unbounded from Proposition 4.17, so we apply Corollary 4.12 to conclude the first result.

To see the second result, notice that

$$\begin{aligned}\deg(\Phi_q^*(\phi)) &= \deg(\Phi_q(\phi)) - \deg(\Phi_1(\phi)) \\ &= ((d^q)^N + \cdots + (d^q) + 1) - (d^N + \cdots + d + 1) \\ &= ((d^q)^N - d^N) + \cdots (d^q - d) > 0.\end{aligned}$$

Hence, we apply Theorem 4.9 to conclude the result. □

References

[1] Baragar, Arthur. Orbits of points on certain K3 surfaces. To appear.

[2] Baragar, Arthur. Orbits of curves on certain K3 surfaces. *Comp. Math.* **137** (2003) 115–134. MR1985002 (2004d:11050), Zbl 1044.14014.

[3] Becker, Thomas. Standard bases in power series rings:uniqueness and superfluous critical pairs. *J. Symb. Comp.* **15** (1993) 251–265. MR1229634 (95b:68061), Zbl 0797.13016.

[4] Cox, David; Little, John; O'Shea, Donal. Using algebraic geometry. Second edition. Graduate Texts in Mathematics, 185. *Springer-Verlag, New York*, 1995. xii+572 pp. ISBN: 0-387-20706-6. MR2122859 (2005i:13037), Zbl 1079.13017.

[5] Call, Gregory S.; Silverman, Joseph. Computing the canonical height on K3 surfaces. *Math. Comp.* **65** (1996) 259–290. MR1322885 (96g:11067), Zbl 0865.14020.

[6] Dold, Albrecht. Fixed points indices of iterated maps. *Invent. Math.* **74** (1983) 419–435. MR0724012 (85c:54077), Zbl 0583.55001.

[7] Fakhruddin, Najmuddin. Questions on self maps of algebraic varieties. *J. Ramanujan Math. Soc.* **18**(2) (2003) 109–122. MR1995861 (2004f:14038), Zbl 1053.14025.

[8] Fagella, Núria; Llibre, Jaume. Periodic points of holomorphic maps via Lefschetz numbers. *Trans. Amer. Math. Soc.* **352**(10) 2000. MR1707699 (2001b:55003), Zbl 0947.55001.

[9] Hartshorne, Robin. Algebraic geometry. Graduate Texts in Mathematics, 52. *Springer-Verlag, New York*, 1977. xvi+496 pp. ISBN: 0-387-90244-9. MR0463157 (57 #3116), Zbl 0531.14001.

[10] Hutz, Benjamin. Arithmetic dynamics on varieities of dimension greater than 1. PhD thesis, Brown University, 2007.

[11] Ireland, Kenneth; Rosen, Michael. A classical introduction to modern number theory. Second edition. Graduate Texts in Mathematics, 84. *Springer-Verlag, New York*, 1990. xiv+389 pp. ISBN: 0-387-97329-X. MR1070716 (92e:11001), Zbl 0712.11001.

[12] Lang, Serge. Algebra. Revised third edition. Graduate Texts in Mathematics, 211. *Springer-Verlag, New York*, 2002. xvi+914 pp. ISBN: 0-387-95385-X. MR1878556 (2003e:00003), Zbl 0984.00001.

[13] Llibre, Jaume. Lefschetz numbers for periodic points. *Cont. Math.* **152** (1993) 215–227. MR1243477 (94g:55002), Zbl 0793.58030.

[14] Morton, Patrick. Arithmetic properties of periodic points of quadratic maps. *Acta Arith.* **62** (1992) 343–72. MR1199627 (93k:12004), Zbl 0767.11016.

[15] Morton, Patrick. Arithmetic properties of periodic points of quadratic maps, II. *Acta Arith.* **87** (1998) 89–102. MR1665198 (2000c:11103), Zbl 1029.12002.

[16] Morton, Patrick. On certain algebraic curves related to polynomial maps. *Comp. Math.* **103** (1996) 319–350. MR1414593 (97m:14030), Zbl 0860.11065.

[17] Morton, Patrick; Patel, Pratiksha. The Galois theory of periodic points of polynomial maps. *Proc. London Math. Soc.* **68** (1994) 225–263. MR1253503 (94i:11090), Zbl 0792.11043.

[18] Morton, Patrick; Silverman, Joseph H. Rational periodic points of rational functions. *Int. Math. Res. Not.* **1994**, no. 2, 97–110. MR1264933 (95b:11066), Zbl 0819.11045.

[19] Morton, Patrick; Silverman, Joseph H. Periodic points, multiplicities, and dynamical units. *J. Reine Angew. Math.* **461** (1995) 81–122. MR1324210 (96b:11090), Zbl 0813.11059.

[20] Morton, Patrick; Vivaldi, Franco. Bifurcations and discriminants for polynomial maps. *Nonlinearity* **8** (1995) 571–584. MR1342504 (96k:11028), Zbl 0827.12001.

[21] Serre, Jean-Pierre. Local algebra. Translated from the French by CheeWhye Chin and revised by the author. Springer Monographs in Mathematics. *Springer-Verlag, Berlin*, 2000. xiv+128 pp. ISBN: 3-540-66641-9. MR1771925 (2001b:13001), Zbl 0959.13010.

[22] Shafarevich, Igor. Basic algebraic geometry. 1. Varieties in projective space. Second edition. Translated from the 1988 Russian edition and with notes by Miles Reid. *Springer-Verlag, Berlin*, 1994. xx+303 pp. ISBN: 3-540-54812-2. MR1328833 (95m:14001), Zbl 0797.14001.

[23] Silverman, Joseph H. Rational points on K3 surfaces: a new cannonical height. *Invent. Math.* **105** (1991) 347–373. MR1115546 (92k:14025), Zbl 0754.14023.

[24] Silverman, Joseph H. The arithmetic of dynamical systems. Graduate Texts in Mathematics, 241. *Springer-Verlag, New York*, 2007. x+511 pp. ISBN: 978-0-387-69903-5. MR2316407 (2008c:11002), Zbl 1130.37001.

[25] Shub, Michael; Sullivan, Dennis. A remark on the lefschetz fixed point formula for differentiable maps. *Topology* **13** (1974) 189–191. MR0350782 (50 #3274), Zbl 0291.58014.

[26] Vivaldi, Franco; Hatjispyros, Spyros. Galois theory of periodic orbits of rational maps. *Nonlinearity* **5** (1992) 961–978. MR1174226 (93i:11153), Zbl 0767.11049.

[27] Wehler, Joachim. K3-surfaces with Picard number 2. *Arch. Math.* **50** (1988) 73–82. MR0925498 (89b:14054), Zbl 0602.14038.

[28] Zariski, Oscar; Samuel, Pierre. Commutative algebra. II. Reprint of the 1960 edition. Graduate Texts in Mathematics, 29. *Springer-Verlag, New York-Heidelberg*, 1975. x+414 pp. ISBN: 978-0387901718. MR0389876 (52 #10706), Zbl 0322.13001.

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