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Complete growth series and products of groups

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Abstract. The property of having a rational complete growth series is shown to be preserved by direct and graph products, as well as certain free products with amalgamation. These results extend earlier work of Alonso, Chiswell and Lewin.

Contents

1. Introduction

There is a long history of studying combinatorial structures in the context of infinite groups. One example is growth series, where for a given set of generators, one counts the number of elements of length n, and converts this sequence into a formal power series. If the group is $\mathbb{Z}$, generated by a single element a, one gets the following series, which happens to be rational:

$$\varphi_{\mathbb{Z}} = 1 + 2z + 2z^2 + \cdots = \frac{1+z}{1-z}.$$

Recently there has been interest in *complete* growth series, where instead of counting the elements of G of length n, one adds them together, forming an element of a group ring. In the same situation as above, where $G = \mathbb{Z}$ and where A denotes the inverse of the generator a, one gets:

$$\psi_{\mathbb{Z}} = 1 + (a+A)z + (a^2 + A^2)z^2 + \cdots = \frac{1-z^2}{1-(a+A)z+z^2}.$$

Even in the context of complete growth series, the formal power series associated to $\mathbb{Z}$ is rational.

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When the group G is non-Abelian the notion of rationality of the corresponding series is less obvious. There is a large literature on noncommutative power series, rationality and automata theory dating back at least to the work of Schützenberger in the 1960s (see the discussion in Chapter 1 of [2]). We give background information on complete growth series for groups, and the notion of rationality, in §2.

Foundational work of Alonso, Chiswell and Lewin showed that many of the products commonly studied in geometric group theory preserve the rationality of standard growth series (see [1, 3, 6, 7]). Here we extend these results to the context of complete growth series. In particular, we show that the property of having a rational complete growth series is preserved by direct, free and graph products of groups. Moreover, it is preserved by free products with amalgamation, assuming that the amalgamated subgroup embeds in a nice fashion into both vertex groups. Since the work of Stoll on the Heisenberg groups [12], it has been known that rationality can depend on the choice of generating set, and so the formal statements of these results are explicit about the generating sets being used. These statements can be found in Lemma 3.1, Theorem 3.4 and Theorem 3.8.

2. Complete growth series

In this section we briefly discuss key terminology from the theory of formal power series; additional information can be found in [2] and [9].

We let $R\langle\langle z\rangle\rangle$ denote the ring of formal power series whose variable is z and whose coefficients come from a ring R. If $\rho = \sum r_i z^i$ and $\rho' = \sum r'_i z^i$ are two elements in $R\langle\langle z\rangle\rangle$, then $\rho + \rho' = \sum (r_i + r'_i) z^i$ and $\rho \cdot \rho'$ is the Cauchy product

$$\rho \cdot \rho' = \sum_{n=0}^{\infty} \left(\sum_{i+j=n} r_i r'_j \right) z^n.$$

An element $\rho \in R\langle\langle z\rangle\rangle$ is a *polynomial* if all but finitely many of the coefficients are zero.

Let G be a group and let $S_G \subset G$ be a finite generating set for G. Let $|g| = |g|_S$ denote the length of a minimal length expression of g as a product of elements in $S \cup S^{-1}$. If c_n is the number of elements in G of length n, then the *standard growth series* of G is

$$\varphi_G = \sum_{g \in G} z^{|g|} = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{Z}\langle\langle z\rangle\rangle.$$

Similarly if $\mathcal{C}_n$ is the element of the group ring $\mathbb{Z}[G]$ formed by summing all of the elements in G of length n, i.e.,

$$\mathcal{C}_n = \sum_{g \in G, |g|=n} g,$$

then the *complete growth series* of G is the formal power series

$$\psi_G = \sum_{g\in G} g z^{|g|} = \sum_{n=0}^{\infty} \mathcal{C}_n z^n \in \mathbb{Z}[G]\langle\langle z\rangle\rangle.$$

More generally, if H is any subset of G, then we let

$$\psi_H = \sum_{g\in H} g z^{|g|}.$$

In particular we will need this notation for sets of coset representatives in §3.

We note that the counting map from $\mathbb{Z}[G]$ to $\mathbb{Z}$ defined by sending each $g \in G$ to 1, sends each $\mathcal{C}_n$ to c_n, hence the extension of the counting map to a map from $\mathbb{Z}[G]\langle\langle z\rangle\rangle$ to $\mathbb{Z}\langle\langle z\rangle\rangle$ sends the complete growth series to the standard growth series.

A standard growth series is said to be rational if it is the series associated to a rational function. This notion of rationality extends to complete growth series, via the process of quasi-inversion. An element $\rho = r_0 + r_1 z + \cdots \in R\langle\langle z\rangle\rangle$ is *quasi-regular* if $r_0 = 0$. For a quasi-regular ρ, the *quasi-inverse* is

$$\rho^+ = \sum_{n=1}^{\infty} \rho^n = \lim_{N\to\infty} \sum_{n=1}^{N} \rho^n.$$

The quasi-inverse of a quasi-regular element is uniquely determined by the equation

$$\rho + \rho \cdot \rho^+ = \rho + \rho^+ \cdot \rho = \rho^+$$

or in other words

$$(1-\rho)\cdot(1+\rho^+) = (1+\rho^+)\cdot(1-\rho) = 1.$$

Thus one can think of $1+\rho^+$ as "$\frac{1}{1-\rho}$", which motivates the following definition.

Definition 2.1. A formal power series $\rho \in R\langle\langle z\rangle\rangle$ is *rational* if it can be created from a finite number of polynomials in $R\langle\langle z\rangle\rangle$, via the application of a finite number of the operations of addition, multiplication, and quasi-inversion.

The complete growth series of some important classes of groups are known to be rational. The foundational work of Grigorchuk and Nagnibeda showed that hyperbolic groups have rational complete growth series with respect to any finite generating set [5]. The complete growth series for Coxeter groups, with respect to their standard generators, are known to be rational, and recent work of Scott shows that in the right-angled case these complete growth series satisfy reciprocity formulas similar to the formulas known in the ordinary setting (see [10] and the references cited there).

Example 2.2. The complete growth series contains substantially more information than the ordinary growth series of a group. A right-angled Artin group is a group whose presentation can be represented by a finite, simple graph Γ:

$$A_\Gamma = \langle v \in \mathrm{V}(\Gamma) \mid vw = wv \text{ when } \{v, w\} \in \mathrm{E}(\Gamma)\rangle.$$

Two right-angled Artin groups are isomorphic if and only if they have the same defining graph [4]. However, the standard growth series does not distinguish distinct right-angled Artin groups [6]. In particular, the standard growth series of any two right-angled Artin groups based on trees with n vertices will be the same.

On the other hand, it is easy to see that the complete growth series do distinguish right-angled Artin groups, since one can reconstruct the graph Γ from the coefficients of z and z^2 in ψ_{A_Γ}. The coefficient of z is the sum of the generators and their inverses, which identifies $\mathrm{V}(\Gamma)$, and both vw and wv appear in the coefficient of z^2 if and only if $\{v, w\} \notin \mathrm{E}(\Gamma)$.

3. Products of groups

In this section we establish that direct products, certain free products with amalgamation, and graph products preserve the property of having rational complete growth series. (Lemma 3.1 on direct products is certainly known, although we could not find it explicitly presented in the literature.)

Lemma 3.1. *Let A and B be groups with finite generating sets S_A and S_B, respectively. The complete growth series of $A \oplus B$, with respect to the generating set $\{S_A \cup S_B\}$, is the product*

$$\psi_{A\oplus B}(z) = \psi_A(z) \cdot \psi_B(z).$$

This proof of this lemma is analogous to the proof of Theorem 9.14 in [8]. The key observation is that

$$\mathcal{C}_{A\oplus B}(n) = \sum_{i=0}^{n} \mathcal{C}_A(i) \cdot \mathcal{C}_B(n-i)$$

where $\mathcal{C}_G(i)$ is the coefficient of z^i in the complete growth series for the group G.

One cannot hope for a simple formula, like that given in Lemma 3.1, for all free products with amalgamation. As was discussed by both Alonso and Lewin, one needs to assume that the subgroup being amalgamated sits 'nicely' inside of the vertex groups. The notion of 'nice' in the context of complete growth series is exactly the same as for standard growth series, and it is given in the definition of admissibility below.

Definition 3.2. Let C be a subgroup of a group G and let S_C and S_G be finite sets of generators for C and G where $S_C \subset S_G$. Denote the length function $|\cdot|_{S_C}$ by $|\cdot|_C$, and similarly abbreviate $|\cdot|_{S_G}$ to $|\cdot|_G$.

The inclusion $(C, S_C) \subset (G, S_G)$ is *admissible* if there exists a set of coset representatives G/C of C in G, such that for all $g \in G/C$, and all $c \in C$:

$$|gc|_G = |g|_G + |c|_C.$$

We refer to this metric condition as the *admissibility length condition.*

Note the following consequences of this definition:

(1) The identity element must be one of the coset representatives in G/C.
Proof. The coset representative for C must be some element of C, call it c. Then by the formula we have

$$0 = |e|_G = |cc^{-1}|_G = |c|_G + |c^{-1}|_C,$$

which is only possible if $c = e$.

(2) If c is any element of C then $|c|_C = |c|_G$. In other words, the embedding of the Cayley graph of C into the Cayley graph of G is 'totally geodesic'.
Proof. Since $e \in G/C$ we know

$$|c|_G = |e \cdot c|_G = |e|_G + |c|_C.$$

Lemma 3.3. *Let C be an admissible subgroup of G, with G/C the corresponding set of coset representatives. Then*

$$\psi_{G/C}(z) \cdot \psi_C(z) = \psi_G(z).$$

Proof. We may express the coset representatives as $G/C = \{1, g_1, g_2, \ldots\}$. Since $G = \bigsqcup g_n C$, each $g \in G$ will appear in exactly one coefficient in the product $\psi_{G/C} \cdot \psi_C$. In fact, if $g = g_i c$, then g will be a term in the coefficient of $z^{|g_i|_G + |c|_C}$. The admissibility length condition tells us that $|gc|_G = |g|_G + |c|_C$, hence g is a term in the coefficient of $z^{|g|_G}$. □

The following theorem extends Theorem 2 in [1] and Corollary 6 in [7] in which Alonso and Lewin independently prove the result for standard growth series.

Theorem 3.4. *Let $G = A *_C B$ be a free product of groups A and B amalgamating C. Assume that $(C, S_C) \subset (A, S_A)$ and $(C, S_C) \subset (B, S_B)$ are both admissible. Assume further that $S_G = \{S_A \cup S_B\}$. Then the complete growth series of G is given by*

$$\frac{1}{\psi_G(z)} = \frac{1}{\psi_A(z)} + \frac{1}{\psi_B(z)} - \frac{1}{\psi_C(z)}.$$

Notice that if H is a finitely generated subgroup of a finitely generated group G, then $\psi_H - 1 = \sum_{h \neq e} |h|_G z^{|h|_G}$ is a quasi-regular element of $\mathbb{Z}[G]\langle\langle z\rangle\rangle$. It follows that ψ_H has an inverse in $\mathbb{Z}[G]\langle\langle z\rangle\rangle$, hence the notation used in the statement of this theorem is valid.

Proof of Theorem 3.4. By the normal form theorem for free products with amalgamation (see [11]), an element $g \in A *_C B$ can be expressed (uniquely) as

$$g = a_0 \cdot b_1 a_1 b_2 a_2 \cdots b_{n-1} a_{n-1} b_n a_n \cdot b_{n+1} \cdot c$$

where the a_i and b_i are taken from a fixed set of coset representatives for C in A and B, $c \in C$, and only a_0, b_{n+1}, and/or c may be the identity. Further, the length of g is

$$|g|_G = |a_0|_G + |b_1|_G + |a_1|_G + \cdots + |b_{n+1}|_G + |c|_G$$

which by the admissibility length condition is the same as

$$|g|_G = \sum_{i=0}^{n} |a_i|_A + \sum_{i=1}^{n+1} |b_i|_B + |c|_C.$$

Therefore we know

$$\begin{aligned}\psi_G(z) &= \psi_{A/C}(z) \cdot \sum_{n\geq 0}^{\infty} [(\psi_{B/C}(z) - 1)(\psi_{A/C}(z) - 1)]^n \cdot \psi_{B/C}(z) \cdot \psi_C(z) \\ &= \psi_{A/C}(z) \cdot [1 - (\psi_{B/C}(z) - 1)(\psi_{A/C}(z) - 1)]^{-1} \cdot \psi_{B/C}(z) \cdot \psi_C(z).\end{aligned}$$

Hence

$$\begin{aligned}\psi_G^{-1}(z) &= \psi_C^{-1}(z) \cdot \psi_{B/C}^{-1}(z) \cdot [1 - (\psi_{B/C}(z) - 1)(\psi_{A/C}(z) - 1)] \cdot \psi_{A/C}^{-1}(z) \\ &= \psi_C^{-1}(z) \cdot \psi_{A/C}^{-1}(z) + \psi_C^{-1}(z) \cdot \psi_{B/C}^{-1}(z) - \psi_C^{-1}(z) \\ &= (\psi_{A/C}(z) \cdot \psi_C(z))^{-1} + (\psi_{B/C}(z) \cdot \psi_C(z))^{-1} - \psi_C^{-1}(z).\end{aligned}$$

Lemma 3.3 says that $\psi_{A/C} \cdot \psi_C = \psi_A$ and $\psi_{B/C} \cdot \psi_C = \psi_B$. Thus we have

$$\frac{1}{\psi_G(z)} = \frac{1}{\psi_A(z)} + \frac{1}{\psi_B(z)} - \frac{1}{\psi_C(z)}. \qquad \square$$

Corollary 3.5. *If $G = A * B$, then*

$$\frac{1}{\psi_G(z)} = \frac{1}{\psi_A(z)} + \frac{1}{\psi_B(z)} - 1.$$

Corollary 3.6. *If the complete growth series for A, B, and C are all rational, then the complete growth series for G is rational.*

Example 3.7. Consider the group $\mathrm{PSL}_2(\mathbb{Z}) \approx \mathbb{Z}_2 * \mathbb{Z}_3 \approx \langle a, b \mid a^2 = b^3 = 1 \rangle$. Applying Corollary 3.5 we get

$$\frac{1}{\psi_{\mathbb{Z}_2 * \mathbb{Z}_3}(z)} = \frac{1}{1 + az} + \frac{1}{1 + (b + B)z} - 1.$$

However, one should avoid the temptation to "simplify-and-solve" for $\psi_{\mathbb{Z}_2 * \mathbb{Z}_3}$, since the expression

$$\frac{(1 + az)(1 + (b + B)z)}{1 - (ba + Ba)z^2}$$

can be interpreted as

$$\frac{1}{1-(ba+Ba)z^2}(1+az)(1+(b+B)z)$$
$$\text{or } (1+az)\frac{1}{1-(ba+Ba)z^2}(1+(b+B)z)$$
$$\text{or } (1+az)(1+(b+B)z)\frac{1}{1-(ba+Ba)z^2},$$

which are not necessarily equal in the world of noncommuting coefficients. In other words, one needs to remember that $\mathbb{Z}[\mathbb{Z}_2 * \mathbb{Z}_3]$ is not commutative.

Let Γ be a finite simple graph (that is, no loops and no multiple edges) and let $\{G_v \mid v \in \mathrm{V}(\Gamma)\}$ be a collection of groups associated to the vertices v of Γ. The resulting *graph product* is the quotient of the free product of the G_v's, modulo "adjacent groups commute":

$$\Pi_\Gamma = (*G_v)\,/\{[G_u, G_w] = 1 \text{ when } \{u,w\} \in \mathrm{E}(\Gamma)\}.$$

When each $G_v \approx \mathbb{Z}$, the resulting graph product is commonly called a right-angled Artin group. (The groups formerly known as graph groups.)

The proof of the following theorem mimics the proof of Proposition 1 in [3], where Chiswell established the analogous formula for standard growth series.

Theorem 3.8. *Let Π_Γ be a graph product where ψ_v denotes the complete growth series of the group G_v. Let*

$$\Lambda = \{\Delta \in \Gamma \mid \Delta \text{ is a complete subgraph of } \Gamma\}$$

and let $\pi_\Delta = \prod_{v \in V(\Delta)} (\frac{1}{\psi_v(z)} - 1)$ where $\pi_\emptyset = 1$. Then

$$\frac{1}{\psi_{\Pi_\Gamma}} = \sum_{\Delta \in \Lambda} \pi_\Delta$$

where the generating set for Π_Γ is the union of the generating sets of the $\{G_v\}$.

Proof. We proceed by induction on the number of vertices in Γ. The statement is obvious for a graph with one vertex, and the two 2-vertex cases follow from Lemma 3.1 and Corollary 3.6. Suppose Γ has n vertices. Assume the statement holds for graphs with strictly less than n vertices. Define Z to be $\Gamma - \{v\}$ and let E be the full subgraph of Γ induced by vertices adjacent to v. Then

$$\Pi_\Gamma = (G_v \oplus \Pi_E) *_{\Pi_E} \Pi_Z.$$

By Lemma 1 in [3], Π_E is admissible in Π_Z and $G_v \oplus \Pi_E$. Therefore Theorem 3.4 and Lemma 3.1 imply

$$\begin{aligned}\frac{1}{\psi_{\Pi_\Gamma}(z)} &= \frac{1}{\psi_{\Pi_E}(z)\cdot\psi_v(z)} + \frac{1}{\psi_{\Pi_Z}(z)} - \frac{1}{\psi_{\Pi_E}(z)} \\ &= \frac{1}{\psi_{\Pi_E}(z)}\left(\frac{1}{\psi_v(z)} - 1\right) + \frac{1}{\psi_{\Pi_Z}(z)} \\ &= \left(\sum_{\Delta' \subset E} \pi_{\Delta'}\right)\left(\frac{1}{\psi_v(z)} - 1\right) + \sum_{\Delta'' \subset Z} \pi_{\Delta''}\end{aligned}$$

where Δ' is the set of complete subgraphs contained in E and Δ'' is the set of complete subgraphs contained in Z. A complete subgraph containing the vertex v will be represented in the first term while the complete subgraphs that do not contain v are represented in the second term. □

Corollary 3.9. *If every G_v has a rational, complete growth series, then so does the graph product Π_Γ.*

References

[1] Alonso, Juan M. Growth functions of amalgams. In *Arboreal group theory* (Berkeley, CA, 1988), 1–34. Math. Sci. Res. Inst. Publ., 19. *Springer, New York*, 1991. ISBN 0-387-97518-7. MR1105328 (93a:20037), Zbl 0796.20021.

[2] Berstel, Jean; Reutenauer, Christophe. Noncommutative rational series with applications. Encyclopedia of Mathematics and its Applications, 137. *Cambridge University Press, Cambridge*, 2011. xiv+248 pp. ISBN: 978-0-521-19022-0. MR2760561, Zbl pre05815123.

[3] Chiswell, I. M. The growth series of a graph product. *Bull. London Math. Soc.* **26** (1994), no. 3, 268–272. MR1289045 (95f:20050), Zbl 0834.20027.

[4] Droms, Carl. Isomorphisms of graph groups. *Proc. Amer. Math. Soc.* **100** (1987), no. 3, 407–408. MR0891135 (88e:20033), Zbl 0619.20015.

[5] Grigorchuk, Rostislav; Nagnibeda, Tatiana. Complete growth functions of hyperbolic groups. *Invent. Math.* **130** (1997), no. 1, 159–188. MR1471889 (98i:20038), Zbl 0880.20024.

[6] Lewin, Jacques. The growth function of a graph group. *Comm. Algebra* **17** (1989), no. 5, 1187–1191. MR0993397 (90c:20040), Zbl 0673.20013.

[7] Lewin, Jacques. The growth function of some free products of groups. *Comm. Algebra* **19** (1991), no. 9, 2405–2418. MR1125179 (92k:20042), Zbl 0742.20027.

[8] Meier, John. Groups, graphs and trees. An introduction to the geometry of infinite groups. London Mathematical Society Student Texts, 73. *Cambridge University Press, Cambridge*, 2008. xii+231 pp. ISBN: 978-0-521-71977-3. MR2498449 (2010e:20066), Zbl pre05306923.

[9] Salomaa, Arto; Soittola, Matti. Automata-theoretic aspects of formal power series. Texts and Monographs in Computer Science. *Springer-Verlag, New York-Heidelberg*, 1978. x+171 pp. ISBN: 0-387-90282-1. MR0483721 (58 #3698), Zbl 0377.68039.

[10] Scott, Richard. Rationality and reciprocity for the greedy normal form of a Coxeter group. *Trans. Amer. Math. Soc.* **363** (2011), no. 1, 385–415. MR2719687, Zbl pre05863880.

[11] Jean-Pierre Serre. Trees. Translated from the French original by John Stillwell. Corrected 2nd printing of the 1980 English translation. Springer Monographs in Mathematics. *Springer-Verlag, Berlin*, 2003. x+142 pp. ISBN: 3-540-44237-5. MR1954121 (2003m:20032), Zbl 1013.20001.

[12] Stoll, Michael. Rational and transcendental growth series for the higher Heisenberg groups. *Invent. Math.* **126** (1996), no. 1, 85–109. MR1408557 (98d:20033), Zbl 0869.20018.

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