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Homogeneous SK_1 of simple graded algebras

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Abstract. For a simple graded algebra $\mathsf{S} = \mathbb{M}_n(\mathsf{E})$ over a graded division algebra E, a short exact sequence is established relating the reduced Whitehead group of the homogeneous part of S to that of E. In particular it is shown that the homogeneous SK_1 is not in general Morita invariant. Along the way we prove the existence and multiplicativity of a Dieudonné determinant for homogeneous elements of S.

Contents

Graded methods in the theory of valued division algebras have proved to be extremely useful. A valuation v on a division algebra D induces a filtration on D which yields an associated graded ring $\mathrm{gr}(D)$. Indeed, $\mathrm{gr}(D)$ is a graded division algebra, i.e., every nonzero homogeneous element of $\mathrm{gr}(D)$ is a unit. While $\mathrm{gr}(D)$ has a much simpler structure than D, nonetheless $\mathrm{gr}(D)$ provides a remarkably good reflection of D in many ways, particualrly when the valuation on the center $Z(D)$ is Henselian. The approach of making calculations in $\mathrm{gr}(D)$, then lifting back to get nontrivial information about D has been remarkably successful. See [JW, W_1] for background on valued division algebras, and [HwW, TW_1, TW_2] for connections between valued and graded division algebras. The recent papers [HW_1, HW_2, WY, W_2] on the reduced Whitehead group SK_1 and its unitary analogue have provided good illustrations of the effectiveness of this approach. Notably it was proved in [HW_1, Th. 4.8, Th. 5.7] that if v on $Z(D)$ is Henselian and D is tame over $Z(D)$, then $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathrm{gr}(D))$ and $\mathrm{SK}_1(\mathrm{gr}(D)) \cong \mathrm{SK}_1(q(\mathrm{gr}(D)))$, where $q(\mathrm{gr}(D))$ is the division ring of quotients of $\mathrm{gr}(D)$. This has allowed

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recovery of many of the known calculations of $\mathrm{SK}_1(D)$ with much easier proofs, as well as leading to determinations of $\mathrm{SK}_1(D)$ in some new cases.

By the graded Wedderburn theorem (see [HwW, Prop. 1.3(a)] and [NvO, Thm 2.10.10]), any simple graded algebra S finite-dimensional over its center T has the form $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\delta_1, \ldots, \delta_n)$, where the δ_i lie in an abelian group Γ containing the grade group Γ_{E}. That is S is the $n \times n$ matrix algebra over a graded division algebra E with its grading shifted by $(\delta_1, \ldots, \delta_n)$. Since S is known to be Azumaya algebra over T, there is a reduced norm map on the group of units, $\mathrm{Nrd}_{\mathsf{S}} \colon \mathsf{S}^* \to \mathsf{T}^*$; one can then define the reduced Whitehead group $\mathrm{SK}_1(\mathsf{S})$ in the usual manner as the kernel of the reduced norm of S modulo the commutator subgroup of S^* (see Definition 2.1). However SK_1 is not a "graded functor", i.e., it does not take into account the grading on S.

To factor in the grading on S, we introduce in this paper the *homogeneous* reduced Whitehead group $\mathrm{SK}_1^h(\mathsf{S})$ (see Definition 2.2), which treats only the homogeneous units of S. We establish a short exact sequence relating $\mathrm{SK}_1^h(\mathsf{S})$ to $\mathrm{SK}_1(\mathsf{E})$ (see Theorem 2.4) which allows us to calculate $\mathrm{SK}_1^h(\mathsf{S})$ in many cases. In particular we show that SK_1^h is not in general Morita invariant for E, and indeed can behave quite badly when the semisimple ring S_0 is not simple (see Example 2.6). As a prelude to this, in §1 we prove the existence and multiplicativity of a Dieudonné determinant for homogeneous elements of $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\delta_1, \ldots, \delta_n)$. This was originally needed for the work on SK_1^h, but later it turned out that the ungraded Dieutdonné determinant for the semisimple algebra S_0 was all that was needed. We have nonetheless included the development of the homogeneous Dieudonné determinant, since we feel that it is of some interest in its own right. Throughout the paper we assume that the grade group Γ is abelian. From §2 on we are interested in graded division algebras arising from valued division algebras, and we then make the further assumption that the abelian group Γ is torsion free.

1. Dieudonné determinant

Throughout this paper we will be working with matrices over graded division rings. Recall that a graded ring $\mathsf{E} = \bigoplus_{\gamma \in \Gamma} \mathsf{E}_\gamma$ is called a *graded division ring* if every nonzero homogeneous element of E is a unit, i.e. it has a (two-sided) multiplicative inverse. We assume throughout that the index set Γ is an abelian group. Note that the hypothesis on E implies that the grade set $\Gamma_{\mathsf{E}} = \{\gamma \in \Gamma \mid \mathsf{E}_\gamma \neq \{0\}\}$ is actually a subgroup of Γ. We write E_h^* for the group of homogeneous units of E, which consists of all the nonzero homogeneous elements of E, and can be a proper subgroup of the group E^* of all units of E.

Let $\mathbb{M}_n(\mathsf{E})$ be the $n \times n$ matrix ring over the graded division ring E. For any $x \in \mathsf{E}$, let $E_{ij}(x)$ be the matrix in $\mathbb{M}_n(\mathsf{E})$ with x in (i,j)-position and 0's otherwise. Take any $\delta_1, \ldots, \delta_n \in \Gamma$. The shifted grading on $\mathbb{M}_n(\mathsf{E})$

determined by $(\delta_1, \dots, \delta_n)$ is defined by setting,

$$\deg(E_{ij}(x)) = \deg(x) + \delta_i - \delta_j, \quad \text{for any homogeneous } x \text{ in } \mathsf{E}. \tag{1.1}$$

Now extend this linearly to all of $\mathbb{M}_n(\mathsf{E})$. One can then see that for $\lambda \in \Gamma$, the λ-component $\mathbb{M}_n(\mathsf{E})_\lambda$ consists of those matrices with homogeneous entries, with the degrees shifted as follows:

$$\mathbb{M}_n(\mathsf{E})_\lambda = \begin{pmatrix} \mathsf{E}_{\lambda+\delta_1-\delta_1} & \mathsf{E}_{\lambda+\delta_2-\delta_1} & \cdots & \mathsf{E}_{\lambda+\delta_n-\delta_1} \\ \mathsf{E}_{\lambda+\delta_1-\delta_2} & \mathsf{E}_{\lambda+\delta_2-\delta_2} & \cdots & \mathsf{E}_{\lambda+\delta_n-\delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{E}_{\lambda+\delta_1-\delta_n} & \mathsf{E}_{\lambda+\delta_2-\delta_n} & \cdots & \mathsf{E}_{\lambda+\delta_n-\delta_n} \end{pmatrix}. \tag{1.2}$$

That is, $\mathbb{M}_n(\mathsf{E})_\lambda$ consists of matrices with each ij-entry lying in $\mathsf{E}_{\lambda+\delta_j-\delta_i}$. We then have

$$\mathbb{M}_n(\mathsf{E}) = \bigoplus_{\lambda\in\Gamma} \mathbb{M}_n(\mathsf{E})_\lambda \quad \text{and} \quad \mathbb{M}_n(\mathsf{E})_\lambda \cdot \mathbb{M}_n(\mathsf{E})_\mu \subseteq \mathbb{M}_n(\mathsf{E})_{\lambda+\mu} \text{ for all } \lambda, \mu \in \Gamma,$$

which shows that $\mathbb{M}_n(\mathsf{E})$ is a graded ring. We denote the matrix ring with this grading by $\mathbb{M}_n(\mathsf{E})(\delta_1, \dots, \delta_n)$ or $\mathbb{M}_n(\mathsf{E})(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots, \delta_n)$. It is not hard to show that $\mathbb{M}_n(\mathsf{E})(\overline{\delta})$ is a simple graded ring, i.e., it has no nontrivial homogeneous two-sided ideals. Observe that for $\mathsf{S} = \mathbb{M}_n(\mathsf{E}(\overline{\delta}))$, the grade set is

$$\Gamma_{\mathsf{S}} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (\delta_j - \delta_i) + \Gamma_{\mathsf{E}}, \tag{1.3}$$

which need not be a group. However, if we let

$$\mathsf{S}_h^* = \{A \in \mathsf{S} \mid A \text{ is homogeneous and } A \text{ is a unit of } \mathsf{S}\},$$

which is a subgroup of the group of units S^* of S, and set

$$\Gamma_{\mathsf{S}}^* = \{\deg(A) \mid A \in \mathsf{S}_h^*\},$$

then Γ_{S}^* is a subgroup of Γ, with $\Gamma_{\mathsf{E}} \subseteq \Gamma_{\mathsf{S}}^* \subseteq \Gamma_{\mathsf{S}}$.

Note that when $\delta_i = 0$, $1 \le i \le n$, then $\mathbb{M}_n(\mathsf{E})_\lambda = \mathbb{M}_n(\mathsf{E}_\lambda)$. We refer to this case as the *unshifted grading* on $\mathbb{M}_n(\mathsf{E})$.

For any graded rings B and C, we write $\mathsf{B} \cong_{\mathrm{gr}} \mathsf{C}$ if there is graded ring isomorphism $\mathsf{B} \to \mathsf{C}$, i.e., a ring isomorphism that maps B_λ onto C_λ for all $\lambda \in \Gamma_{\mathsf{B}} = \Gamma_{\mathsf{C}}$.

The following two statements can be proved easily (see [NvO, pp. 60-61]):

- If $\alpha \in \Gamma$ and $\pi \in S_n$ is a permutation, then

$$\mathbb{M}_n(\mathsf{E})(\delta_1, \dots, \delta_n) \cong_{\mathrm{gr}} \mathbb{M}_n(\mathsf{E})(\delta_{\pi(1)} + \alpha, \dots, \delta_{\pi(n)} + \alpha). \tag{1.4}$$

- If $\alpha_1, \dots, \alpha_n \in \Gamma$ with $\alpha_i = \deg(u_i)$ for some units $u_i \in \mathsf{E}_h^*$, then

$$\mathbb{M}_n(\mathsf{E})(\delta_1, \dots, \delta_n) \cong_{\mathrm{gr}} \mathbb{M}_n(\mathsf{E})(\delta_1 + \alpha_1, \dots, \delta_n + \alpha_n). \tag{1.5}$$

Take any $\delta_1, \dots, \delta_n \in \Gamma$. In the factor group $\Gamma/\Gamma_{\mathsf{E}}$, let $\varepsilon_1 + \Gamma_{\mathsf{E}}, \dots, \varepsilon_k + \Gamma_{\mathsf{E}}$ be the distinct cosets in $\{\delta_1 + \Gamma, \dots, \delta_n + \Gamma\}$. For each ε_ℓ, let r_ℓ be the number of i with $\delta_i + \Gamma = \varepsilon_\ell + \Gamma$. It was observed in [HwW, Prop. 1.4] that

$$\mathbb{M}_n(\mathsf{E})_0 \cong \mathbb{M}_{r_1}(\mathsf{E}_0) \times \dots \times \mathbb{M}_{r_k}(\mathsf{E}_0). \tag{1.6}$$

Thus $\mathbb{M}_n(\mathsf{E})_0$ is a a semisimple ring; it is simple if and only if $k = 1$. Indeed, (1.6) follows easily from the observations above. For, using (1.4) and (1.5) we get

$$\mathbb{M}_n(\mathsf{E})(\delta_1, \dots, \delta_n) \cong_{\mathrm{gr}} \mathbb{M}_n(\mathsf{E})(\varepsilon_1, \dots, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k), \tag{1.7}$$

with each ε_ℓ occurring r_ℓ times. Now (1.2) for $\lambda = 0$ and $(\delta_1, \dots, \delta_n) = (\varepsilon_1, \dots \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k)$ immediately gives (1.6).

If the graded ring E is commutative then the usual determinant map is available, and $\det\big(\mathbb{M}_n(\mathsf{E})_\lambda\big) \subseteq \mathsf{E}_{n\lambda}$. Indeed, if $a = (a_{ij}) \in \mathbb{M}_n(\mathsf{E})_\lambda$, then $\det(a) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma1} a_{2\sigma2} \dots a_{n\sigma n} \in \mathsf{E}$. But by (1.2)

$$\deg(a_{1\sigma1} a_{2\sigma2} \dots a_{n\sigma n}) = n\lambda + \sum_{i=1}^{n} \delta_{\sigma(i)} - \sum_{i=1}^{n} \delta_i = n\lambda. \tag{1.8}$$

When E is not commutative, there is no well-defined determinant available in general. For a division ring D, Dieudonné constructed a determinant map which reduces to the usual determinant when D is commutative. This is a group homomorphism $\det\colon \mathrm{GL}_n(D) \to D^*/[D^*, D^*]$. The kernel of det is the subgroup $E_n(D)$ of $\mathrm{GL}_n(D)$ generated by elementary matrices, which coincides with the commutator group $[\mathrm{GL}_n(D), \mathrm{GL}_n(D)]$ unless $\mathbb{M}_n(D) = \mathbb{M}_2(\mathbb{F}_2)$ (see Draxl [D, §20]). Note that the construction of a Dieudonné determinant has been carried over to (noncommutative) local and semilocal rings in [V].

Since graded division rings behave in many ways like local rings, one may ask whether there is a map like the Dieudonné determinant in the graded setting. We will show that this is indeed the case, so long as one restricts to homogeneous elements. Specifically, let E be a graded division ring with grade group $\Gamma_E \subseteq \Gamma$ with Γ abelian, and let $\overline{\delta} = (\delta_1, \dots, \delta_n)$, where $\delta_i \in \Gamma$. Let $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$ be the matrix ring over E with grading shifted by $\overline{\delta}$. Denote by S_h the set of homogeneous elements of S and by S_h^* or $\mathrm{GL}_n^h(\mathsf{E})(\overline{\delta})$ the group of homogeneous units of S. We will show in Theorem 1.2 that there is a determinant-like group homomorphism $\det_{\mathsf{E}}\colon \mathsf{S}_h^* \to \mathsf{E}_h^*/[\mathsf{E}_h^*, \mathsf{E}_h^*]$ which is compatible with the Dieudonné determinant on the semisimple ring S_0 (see commutative diagram (1.18)).

We first show that every matrix in $\mathrm{GL}_n^h(\mathsf{E})(\overline{\delta})$ can be decomposed into strict Bruhat normal form. In this decomposition, a triangular matrix is said to be *unipotent* triangular if all its diagonal entries are 1's.

Proposition 1.1 (Bruhat normal form)**.** *Let E be a graded division ring with grade group $\Gamma_{\mathsf{E}} \subseteq \Gamma$. Let $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$ be a matrix ring over E with* *grading shifted by* $\overline{\delta} = (\delta_1, \ldots, \delta_n)$, $\delta_i \in \Gamma$. *Then every* $A \in \mathsf{S}_h^*$ *has a unique strict Bruhat normal form, i.e.,* A *can be decomposed uniquely as*

$$A = TUP_\pi V$$

for matrices T, U, P_π, V *in* S *such that* T *is unipotent lower triangular,* U *is diagonal and invertible,* P_π *is a permutation matrix, and* V *is unipotent upper triangular with* $P_\pi V P_\pi^{-1}$ *also unipotent upper triangular. Moreover,* T, UP_π, *and* V *are homogeneous matrices, with* $\deg(T) = \deg(V) = 0$ *and* $\deg(UP_\pi) = \deg(A)$. *Also,* T *is a product of homogeneous elementary matrices* (*of degree* 0).

Proof. The construction follows closely that in Draxl [D, §19, Thm 1], with extra attention given to degrees of the homogeneous matrices in the graded ring $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$. We will carry out elementary row operations on homogeneous invertible matrices, which corresponds to left multiplication by elementary matrices. But, we use only homogeneous elementary matrices thereby preserving homogeneity of the matrices being reduced. For $x \in \mathsf{E}$ and $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, let $e_{ij}(x) = \mathbb{I}_n + E_{ij}(x)$, which is the elementary matrix with all 1's on the main diagonal, x in the (i, j)-position and all other entries 0. Note that if $e_{ij}(x)$ is homogeneous, it must have degree 0 because of the 1's on the main diagonal. So, in view of (1.2), $e_{ij}(x)$ is homogeneous if and only if x is homogeneous with $\deg(x) = \delta_j - \delta_i$ or $x = 0$. Let

$$\begin{aligned} \mathcal{E}\ell_h &= \{\text{homogeneous elementary matrices in } \mathsf{S}\} \\ &= \{e_{ij}(x) \mid i \neq j \text{ and } x \in \mathsf{E}_{\delta_j - \delta_i}\}. \end{aligned} \tag{1.9}$$

Let $A \in \mathsf{S}_h^*$. Since A is homogeneous, every nonzero entry of A is a homogeneous element of the graded division ring E (see (1.2)), and so is a unit of E. Since A is an invertible matrix, each row must have at least one nonzero entry. Write the (i, j)-entry of A as a^1_{ij}; so $A = (a^1_{ij})$. Let $a^1_{1\rho(1)}$ be the first nonzero entry in the first row, working from the left. For $i > 1$, multiplying A on the left by the elementary matrix $e_{i1}(-a^1_{i\rho(1)}(a^1_{1\rho(1)})^{-1})$ amounts to adding the left multiple $-a^1_{i\rho(1)}(a^1_{1\rho(1)})^{-1}$ times the first row to the i-th row; it makes the $(i, \rho(1))$-entry zero, without altering any other rows besides the i-th. By iterating this for each row below the first row, we obtain a matrix $A^{(1)} = \prod_{i=n}^{2} e_{i1}(-a^1_{i\rho(1)}(a^1_{1\rho(1)})^{-1})A$, which has the form

$$A^{(1)} = \begin{pmatrix} 0 & 0 & \cdots & a^1_{1\rho(1)} & a^1_{1,\rho(1)+1} & \cdots & a^1_{1n} \\ a^1_{21} & a^1_{22} & \cdots & 0 & b_{2,\rho(1)+1} & \cdots & b_{2n} \\ a^1_{31} & a^1_{32} & \cdots & 0 & b_{3,\rho(1)+1} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^1_{n1} & a^1_{n2} & \cdots & 0 & b_{n,\rho(1)+1} & \cdots & b_{nn} \end{pmatrix}. \tag{1.10}$$

Let $\lambda = \deg(A)$. From the definition of the grading on $\mathbb{M}_n(A)(\overline{\delta})$ we have $\deg(a^1_{i\rho(1)}) = \lambda + \delta_{\rho(1)} - \delta_i$ (see (1.2)). Thus $\deg(-a^1_{i\rho(1)}(a^1_{1\rho(1)})^{-1}) = \delta_1 - \delta_i$, which shows that $e_{i1}(-a^1_{i\rho(1)}(a^1_{1\rho(1)})^{-1}) \in \mathcal{E}\ell_h$ for $i = 2,3,\dots,n$. Since homogeneous elementary matrices have degree 0, $A^{(1)}$ is homogeneous with $\deg(A^{(1)}) = \deg(A) = \lambda$.

Write $A^{(1)} = (a^2_{ij})$. Since $A^{(1)}$ is invertible, not all the entries of its second row can be zero. Let $a^2_{2\rho(2)}$ be the first nonzero entry in the second row working from the left (clearly $\rho(1) \neq \rho(2)$), and repeat the process above with $A^{(1)}$ to get a homogeneous invertible matrix $A^{(2)}$ with all entries below $a^2_{2\rho(2)}$ zero. In doing this, the entries in the $\rho(1)$ column are unchanged. By iterating this process, working down row by row, we obtain a matrix $A^{(n-1)} = \big(a^n_{ij}\big) = T'A$, where

$$T' = \prod_{j=n-1}^{1} \prod_{i=n}^{j+1} e_{ij}\big(-{a^j}_{i\rho(j)}({a^j}_{j\rho(j)})^{-1}\big). \tag{1.11}$$

Note that

$$\deg\big(-{a^j}_{i\rho(j)}(a^j_{j\rho(j)})^{-1}\big) = \lambda + \delta_{\rho(j)} - \delta_i - (\lambda + \delta_{\rho(j)} - \delta_j) = \delta_j - \delta_i.$$

Therefore, in the product for T' each $e_{ij}\big(-{a^j}_{i\rho(j)}(a^j_{j\rho(j)})^{-1}\big) \in \mathcal{E}\ell_h$; it is also unipotent lower triangular, as $i > j$. Hence, T' is homgeneous of degree 0 and is unipotent lower triangular. Set

$$T \;=\; T'^{-1} \;=\; \prod_{j=1}^{n-1} \prod_{i=j+1}^{n} e_{ij}\big({a^j}_{i\rho(j)}({a^j}_{j\rho(j)})^{-1}\big),$$

which is again a homogeneous unipotent lower triangular matrix of degree zero. Our construction shows that in the matrix $A^{(n-1)} = T^{-1}A$ the leftmost nonzero entry in the i-th row is $a^n_{i\rho(i)}$ which is homogeneous in E, hence a unit. Furthermore, every entry below $a^n_{i\rho(i)}$ is zero. The function ρ of the indices is a permutation of $\{1,\dots,n\}$. Set

$$U = \operatorname{diag}(a^n_{1\,\rho(1)}, \dots, a^n_{n\,\rho(n)}), \tag{1.12}$$

where $\operatorname{diag}(u_1,\dots,u_n)$ denotes the $n \times n$ diagonal matrix with successive diagonal entries $u_1,\dots,u_n$. While U need not be homogeneous, its diagonal entries are all nonzero and homogeneous, hence units of E; so, U is invertible in S.

Clearly $U^{-1}A^{(n-1)} = U^{-1}T^{-1}A$ is a matrix whose leftmost nonzero entry in the i-th row is the 1 in the $(i,\rho(i))$-position. Furthermore, every entry below the $(i,\rho(i))$-entry is zero. Let $\pi = \rho^{-1}$, and let P_π be the permutation matrix of π. Since left multiplication by P_ρ $(= P_\pi^{-1})$ moves the i-th row to the $\rho(i)$-th row the matrix

$$V \;=\; P_\pi^{-1}U^{-1}T^{-1}A$$

is unipotent upper triangular. We have $A = TUP_\pi V$ which we show has the form asserted in the proposition.

As to the homogeneity of these matrices, we have seen that T is homogeneous with $\deg(T) = 0$. Observe next that U and P_π need not be homogeneous but UP_π is homogeneous. For, UP_π has its only nonzero entries $a^n_{i\rho(i)}$ in the $(i, \rho(i))$-position for $1 \leq i \leq n$ Thus, UP_π is obtainable from the homogeneous matrix $A^{(n-1)}$ by replacing some entries in $A^{(n-1)}$ by 0's. Hence, UP_π is homogeneous with

$$\deg(UP_\pi) = \deg\left(A^{(n-1)}\right) = \lambda = \deg(A).$$

Therefore, $V = (UP_\pi)^{-1}T^{-1}A$ is also homogeneous, with

$$\deg(V) = \deg\left((UP_\pi)^{-1}\right) + \deg(T^{-1}) + \deg(A) = 0.$$

Next we show that, $P_\pi V P_\pi^{-1}$ is also unipotent upper triangular, so $A = TUP_\pi V$ is in *strict* Bruhat normal form. We have

$$P_\pi V P_\pi^{-1} = U^{-1}T^{-1}AP_\pi^{-1}. \tag{1.13}$$

Recall the arrangement of entries in the columns of $U^{-1}T^{-1}A = U^{-1}A^{(n-1)}$. Since right multiplication of this matrix by $P_\pi^{-1} = P_\rho$ moves the $\rho(i)$-th column to the i-th column, $U^{-1}T^{-1}AP_\pi^{-1}$ is unipotent upper triangular. Thus, $P_\pi V P_\pi^{-1}$ is unipotent upper triangular by (1.13).

It remains only to show that this strict Bruhat decomposition is unique. (This uniqueness argument is valid for matrices over any ring.) Suppose $T_1U_1P_{\pi_1}V_1 = T_2U_2P_{\pi_2}V_2$, are two strict Bruhat normal forms for the same matrix. Then

$$U_1^{-1}T_1^{-1}T_2U_2 = P_{\pi_1}V_1V_2^{-1}P_{\pi_2}^{-1}. \tag{1.14}$$

Since $V_1V_2^{-1}$ is unipotent upper triangular, we can write $V_1V_2^{-1} = \mathbb{I}_n + N$, where $\mathbb{I}_n$ is the identity matrix and N is nilpotent upper triangular (i.e., an upper triangular matrix with zeros on the diagonal). Note that there is no position (i, j) where the matrices $\mathbb{I}_n$ and N both have a nonzero entry. Writing

$$P_{\pi_1}V_1V_2^{-1}P_{\pi_2}^{-1} = P_{\pi_1}P_{\pi_2}^{-1} + P_{\pi_1}NP_{\pi_2}^{-1}, \tag{1.15}$$

the summands on the right again have no overlapping nonzero entries. Therefore, as $P_{\pi_1}V_1V_2^{-1}P_{\pi_2}^{-1}$ is lower triangular by (1.14), each of $P_{\pi_1}P_{\pi_2}^{-1}$ and $P_{\pi_1}NP_{\pi_2}^{-1}$ must be lower triangular. Since $P_{\pi_1}P_{\pi_2}^{-1} = P_{\pi_1\pi_2^{-1}}$ is a lower triangular permutation matrix, it must be $\mathbb{I}_n$; thus, $\pi_1 = \pi_2$. Because of the nonoverlapping nonzero entries noted in (1.15), the diagonal entries of $P_{\pi_1}V_1V_2^{-1}P_{\pi_2}^{-1}$ must be 1's. But because the T_i are unipotent lower triangular and the U_i are diagonal, (1.14) shows that the diagonal entries of $P_{\pi_1}V_1V_2^{-1}P_{\pi_2}^{-1}$ coincide with those of the diagonal matrix $U_1^{-1}U_2$. Hence, $U_1^{-1}U_2 = \mathbb{I}_n$, i.e., $U_1 = U_2$.

Since $\pi_2 = \pi_1$, we can rewrite (1.14) as

$$U_1^{-1}T_1^{-1}T_2U_2 \; = \; P_{\pi_1}V_1P_{\pi_1}^{-1}\left(P_{\pi_2}V_2P_{\pi_2}^{-1}\right)^{-1}. \tag{1.16}$$

Since the decompositions are strict Bruhat, the right side of (1.16) is unipotent upper triangular while the left is lower triangular. This forces each side to be $\mathbb{I}_n$. Hence, $V_1 = V_2$, $U_1 = U_2$ (as we have seeen already), and $T_1 = T_2$. This proves the uniqueness. □

Remark. The first part of the uniqueness proof above (preceding (1.16)) shows that if A admits a Bruhat decomposition $A = TUP_\pi V$ (without the assumption on $P_\pi V P_\pi^{-1}$), then π and U are uniquely determined.

Theorem 1.2. *Let E be a graded division ring. Let $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$ where $\overline{\delta} = (\delta_1, \ldots, \delta_n)$, $\delta_i \in \Gamma$. Then there is a Dieudonnné determinant group homomorphism*

$$\det_{\mathsf{E}} \colon \mathrm{GL}_n^h(\mathsf{E})(\overline{\delta}) \longrightarrow \mathsf{E}_h^*/[\mathsf{E}_h^*, \mathsf{E}_h^*].$$

If $A \in \mathrm{GL}_n^h(\mathsf{E})(\overline{\delta}) = \mathsf{S}_h^*$ has strict Bruhat decomposition $A = TUP_\pi V$ with $U = \operatorname{diag}(u_1, \ldots, u_n)$ as in Proposition 1.1, then

$$\det_{\mathsf{E}}(A) = \operatorname{sgn}(\pi)\, u_1 \ldots u_n\, [\mathsf{E}_h^*, \mathsf{E}_h^*]. \tag{1.17}$$

Moreover, if $\det_0 \colon \mathsf{S}_0^* \to \mathsf{E}_0^*/[\mathsf{E}_0^*, \mathsf{E}_0^*]$ is the Dieudonné determinant for the semisimple ring S_0, then there is a commutative diagram

$$\begin{array}{ccc} \mathsf{S}_0^* & \xrightarrow{\det_0} & \mathsf{E}_0^*/[\mathsf{E}_0^*, \mathsf{E}_0^*] \\ \downarrow & & \downarrow \\ \mathsf{S}_h^* & \xrightarrow{\det_{\mathsf{E}}} & \mathsf{E}_h^*/[\mathsf{E}_h^*, \mathsf{E}_h^*]. \end{array} \tag{1.18}$$

Proof. Throughout the proof we assume that

$$(\delta_1, \ldots, \delta_n) = (\varepsilon_1, \ldots, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_2, \ldots, \varepsilon_k, \ldots, \varepsilon_k)$$

with each ε_ℓ occurring r_ℓ times and the cosets $\varepsilon_1 + \Gamma_{\mathsf{E}}, \ldots, \varepsilon_k + \Gamma_{\mathsf{E}}$ distinct in $\Gamma/\Gamma_{\mathsf{E}}$. There is no loss of generality with this assumption, in view of (1.7). Thus, any matrix B in S_0 is in block diagonal form, say with diagonal blocks $B_1, \ldots, B_k$, with each $B_\ell \in M_{r_\ell}(\mathsf{E}_0)$; we will identify

$$\mathsf{S}_0 = \mathbb{M}_{r_1}(\mathsf{E}_0) \times \cdots \times \mathbb{M}_{r_\ell}(\mathsf{E}_0),$$

by identifying B with $(B_1, \ldots, B_k)$, which we call the block decomposition of B.

We first assume that $\mathsf{E}_0 \neq \mathbb{F}_2$, the field with two elements; the exceptional case will be treated toward the end of the proof.

It is tempting to use formula (1.17) as the definition of $\det(A)$. But since it is difficult to show that the resulting function is a group homomorphism, we take a different tack.

We call a matrix M in S a *monomial matrix* if M has exactly one nonzero entry in each row and in each column, and if each nonzero entry lies in E^*. Clearly, M is a monomial matrix if and only if $M = UP$ where U is a diagonal matrix with every diagonal entry a unit, and P is a permutation matrix. Moreover, P and U are uniquely determined by M. The set $\mathcal{M}$

of all monomial matrices in S is a subgroup of S^*, and the set $\mathcal{M}^h$ of all homogeneous monomial matrices is a subgroup of S_h^*. Define a function

$$\Delta\colon \mathcal{M}^h \longrightarrow \mathsf{E}_h^*/[\mathsf{E}_h^*, \mathsf{E}_h^*]$$

by

$$\Delta(UP_\pi) = \operatorname{sgn}(\pi)\, u_1 u_2 \dots u_n\, [\mathsf{E}_h^*, \mathsf{E}_h^*], \quad \text{where } U = \operatorname{diag}(u_1, \dots, u_n).$$

This Δ is clearly well-defined, since UP_π determines U and the permutation matrix P_π for π in the symmetric group S_n. Note also that Δ is a group homomorphism. For, if $M = UP_\pi$ and $M' = U'P_{\pi'}$ with $U = \operatorname{diag}(u_1, \dots, u_n)$ and $U' = \operatorname{diag}(u_1', \dots, u_n')$, then

$$\begin{aligned} MM' &= \big(UP_\pi U' P_\pi^{-1}\big) P_{\pi\pi'} \\ &= \operatorname{diag}\big(u_1 u'_{\pi^{-1}(1)}, \dots, u_n u'_{\pi^{-1}(n)}\big) P_{\pi\pi'}. \end{aligned}$$

It follows immediately that $\Delta(MM') = \Delta(M)\Delta(M')$.

Recall that $\mathsf{S}_0 = \mathbb{M}_{r_1}(\mathsf{E}_0) \times \dots \times \mathbb{M}_{r_k}(\mathsf{E}_0)$. Each component $\mathrm{GL}_{r_\ell}(\mathsf{E}_0)$ of S_0^* has a Dieudonné determinant function $\det_\ell$ mapping it to $\mathsf{E}_0^*/[\mathsf{E}_0^*, \mathsf{E}_0^*]$, and these maps are used to define the Dieudonné determinant

$$\det_0\colon \mathsf{S}_0^* \to \mathsf{E}_0^*/[\mathsf{E}_0^*, \mathsf{E}_0^*]$$

for the semisimple ring S_0 by

$$\det_0(B_1, \dots, B_k) \,=\, \prod_{\ell=1}^k \det_\ell(B_\ell). \tag{1.19}$$

Set $\overline{\det_0}$ to be the composition $\mathsf{S}_0^* \xrightarrow{\det_0} \mathsf{E}_0^*/[\mathsf{E}_0^*, \mathsf{E}_0^*] \longrightarrow \mathsf{E}_h^*/[\mathsf{E}_h^*, \mathsf{E}_h^*]$. We claim that if $M \in \mathcal{M}^h$ has degree 0, then

$$\Delta(M) \,=\, \overline{\det_0}(M). \tag{1.20}$$

For, as $M \in \mathsf{S}_0$, it follows that U and P lie in S_0, and when we view $M = (M_1, \dots, M_k)$, $U = (U_1, \dots, U_k)$, $P = (P_1, \dots, P_k)$, we have: in each $M_{r_\ell}(\mathsf{E}_0)$, U_ℓ is a diagonal, say $U_\ell = \operatorname{diag}(u_{\ell 1}, \dots, u_{\ell r_\ell})$, and P_ℓ a permutation matrix, say $P_\ell = P_{\pi_\ell}$ for some $\pi_\ell \in S_{r_\ell}$, and $M_\ell = U_\ell P_\ell$. So, M_ℓ is a monomial matrix in $\mathbb{M}_{r_\ell}(\mathsf{E}_0)$. Since each M_ℓ has (nonstrict) Bruhat decomposition $M_\ell = \mathbb{I}_{r_\ell} U_\ell P_{\pi_\ell} \mathbb{I}_{r_\ell}$ in $\mathrm{GL}_{r_\ell}(\mathsf{E}_0)$,[D, §20, Def. 1, Cor. 1] yields $\det_\ell(M_\ell) = \operatorname{sgn}(\pi_\ell)\, u_{\ell 1} \dots u_{\ell r_\ell}\, [\mathsf{E}_0^*, \mathsf{E}_0^*]$. Moreover, as $P = P_\pi$, where $\pi = (\pi_1, \dots, \pi_k)$ when we view $S_{r_1} \times \dots \times S_{r_k} \subseteq S_n$, we have $\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi_1) \dots \operatorname{sgn}(\pi_k)$. Thus,

$$\det_0(M) = \prod_{\ell=1}^k \big(\operatorname{sgn}(\pi_\ell)\, u_{\ell 1} \dots u_{\ell r_\ell}\big)[\mathsf{E}_0^*, \mathsf{E}_0^*] = \operatorname{sgn}(\pi) \prod_{\ell=1}^k \big(u_{\ell 1} \dots u_{\ell r_\ell}\big)[\mathsf{E}_0^*, \mathsf{E}_0^*],$$

which yields (1.20).

We next claim that every matrix A in S_h^* is expressible (not uniquely) in the form $A = CM$, where $C \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$ and $M \in \mathcal{M}^h$. For this, consider first an elementary matrix $e \in \mathcal{E}\ell_h$. The block form of e is $(e_1, \dots, e_k)$, where clearly one e_ℓ is an elementary matrix in $\mathbb{M}_{r_\ell}(\mathsf{E}_0)$ and all the other blocks are identity matrices. Since every elementary matrix in $\mathbb{M}_{r_\ell}(\mathsf{E}_0)$ lies in $[\mathrm{GL}_{r_\ell}(\mathsf{E}_0), \mathrm{GL}_{r_\ell}(\mathsf{E}_0)]$ by [D, §20, Th. 3, Th. 4(i)] (as $\mathsf{E}_0 \neq \mathbb{F}_2$ by assumption) it follows thet $e \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$. Now, take any $A \in \mathsf{S}_h^*$, with its strict Bruhat decomposition $A = TUP_\pi V$ as in Proposition 1.1. Then, T is a product of elementary matrices in S_0^*; so $T \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$. Moreover the transpose V^t of V is unipotent lower triangular of degree 0. The unique strict Bruhat normal form of V^t is clearly $V^t = V^t \mathbb{I}_n P_{\mathrm{id}} \mathbb{I}_n$. Hence, Proposition 1.1 shows that V^t is a product of matrices in $\mathcal{E}\ell_h$. Therefore, $V^t \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$, which implies that $V \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$. Now, let $M = UP_\pi \in \mathcal{M}^h$, and let $C = TMVM^{-1} = AM^{-1}$. Because $V \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$ and M is homogeneous, $MVM^{-1} \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$. (For take any $Z_1, Z_2 \in \mathsf{S}_0^*$. Then, $M[Z_1, Z_2]M^{-1} = [MZ_1M^{-1}, MZ_2M^{-1}] \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$, as each $MZ_iM^{-1} \in \mathsf{S}_0^*$.) Hence, $C \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$, so $A = CM$, as claimed.

Define $\det_{\mathsf{E}} \colon \mathsf{S}_h^* \to \mathsf{E}_h^*/[\mathsf{E}_h^*, \mathsf{E}_h^*]$ by

$$\det_{\mathsf{E}}(CM) = \Delta(M), \quad \text{for any } C \in [\mathsf{S}_0^*, \mathsf{S}_0^*],\ M \in \mathcal{M}^h.$$

To see that $\det_{\mathsf{E}}$ is well-defined, suppose $C_1M_1 = C_2M_2$ with $C_1, C_2 \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$ and $M_1, M_2 \in \mathcal{M}^h$. Then,

$$M_1M_2^{-1} = C_1^{-1}C_2 \in [\mathsf{S}_0^*, \mathsf{S}_0^*].$$

Hence, $\deg(M_1M_2^{-1}) = 0$ and $\det_0(M_1M_2^{-1}) = \det_0(C_1^{-1}C_2) = 1$, which implies that also $\overline{\det_0}(M_1M_2^{-1}) = 1$. So, by (1.20) $\Delta(M_1M_2^{-1}) = 1$. Since Δ is a group homomorphism, it follows that $\Delta(M_1) = \Delta(M_2)$. Thus, $\det_{\mathsf{E}}$ is well-defined. To see that it is a group homomorphism, take any $C, C' \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$, and $M, M' \in \mathcal{M}^h$. Then,

$$(CM)(C'M') = \big(C(MC'M^{-1})\big)(MM').$$

Since $C' \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$, we have $MC'M^{-1} \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$, as noted above; so, $C(MC'M^{-1}) \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$. Also, $MM' \in \mathcal{M}^h$. Hence,

$$\det_{\mathsf{E}}\big((CM)(C'M')\big) = \Delta(MM') = \Delta(M)\Delta(M') = \det_{\mathsf{E}}(CM)\det_{\mathsf{E}}(C'M');$$

so, $\det_{\mathsf{E}}$ is a group homomorphism. For $A \in \mathsf{S}_h^*$ with strict Bruhat decomposition $A = TUP_\pi V$, we have seen that $A = CM$ with $M = UP_\pi \in \mathcal{M}^h$ and $C = TMVM^{-1} \in [\mathsf{S}_0^*, \mathsf{S}_0^*]$, so $\det_{\mathsf{E}}(A) = \Delta(M)$, which yields formula (1.17).

We now dispose of the exceptional case where $\mathsf{E}_0 = \mathbb{F}_2$. When this holds, replace $[\mathsf{S}_0^*, \mathsf{S}_0^*]$ in the proof by S_0^*, and the argument goes through. Observe that now if $M \in \mathcal{M}^h$ with $\deg(M) = 0$, then $\Delta(M) = 1$. For, all nonzero entries of M then lie in $\mathsf{E}_0^* = \{1\}$ and the $\mathrm{sgn}(\pi)$ term in the formula for $\Delta(M)$ drops out as $\mathrm{char}(\mathsf{E}_0) = 2$. This replaces use of (1.20) in the proof. There is no need to invoke $\det_0$, which is in fact trivial here as $|\mathsf{E}_0^*| = 1$. The argument that a homogeneous elementary matrix e lies in $[\mathsf{S}_0^*, \mathsf{S}_0^*]$ is replaced by the tautology that $e \in \mathsf{S}_0^*$.

Turning to diagram (1.18), take any $A \in \mathsf{S}_0^*$, with strict Bruhat decomposition $A = TUP_\pi V$. Then, $\det(UP_\pi) = \deg(A) = 0$, so U and P_π lie in S_0^*. Take the block decomposition $A = (A_1, \ldots, A_k)$ and likewise for

T, U, P, V. Then, $P_\pi = (P_{\pi_1}, \ldots, P_{\pi_k})$, where $\pi = (\pi_1, \ldots, \pi_k)$ when we view $S_{r_1} \times \cdots \times S_{r_k} \subseteq S_n$. Note that $A_\ell = T_\ell U_\ell P_{\pi_\ell} V_\ell$ is the strict Bruhat decomposition of A_ℓ in $\mathrm{GL}_{r_\ell}(\mathsf{E}_0)$ for $\ell = 1, 2, \ldots, k$. So,

$$\det_0(A) = \prod_{\ell=1}^{k} \det_\ell(A_\ell) = \prod_{\ell=1}^{k} \det_\ell(U_\ell P_{\pi_\ell}) = \det_0(UP_\pi).$$

Hence, invoking (1.20) for $UP_\pi \in \mathcal{M}^h$,

$$\overline{\det_0}(A) = \overline{\det_0}(UP_\pi) = \det_{\mathsf{E}}(UP_\pi) = \det_{\mathsf{E}}(A),$$

showing that diagram (1.18) is commutative. $\square$

In a matrix ring $\mathbb{M}_r(R)$ over any ring R, for any $a \in R$ we write $\mathbb{D}_r(a)$ for the diagonal matrix $\mathrm{diag}(1, \ldots, 1, a)$.

Proposition 1.3. *Let* $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$ *with*

$$(\delta_1, \ldots, \delta_n) = (\varepsilon_1, \ldots, \varepsilon_1, \ldots, \varepsilon_k, \ldots, \varepsilon_k)$$

as in the proof of Theorem 1.2. If Γ_{E} *is* n*-torsion free, then*

$$\ker(\det_{\mathsf{E}}) = \big\langle \mathcal{E}\ell_h \big\rangle \cdot \big\{ \big(\mathbb{D}_{r_1}(c_1), \ldots, \mathbb{D}_{r_k}(c_k)\big) \mid \textit{each } c_i \in \mathsf{E}_0^* \textit{ and } c_1 \ldots c_k \in [\mathsf{E}_h^*, \mathsf{E}_h^*] \big\}.$$

Here, $\mathcal{E}\ell_h$ denotes the group of homogeneous elementary matrices, as in (1.9), and $\big(\mathbb{D}_{r_1}(c_1), \ldots, \mathbb{D}_{r_k}(c_k)\big)$ denotes the block diagonal matrix with diagonal blocks $\mathbb{D}_{r_1}(c_1), \ldots, \mathbb{D}_{r_k}(c_k)$.

Proof. Suppose $A \in \mathsf{S}_h^*$ and $\deg(A) = \lambda \neq 0$, and let $A = TUP_\pi V$ be the strict Bruhat decomposition of A, with $U = \mathrm{diag}(u_1, \ldots, u_n)$. Since the monomial matrix UP_π is homogeneous of degree λ with $(i, \pi^{-1}(i))$-entry u_i, we have $\deg(u_i) = \lambda + \delta_i - \delta_{\pi^{-1}(i)}$. So $\deg(\mathrm{sgn}(\pi) u_1 \ldots u_n) = n\lambda \neq 0$, as Γ_{E} is n-torsion free. But, $[\mathsf{S}_h^*, \mathsf{S}_h^*] \subseteq \mathsf{S}_0^*$, as every commutator of homogeneous matrices has degree 0. Hence, $\det_{\mathsf{E}}(A) \neq 1$. Thus, $\ker(\det_{\mathsf{E}}) \subseteq \mathsf{S}_0^*$.

Note that every homogeneous elementary matrix e has stict Bruhat decomposition $e = e\mathbb{I}_n P_{\mathrm{id}} \mathbb{I}_n$ or $e = \mathbb{I}_n \mathbb{I}_n P_{\mathrm{id}} e$. In either case, $\det_{\mathsf{E}}(e) = 1$. This shows that $\langle \mathcal{E}\ell_h \rangle \subseteq \ker(\det_{\mathsf{E}})$.

Now take $A \in \mathsf{S}_0^*$ with block decomposition $(A_1, \ldots, A_k)$. By [D, §20, Th. 2], each A_ℓ is expressible in $\mathrm{GL}_{r_\ell}(\mathsf{E}_0)$ as $A_\ell = B_\ell \mathbb{D}_{r_l}(c_\ell)$ for some $c_\ell \in \mathsf{E}_0^*$, where B_ℓ is a product of elementary matrices in $M_{r_\ell}(\mathsf{E}_0)$. So, $(\mathbb{I}_{r_1}, \ldots, \mathbb{I}_{r_{\ell-1}}, B_\ell, \mathbb{I}_{r_{\ell+1}}, \ldots, \mathbb{I}_{r_k})$ is a product of the corresponding homogeneous elementary matrices in S_0. Hence $A = BD$, with

$$B = (B_1, \ldots, B_k) \in \langle \mathcal{E}\ell_h \rangle \quad \text{and} \quad D = (\mathbb{D}_{r_1}(c_1), \ldots \mathbb{D}_{r_k}(c_k)),$$

which is a diagonal matrix in S_0. Thus,

$$\det_{\mathsf{E}}(A) = \det_{\mathsf{E}}(B) \det_{\mathsf{E}}(D) = c_1 \ldots c_k \, [\mathsf{E}_h^*, \mathsf{E}_h^*].$$

So, $A \in \ker(\det_{\mathsf{E}})$ if and only if $c_1 \ldots c_k \in [\mathsf{E}_h^*, \mathsf{E}_h^*]$, which yields the proposition. $\square$

Recall that a graded division ring E with center T is said to be *unramified* if $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{T}}$. In Theorem 2.4(iv) below we will show that homogeneous SK_1 of unramified graded division algebras is Morita invariant. For nonstable K_1, we have the following:

Corollary 1.4. *Let E be a graded division ring and let $\mathsf{S} = \mathbb{M}_n(\mathsf{E})$ with unshifted grading. Suppose Γ_{E} is n-torsion free, E is unramified, and $\mathbb{M}_n(\mathsf{E}_0) \neq \mathbb{M}_2(\mathbb{F}_2)$. Then $\det_{\mathsf{E}}$ induces a group monomorphism*

$$\mathrm{GL}_n^h(\mathsf{E})/[\mathrm{GL}_n^h(\mathsf{E}), \mathrm{GL}_n^h(\mathsf{E})] \ \hookrightarrow\ \mathsf{E}_h^*/[\mathsf{E}_h^*, \mathsf{E}_h^*].$$

Proof. We need to show that $\ker(\det_{\mathsf{E}}) = [\mathrm{GL}_n^h(\mathsf{E}), \mathrm{GL}_n^h(\mathsf{E})]$. The inclusion $\supseteq$ is clear as $\det_{\mathsf{E}}$ is a group homomorphism mapping into an abelian group. For the reverse inclusion, note that as S_0 is simple, Proposition 1.3 says

$$\ker(\det_{\mathsf{E}}) = \big\langle \mathcal{E}\ell_h \big\rangle \cdot \big\{ \mathbb{D}_n(a) \mid a \in [\mathsf{E}_h^*, \mathsf{E}_h^*] \big\}.$$

Because $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{T}}$ where T is the center of E, we have $\mathsf{E}_h^* = \mathsf{T}_h^* \cdot \mathsf{E}_0^*$, hence $[\mathsf{E}_h^*, \mathsf{E}_h^*] = [\mathsf{E}_0^*, \mathsf{E}_0^*]$. Thus,

$$\big\{ \mathbb{D}_n(a) \mid a \in [\mathsf{E}_h^*, \mathsf{E}_h^*] \big\} \ =\ [\mathbb{D}_n(\mathsf{E}_0^*), \mathbb{D}_n(\mathsf{E}_0^*)] \ \subseteq\ [\mathrm{GL}_n^h(\mathsf{E}), \mathrm{GL}_n^h(\mathsf{E})].$$

Also, as $\mathsf{S}_0 = \mathbb{M}_n(\mathsf{E}_0)$, the homogeneous elementary matrices of S, which all have degree 0, are the same as the elementary matrices of $\mathbb{M}_n(\mathsf{E}_0)$; since E_0 is a division ring, by [D, §20, Th. 4, Lemma 4] these all lie in $[\mathrm{GL}_n(\mathsf{E}_0), \mathrm{GL}_n(\mathsf{E}_0)] \ \subseteq\ [\mathrm{GL}_n^h(\mathsf{E}), \mathrm{GL}_n^h(\mathsf{E})]$, as $\mathbb{M}_n(\mathsf{E}_0) \neq \mathbb{M}_2(\mathbb{F}_2)$. Hence, $\ker(\det_{\mathsf{E}}) \subseteq [\mathrm{GL}_n^h(\mathsf{E}), \mathrm{GL}_n^h(\mathsf{E})]$, completing the proof. $\square$

2. Homogeneous $\mathbf{SK_1}$

Throughout this section we consider graded division algebras E, i.e., E is a graded division ring which is finite-dimensional as a graded vector space over its center T. In addition, as we are interested in graded division algebras arising from valued division algebras, we assume that the abelian group Γ (which contains Γ_{E}) is torsion free. The assumption on Γ implies that every unit in E is actually homogeneous, so $\mathsf{E}_h^* = \mathsf{E}^*$. This assumption also implies that E has no zero divisors. (These properties follow easily from the fact that the torsion-free abelian group Γ_{E} can be made into a totally ordered group, see, e.g. [HwW, p. 78].) Hence, E has a quotient division ring obtained by central localization, $q(\mathsf{E}) = \mathsf{E} \otimes_{\mathsf{T}} q(\mathsf{T})$, where $q(\mathsf{T})$ is the quotient field of the integral domain T. In addition, every graded module M over E is a free module with well-defined rank; we thus call M a graded vector space over E, and write $\dim_{\mathsf{E}}(\mathsf{M})$ for $\mathrm{rank}_{\mathsf{E}}(\mathsf{M})$. This applies also for graded modules over T, which is a commutative graded division ring. We write $[\mathsf{E} : \mathsf{T}]$ for $\dim_{\mathsf{T}}(\mathsf{E})$, and $\mathrm{ind}(\mathsf{E}) = \sqrt{[\mathsf{E} : \mathsf{T}]}$. Clearly, $[\mathsf{E} : \mathsf{T}] = [q(\mathsf{E}) : q(\mathsf{T})]$, so $\mathrm{ind}(\mathsf{E}) = \mathrm{ind}(q(\mathsf{E})) \in \mathbb{N}$. Moreover, in ([B, Prop. 5.1] and [HwW, Cor. 1.2] it was observed that E is an Azumaya algebra over T.

In general for an Azumaya algebra A of constant rank m^2 over a commutative ring R, there is a commutative ring S faithfully flat over R which splits A, i.e., $A \otimes_R S \cong \mathbb{M}_m(S)$. For $a \in A$, considering $a \otimes 1$ as an element of $\mathbb{M}_m(S)$, one then defines the *reduced characteristic polynomial*, $\mathrm{char}_A(X, a)$, the *reduced trace*, $\mathrm{Trd}_A(a)$, and the *reduced norm*, $\mathrm{Nrd}_A(a)$, of a by

$$\begin{aligned}\mathrm{char}_A(X, a) &= \det(X\mathbb{I}_m - (a \otimes 1)) \\ &= X^m - \mathrm{Trd}_A(a)X^{m-1} + \cdots + (-1)^m \mathrm{Nrd}_A(a).\end{aligned}$$

in $S[X]$. Using descent theory, one shows that $\mathrm{char}_A(X, a)$ is independent of S and of the choice of R-isomorphism $A \otimes_R S \cong \mathbb{M}_m(S)$, and that $\mathrm{char}_A(X, a)$ lies in $R[X]$; furthermore, the element a is invertible in A if and only if $\mathrm{Nrd}_A(a)$ is invertible in R (see Knus [K, III.1.2] and Saltman [S, Th. 4.3]). Let $A^{(1)}$ denote the multiplicative group of elements of A of reduced norm 1. One then defines the *reduced Whitehead group* of A to be $\mathrm{SK}_1(A) = A^{(1)}/A'$, where $A' = [A^*, A^*]$ denotes the commutator subgroup of the group A^* of units of A. For any integer $n \geq 1$, the matrix ring $\mathbb{M}_n(A)$ is also an Azumaya algebra over R. One says that SK_1 is *Morita invariant* for A if

$$\mathrm{SK}_1(\mathbb{M}_n(A)) \cong \mathrm{SK}_1(A) \quad \text{for all } n \in \mathbb{N}.$$

Specializing to the case of a graded division algebra E and the graded matrix algebra $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \ldots, \delta_n) \in \Gamma^n$, we have the reduced Whitehead group

$$\text{(2.1)} \qquad \mathrm{SK}_1(\mathsf{S}) = \mathsf{S}^{(1)}/[\mathsf{S}^*, \mathsf{S}^*], \text{ where } \mathsf{S}^{(1)} = \big\{x \in \mathsf{S}^* \mid \mathrm{Nrd}_{\mathsf{S}}(x) = 1\big\}.$$

Here S^* is the group of units of the ring $\mathbb{M}_n(\mathsf{E})$ (thus the shifted grading on S does not affect $\mathrm{SK}_1(\mathsf{S})$). Restricting to the homogeneous elements of S we define

$$\text{(2.2)} \qquad \mathrm{SK}_1^h(\mathsf{S}) = \mathsf{S}_h^{(1)}/[\mathsf{S}_h^*, \mathsf{S}_h^*], \quad \text{where} \quad \mathsf{S}_h^{(1)} = \big\{x \in \mathsf{S}_h^* \mid \mathrm{Nrd}_{\mathsf{S}}(x) = 1\big\}.$$

To distinguish these two groups, we call the second one the *homogeneous reduced Whitehead group* of S. These groups coincide for $n = 1$, i.e, $\mathrm{SK}_1^h(\mathsf{E}) = \mathrm{SK}_1(\mathsf{E})$. For, $\mathsf{E}^* = \mathsf{E}_h^*$, as noted above. (See [$\mathrm{HW}_1$] for an extensive study of SK_1 of graded division algebras.)

The question naturally arises whether $\mathrm{SK}_1(A)$ is Morita invariant for an Azumaya algebra A. When A is a central simple algebra this is known to be the case (see, e.g., [D, §22, Cor. 1] or [P, §16.5, Prop. b]). We will answer the analogous question for homogeneous reduced Whitehead groups when A is a graded division algebra E by establishing an exact sequence relating $\mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E}))$ and $\mathrm{SK}_1(\mathsf{E})$ (Theorem 2.4) and producing examples showing that they sometimes differ (Example 2.5); thus, SK_1^h is not Morita invariant. We will see in fact that, as n varies, $\mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E}))$ depends only on the congruence class of n modulo a constant e dividing the ramification index of E over its center. Furthermore, $\mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E})) \cong \mathrm{SK}_1(\mathsf{E})$ whenever n is prime to e.

A major reason why $\mathrm{SK}_1^h(\mathsf{S})$ is more tractable than $\mathrm{SK}_1(\mathsf{S})$ for $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$ is that $\mathsf{S}_h^{(1)}$ consists of homogeneous elements of degree 0, as we next show. This will allow us to use the Dieudonné determinant for the semisimple algebra S_0 to relate $\mathrm{SK}_1^h(\mathsf{S})$ to $\mathrm{SK}_1(\mathsf{E})$.

Lemma 2.1. *With the hypotheses on* E *as above, let* $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$ *for* $\overline{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$*. Let* T *be the center of* E*. Then,* $\mathrm{Nrd}_{\mathsf{S}}(\mathsf{S}_\lambda) \subseteq \mathsf{T}_{ns\lambda}$ *for any* $\lambda \in \Gamma_{\mathsf{S}}$*, where* $s = \mathrm{ind}(\mathsf{E})$*. Hence,* $\mathsf{S}_h^{(1)} \subseteq \mathsf{S}_0^*$*.*

Proof. For calculating $\mathrm{Nrd}_{\mathsf{S}}$, we split E using a graded faithfully flat extension of its center T, in order to preserve the graded structure. For this we employ a maximal graded subfield L of E. Associated to the graded field T there is a graded Brauer group $\mathrm{grBr}(\mathsf{T})$ of equivalence classes of graded division algebras with center T. See [HwW, TW$_1$] for properties of graded Brauer groups. In particular, there is a commutative diagram of scalar extension homomorphisms,

$$\begin{tikzcd} \mathrm{grBr}(\mathsf{T}) \arrow[r] \arrow[d, hook, "-\otimes_{\mathsf{T}} q(\mathsf{T})"] & \mathrm{grBr}(\mathsf{L}) \arrow[d, hook, "-\otimes_{L} q(\mathsf{L})"] \\ \mathrm{Br}(q(\mathsf{T})) \arrow[r] & \mathrm{Br}(q(\mathsf{L})), \end{tikzcd}$$

where the vertical maps are injective. If L is a maximal graded subfield of E, then $[\mathsf{L}:\mathsf{T}] = \mathrm{ind}(\mathsf{E})$ by the graded Double Centralizer Theorem [HwW, Prop. 1.5]. Since $[q(\mathsf{L}):q(\mathsf{T})] = [\mathsf{L}:\mathsf{T}] = \mathrm{ind}(\mathsf{E}) = \mathrm{ind}(q(\mathsf{E}))$, it follows that $q(\mathsf{L})$ is a maximal subfield of the division ring $q(\mathsf{E})$, which is known to be a splitting field for $q(\mathsf{E})$ (see §9, Cor. 5 in [D]). The commutativity of the diagram above and the injectivity of vertical arrows imply that L splits E as well, i.e., $\mathsf{E} \otimes_{\mathsf{T}} \mathsf{L} \cong_{\mathrm{gr}} \mathbb{M}_s(\mathsf{L})(\overline{\gamma})$, for some $\overline{\gamma} = (\gamma_1, \dots, \gamma_s) \in \Gamma^s$, where $s = \mathrm{ind}(\mathsf{E})$. Moreover L is a free, hence faithfully flat, T-module.

The graded field L also splits $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$. Indeed,

$$\begin{aligned} \mathsf{S} \otimes_{\mathsf{T}} \mathsf{L} &\cong_{\mathrm{gr}} \mathbb{M}_n(\mathsf{E})(\overline{\delta}) \otimes_{\mathsf{T}} \mathsf{L} \cong_{\mathrm{gr}} \mathbb{M}_n(\mathsf{E} \otimes_{\mathsf{T}} \mathsf{L})(\overline{\delta}) \\ &\cong_{\mathrm{gr}} \mathbb{M}_n\big(\mathbb{M}_s(\mathsf{L})(\overline{\gamma})\big)(\overline{\delta}) \cong_{\mathrm{gr}} \mathbb{M}_{sn}(\mathsf{L})(\overline{\omega}), \end{aligned}$$

where $\overline{\omega} = (\gamma_i + \delta_j)$, $1 \le i \le s$, $1 \le j \le n$. For a homogeneous element a of S with $\deg(a) = \lambda$, its image $a \otimes 1$ in $\mathsf{S} \otimes_{\mathsf{T}} \mathsf{L}$ is also homogeneous of degree λ, and $\mathrm{Nrd}_{\mathsf{S}}(a) = \det(a \otimes 1)$. But, as noted in (1.8) above, $\det(s \otimes 1) \in \mathsf{T}_{ns\lambda}$. Thus, $\mathrm{Nrd}(\mathsf{S}_\lambda) \subseteq \mathsf{T}_{ns\lambda}$. If $\mathrm{Nrd}_{\mathsf{S}}(a) = 1 \in \mathsf{T}_0$, then $\deg(a) = 0$, as Γ is assumed torsion free. Thus, $\mathsf{S}_h^{(1)} \subseteq \mathsf{S}_0$. □

In order to establish a connection between the homogeneous $\mathrm{SK}_1^h(\mathsf{S})$ and $\mathrm{SK}_1(\mathsf{E})$ we need to relate the reduced norm of S to that of S_0, which we do in the next lemma. Recall that S_0 is a semisimple ring (see (1.6)). For a division algebra D, one defines the reduced norm map on a semisimple algebra $\mathbb{M}_{r_1}(D) \times \cdots \times \mathbb{M}_{r_k}(D)$ finite-dimensional over its center as the product of reduced norms of the simple factors.

Lemma 2.2. *With the hypotheses on the graded division algebra* E *as above, let* $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$ *for* $\overline{\delta} = (\delta_1, \dots, \delta_n) \subseteq \Gamma^n$. *Let* T *be the center of* E. *Then, for* $a \in \mathsf{S}_0$

$$\mathrm{Nrd}_{\mathsf{S}}(a) \;=\; N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{S}_0}(a))^d, \tag{2.3}$$

where $d = \mathrm{ind}(\mathsf{E})/\big(\,\mathrm{ind}(\mathsf{E}_0)\,[Z(\mathsf{E}_0) : \mathsf{T}_0]\big)$.

Here $Z(\mathsf{E}_0)$ denotes the center of E_0, which is a field finite-dimensional and abelian Galois over T_0. Also, $N_{Z(\mathsf{E}_0)/\mathsf{T}_0}$ denotes the field norm from $Z(\mathsf{E}_0)$ to T_0.

Proof. After applying a graded isomorphism, we may assume $(\delta_1, \dots, \delta_n)$ has the form $(\varepsilon_1, \dots, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k)$ as in (1.7) above. Then, $\mathsf{S}_0 = \mathbb{M}_{r_1}(\mathsf{E}_0) \times \cdots \times \mathbb{M}_{r_k}(\mathsf{E}_0)$. Let $a = (a_1, ..., a_k) \in \mathsf{S}_0$ with each $a_i \in \mathbb{M}_{r_i}(\mathsf{E}_0)$. That is, a is in block diagonal form with diagonal blocks $a_1, \dots, a_k$; so, $\mathrm{Nrd}_{\mathsf{S}_0}(a) = \prod_{i=1}^k \mathrm{Nrd}_{\mathbb{M}_{r_i}}(\mathsf{E}_0)(a_i)$. We thus need to prove that:

$$\mathrm{Nrd}_{\mathsf{S}}(a) \;=\; \prod_{i=1}^{k} N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathbb{M}_{r_i}(\mathsf{E}_0)}(a_i))^d. \tag{2.4}$$

Formula (2.3) is known for $n = 1$, i.e., $\mathsf{S} = \mathsf{E}$, by [HW_1, Prop. 3.2]. The further fact needed here is that for any b in $\mathbb{M}_n(\mathsf{E})$ in block triangular form, say with diagonal blocks $b_1, \dots, b_m$, where $b_j \in \mathbb{M}_{t_j}(\mathsf{E})$, and $t_1 + \cdots + t_m = n$, we have

$$\mathrm{Nrd}_{\mathbb{M}_n(\mathsf{E})}(b) \;=\; \prod_{j=1}^{m} \mathrm{Nrd}_{\mathbb{M}_{t_j}(\mathsf{E})}(b_j). \tag{2.5}$$

Indeed, if we split E by extending scalars, say $\mathsf{E} \otimes_{\mathsf{T}} \mathsf{L} \cong \mathbb{M}_s(\mathsf{L})$ for some graded field L, then

$$\mathbb{M}_n(\mathsf{E}) \otimes_{\mathsf{T}} \mathsf{L} \cong \mathbb{M}_{ns}(\mathsf{L});$$

the matrix for $b \otimes 1$ is again in block triangular form with its diagonal blocks coming from the splitting of the diagonal blocks of b. So formula (2.5) follows from the determinant formula for matrices in block triangular form.

Formula (2.5) applied to the block diagonal matrix a shows that it suffices to verify that

$$\mathrm{Nrd}_{\mathbb{M}_{r_i}(\mathsf{E})}(a_i) \;=\; N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathbb{M}_{r_i}(\mathsf{E}_0)}(a_i))^d \tag{2.6}$$

for each i. Formula (2.6) is clearly multiplicative in a_i. Moreover, it holds for any triangular matrix in $\mathbb{M}_{r_i}(\mathsf{E}_0)$ by (2.5) with $t_1 = \cdots = t_m = 1$ and $m = r_i$, since it holds when $\mathsf{S} = \mathsf{E}$. But, we can always write $a_i = e_{i1} c_i e_{i2}$, where e_{i1}, e_{i2} are products of elementary matrices in $\mathbb{M}_{r_i}(\mathsf{E}_0)$ and c_i is a diagonal matrix. This is just another way of saying that we can diagonalize a_i in $\mathbb{M}_{r_i}(\mathsf{E}_0)$ by elementary row and column operations. Thus, formula (2.6) holds for a_i because a_i is a product of triangular matrices. This yields (2.3). $\square$

In producing the first examples of division algebras D with nontrivial reduced Whitehead groups, Platonov worked in [Pl] with division algebras over twice iterated Laurent series over a global field. Ershov later in [E] generalized and systematized Platonov's approach, by working with division algebras over arbitrary Henselian valued fields. Ershov encapsulated his results in a commutative diagram with exact rows and columns which related $\mathrm{SK}_1(D)$ to various quantities involving the residue division algebra $\overline{D}$ and the value group Γ_D for the valuation on D. More recently it was shown in [HW_1, Th. 3.4] that there is a commutative diagram analogous to Ershov's for computing $\mathrm{SK}_1(\mathsf{E})$, where E is a graded division algebra. It was also shown in [HW_1, Th. 4.8] that Ershov's results for D over a Henselian field could be deduced from the corresponding graded ones by proving that $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathrm{gr}(D))$, where $\mathrm{gr}(D)$ is the associated graded division algebra of the valued division algebra D. The diagram for $\mathrm{SK}_1(\mathsf{E})$ is the vertical E-plane in the following diagram (2.7).

(2.7)

$$
\begin{tikzcd}
 & & & 1 \arrow[dd] & & & \\
 & & 1 \arrow[dd] & & & & \\
 & \mathrm{SK}_1(\mathsf{E}_0) \arrow[rr] & & \ker \widetilde{N}_{\mathsf{E}}/[\mathsf{E}_0^*,\mathsf{E}^*] \arrow[rr, "\mathrm{Nrd}_{\mathsf{E}_0}"] \arrow[dd] & & \widehat{H}^{-1}(G,\mathrm{Nrd}_{\mathsf{E}_0}(\mathsf{E}_0^*)) \arrow[r] & 1 \\
 \mathrm{SK}_1(\mathsf{S}_0) \arrow[rr] \arrow[ur, "\cong"] & & \ker \widetilde{N}_{\mathsf{S}}/[\mathsf{S}_h^*,\mathsf{S}_0^*] \arrow[rr] \arrow[ur, "\cong"] \arrow[dd] & & \widehat{H}^{-1}(G,\mathrm{Nrd}_{\mathsf{S}_0}(\mathsf{S}_0^*)) \arrow[r] \arrow[ur, "\cong"] & 1 & \\
 & \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}\wedge\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \arrow[rr] & & \mathsf{E}^{(1)}/[\mathsf{E}_0^*,\mathsf{E}^*] \arrow[rr] \arrow[dd] & & \mathrm{SK}_1(\mathsf{E}) \arrow[r] & 1 \\
 \Gamma_{\mathsf{S}}/\Gamma_{\mathsf{T}}\wedge\Gamma_{\mathsf{S}}/\Gamma_{\mathsf{T}} \arrow[rr] \arrow[ur, "\eta_n"] & & \mathsf{S}_h^{(1)}/[\mathsf{S}_h^*,\mathsf{S}_0^*] \arrow[rr] \arrow[ur, "\cong"] \arrow[dd, "\widetilde{N}"] & & \mathrm{SK}_1^h(\mathsf{S}) \arrow[r] \arrow[ur] & 1 & \\
 & & & \mu_\delta(\mathsf{T}_0)\cap\widetilde{N}(\mathsf{E}_0^*) \arrow[dd] & & & \\
 & & \mu_\delta(\mathsf{T}_0)\cap\widetilde{N}(\mathsf{S}_0^*) \arrow[ur, "\cong"] \arrow[dd] & & & & \\
 & & & 1 & & & \\
 & & 1 & & & &
\end{tikzcd}
$$

This diagram shows the close connections between $\mathrm{SK}_1(\mathsf{E})$ and $\mathrm{SK}_1^h(\mathsf{S})$, where $\mathsf{S} = \mathbb{M}_n(\mathsf{E})$ with unshifted grading. The diagram is commutative with exact rows and columns. The group G in it is $G = \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$, where T is the center of E, and $Z(\mathsf{E}_0)$ is the center of E_0; it is known that $Z(\mathsf{E}_0)$ is Galois over T_0, and that G is a homomorphic image of $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$, so G is abelian. Also, $d = \mathrm{ind}(\mathsf{E})\big/\big(\mathrm{ind}(\mathsf{E}_0)\,[Z(\mathsf{E}_0):\mathsf{T}_0]\big)$, and $\mu_d(\mathsf{T}_0)$ is the group of those d-th roots of unity lying in T_0. The map $\widetilde{N}_{\mathsf{E}}$ is the composition $\widetilde{N}_{\mathsf{E}} = N_{Z(\mathsf{E}_0)/\mathsf{T}_0} \circ \mathrm{Nrd}_{\mathsf{E}_0}\colon \mathsf{E}_0^* \to \mathsf{T}_0^*$; the map $\widetilde{N}_{\mathsf{S}}$ is defined analogously. Exactness of the rows and column in the vertical E-plane is proved in [HW_1, Th. 3.4]; exactness in the S-plane is proved analogously, as the reader can readily verify. The maps from the S-plane to the E-plane are induced by the Dieudonné determinant $\det_{\mathsf{S}_0}$ from $\mathsf{S}_0 = \mathbb{M}_n(\mathsf{E}_0)$ to E_0. By Lemma 2.1, $\mathsf{S}_h^{(1)} \subseteq \mathsf{S}_0^*$; moreover, the images of $\mathsf{S}_h^{(1)}$, $[\mathsf{S}_h^*, \mathsf{S}_h^*]$ and $[\mathsf{S}_h^*, \mathsf{S}_0^*]$ in $\mathsf{S}_0^*/[\mathsf{S}_0^*, \mathsf{S}_0^*]$, under $\det_{\mathsf{S}_0}$, lie in the images of $\mathsf{E}^{(1)}$, $[\mathsf{E}^*, \mathsf{E}^*]$ and $[\mathsf{E}^*, \mathsf{E}_0^*]$ in $\mathsf{E}_0^*/[\mathsf{E}_0^*, \mathsf{E}_0^*]$, respectively— see Proposition 2.3 below, which yields the middle isomorphism in the lower horizontal plane of the diagram. Here, $\Gamma_\mathsf{S} = \Gamma_\mathsf{E}$ since the grading on S is unshifted, and the map η_n on the left is $x \mapsto nx$. This diagram gives some insight into where to look for differences between $\mathrm{SK}_1^h(\mathsf{S})$ and $\mathrm{SK}_1(\mathsf{E})$; the differences are delineated in Theorem 2.4 below.

Let $\mathsf{S} = \mathbb{M}_n(\mathsf{E})$, with unshifted grading. We have the filtration of commutator groups

$$[\mathsf{S}_0^*, \mathsf{S}_0^*] \subseteq [\mathsf{S}_h^*, \mathsf{S}_0^*] \subseteq [\mathsf{S}_h^*, \mathsf{S}_h^*] \subseteq \mathsf{S}_h^{(1)},$$

with $\mathrm{SK}_1^h(\mathsf{S}) = \mathsf{S}_h^{(1)}/[\mathsf{S}_h^*, \mathsf{S}_h^*]$. We relate the factors in this filtration to the corresponding ones for E in order to relate $\mathrm{SK}_1^h(\mathsf{S})$ to $\mathrm{SK}_1(\mathsf{E})$:

Proposition 2.3. *Let $\mathsf{S} = \mathbb{M}_n(\mathsf{E})$ with unshifted grading, and suppose $\mathsf{S}_0 \neq \mathbb{M}_2(\mathbb{F}_2)$. Then,*

$$\mathsf{S}^{(1)} \big/ [\mathsf{S}_0^*, \mathsf{S}_0^*] \cong \mathsf{E}^{(1)} \big/ [\mathsf{E}_0^*, \mathsf{E}_0^*], \tag{2.8}$$

and this isomorphism maps $[\mathsf{S}_h^*, \mathsf{S}_0^*]/[\mathsf{S}_0^*, \mathsf{S}_0^*]$ onto $[\mathsf{E}_h^*, \mathsf{E}_0^*]/[\mathsf{E}_0^*, \mathsf{E}_0^*]$.

Proof. Let $\overline{\mathsf{S}_h^{(1)}} = \mathsf{S}_h^{(1)}/[\mathsf{S}_0^*, \mathsf{S}_0^*]$ and $\overline{\mathsf{E}^{(1)}} = \mathsf{E}^{(1)}/[\mathsf{E}_0^*, \mathsf{E}_0^*]$. Note that $\mathsf{S}_0 = \mathbb{M}_n(\mathsf{E}_0)$, since the grading on S is unshifted. There is a homomorphism

$$\eta \colon \overline{\mathsf{E}^{(1)}} \to \overline{\mathsf{S}_h^{(1)}} \ \text{ induced by } c \mapsto \operatorname{diag}(c, 1, 1, \ldots, 1).$$

This η is well-defined, as $\mathrm{Nrd}_\mathsf{S}(\operatorname{diag}(c, 1, \ldots, 1)) = \mathrm{Nrd}_\mathsf{E}(c)$. Moreover, η is surjective, as $\mathsf{S}_0^* = \operatorname{diag}(\mathsf{E}_0^*, 1, \ldots, 1)[\mathsf{S}_0^*, \mathsf{S}_0^*]$ (see [D, §22, Th. 1]) since $\mathsf{S}_0 \neq \mathbb{M}_2(\mathbb{F}_2)$. To get a map in the other direction we use the Dieudonné determinant for S_0,

$$\det\nolimits_{\mathsf{S}_0} \colon \mathsf{S}_0^* \longrightarrow \mathsf{E}_0^* \big/ [\mathsf{E}_0^*, \mathsf{E}_0^*].$$

Recall (see [D, §22, Th. 1]) that $\det_{\mathsf{S}_0}$ is compatible with reduced norms, i.e., $\mathrm{Nrd}_{\mathsf{S}_0}(a) = \overline{\mathrm{Nrd}_{\mathsf{E}_0}}(\det_{\mathsf{S}_0}(a))$ for all $a \in \mathsf{S}_0^*$, where $\overline{\mathrm{Nrd}_{\mathsf{E}_0}} \colon \mathsf{E}_0^*/[\mathsf{E}_0^*, \mathsf{E}_0^*] \to Z(\mathsf{E}_0)^*$ is induced by $\mathrm{Nrd}_{\mathsf{E}_0}$. Therefore, if $a \in \mathsf{S}^{(1)}$, then $a \in \mathsf{S}_0^*$ by Lemma 2.1, so by Lemma 2.2 (used for S then for E),

$$\begin{aligned} 1 = \mathrm{Nrd}_\mathsf{S}(a) &= N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{S}_0}(a))^d \\ &= N_{Z(\mathsf{E}_0)/\mathsf{T}_0}\big(\overline{\mathrm{Nrd}_{\mathsf{E}_0}}(\det\nolimits_{\mathsf{S}_0}(a))\big)^d \\ &= \overline{\mathrm{Nrd}_\mathsf{E}}(\det\nolimits_{\mathsf{S}_0}(a)). \end{aligned}$$

This shows that there is a well-defined homomorphism

$$\xi \colon \overline{\mathsf{S}_h^{(1)}} \longrightarrow \overline{\mathsf{E}_h^{(1)}} \quad \text{induced by } \det\nolimits_{\mathsf{S}_0}.$$

Since $\det_{\mathsf{S}_0}(\operatorname{diag}(c,1,\ldots,1)) = c\,[\mathsf{E}_0^*,\mathsf{E}_0^*]$, for $c\in\mathsf{E}_0^*$ we have $\xi\eta=\mathrm{id}$. Therefore, as η is surjective, η and ξ are isomorphisms, proving (2.8).

Let $\overline{[\mathsf{S}_h^*,\mathsf{S}_0^*]} = [\mathsf{S}_h^*,\mathsf{S}_0^*]/[\mathsf{S}_0^*,\mathsf{S}_0^*]$ and $\overline{[\mathsf{E}^*,\mathsf{E}_0^*]} = [\mathsf{E}_h^*,\mathsf{E}_0^*]/[\mathsf{E}_0^*,\mathsf{E}_0^*]$. It remains to show that these groups are isomorphic via ξ.

Since $\Gamma_{\mathsf{S}}=\Gamma_{\mathsf{E}}$ as the grading on E is unshifted, we have $\Gamma_{\mathsf{S}}^*=\Gamma_{\mathsf{E}}$. That is, for any $s\in\mathsf{S}_h^*$ there is $e\in\mathsf{E}^*$ with $\deg(e)=\deg(s)$. Then, $s=[s(e^{-1}\mathbb{I}_n)]\,e\mathbb{I}_n$ with $\deg(s(e^{-1}\mathbb{I}_n))=0$. Thus, $\mathsf{S}_h^*=(\mathsf{E}^*\mathbb{I}_n)\mathsf{S}_0^*$. Recall the general commutator identity

$$[ab,c] \;=\; [{}^ab,{}^ac][a,c], \quad \text{where } \ {}^ax=axa^{-1}. \tag{2.9}$$

Since S_0^* is a normal subgroup of S_h^*, this identity shows that $\overline{[\mathsf{S}_h^*,\mathsf{S}_0^*]}$ is generated by the images of commutators of the form $[c\mathbb{I}_n,a]$, where $c\in\mathsf{E}^*$ and $a\in\mathsf{S}_0^*$. Now if φ is any ring automorphism of E_0, then φ induces an automorphism of $\mathsf{S}_0=\mathbb{M}_n(\mathsf{E}_0)$, again called φ, and also an automorphism $\overline{\varphi}$ of $\mathsf{E}_0^*/[\mathsf{E}_0^*,\mathsf{E}_0^*]$. Because φ is compatible with strict Bruhat decompositions of matrices, it is compatible with $\det_{\mathsf{S}_0}$, i.e., $\det_{\mathsf{S}_0}(\varphi(s))=\overline{\varphi}(\det_{\mathsf{S}_0}(s))$ for any $s\in\mathsf{S}_0^*$. By applying this to the automorphism of E_0 given by conjugation by $c\in\mathsf{E}^*$, we obtain, for any $a\in\mathsf{S}_0^*$,

$$\det_{\mathsf{S}_0}([c\mathbb{I}_n,a]) \;=\; \det_{\mathsf{S}_0}(c\mathbb{I}_n\,a\,c^{-1}\mathbb{I}_n)\det_{\mathsf{S}_0}(a^{-1}) \;=\; cdc^{-1}d^{-1}\,[\mathsf{E}_0^*,\mathsf{E}_0^*],$$

where $\det_{\mathsf{S}_0}(a)=d\,[\mathsf{E}_0^*,\mathsf{E}_0^*]$. This shows that $\xi\big(\overline{[\mathsf{S}_h^*,\mathsf{S}_0^*]}\big)=\overline{[\mathsf{E}^*,\mathsf{E}_0^*]}$, and hence $\eta\big(\overline{[\mathsf{E}^*,\mathsf{E}_0^*]}\big)=\overline{[\mathsf{S}_h^*,\mathsf{S}_0^*]}$. □

Theorem 2.4. *Let E be a graded division algebra finite-dimensional over its center T (with Γ_{E} torsion-free). For $n\in\mathbb{N}$ let $\mathsf{S}=\mathbb{M}_n(\mathsf{E})$, with unshifted grading, and assume $\mathbb{M}_n(\mathsf{E}_0)\neq\mathbb{M}_2(\mathbb{F}_2)$. Then there is an exact sequence*
(2.10)

$$0 \longrightarrow [\mathsf{E}^*,\mathsf{E}^*]\,/\big([\mathsf{E}^*,\mathsf{E}^*]^n\,[\mathsf{E}^*,\mathsf{E}_0^*]\big) \longrightarrow \mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E})) \xrightarrow{\;\overline{\xi}\;} \mathrm{SK}_1(\mathsf{E}) \longrightarrow 0,$$

where $\overline{\xi}$ is induced by the Dieudonné determinant

$$\det_{\mathsf{S}_0}\colon \mathsf{S}_0^* \longrightarrow \mathsf{E}^*/[\mathsf{E}_0^*,\mathsf{E}_0^*].$$

Furthermore, let $\Lambda=\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}\wedge\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$, a finite abelian group. and let e be the exponent of Λ. Then,

(i) *The group $[\mathsf{E}^*,\mathsf{E}^*]/[\mathsf{E}^*,\mathsf{E}_0^*]$ is a homomorphic image of Λ. Hence, $[\mathsf{E}^*,\mathsf{E}^*]\,/\big([\mathsf{E}^*,\mathsf{E}^*]^n[\mathsf{E}^*,\mathsf{E}_0^*]\big)$ is a homomorphic image of $\Lambda/n\Lambda$.*

(ii) *As n varies, $[\mathsf{E}^*,\mathsf{E}^*]/[\mathsf{E}^*,\mathsf{E}^*]^n[\mathsf{E}^*,\mathsf{E}_0^*]$ depends only on the congruence class of n mod e.*

(iii) *If $\gcd(n,e)=1$, then $\mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E}))\cong\mathrm{SK}_1(\mathsf{E})$. This holds for all n if Λ is trivial, which occurs, e.g. if $\Gamma_{\mathsf{E}}=\mathbb{Z}$ or more generally if $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ is cyclic.*

(iv) *If E is unramified over T, then $\mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E}))\cong\mathrm{SK}_1(\mathsf{E})\cong\mathrm{SK}_1(\mathsf{E}_0)$.*

(v) *Suppose E is totally ramified over T. Then, $e=\exp(\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}})$, and $\mathrm{SK}_1(\mathsf{E})\cong\mu_s(\mathsf{T}_0)/\mu_e(\mathsf{T}_0)$, where $s=\operatorname{ind}(\mathsf{E})$. Moreover, there is a* *short exact sequence*

$$0 \longrightarrow \mathbb{Z}/(n,e)\mathbb{Z} \longrightarrow \mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E})) \xrightarrow{\overline{\xi}} \mathrm{SK}_1(\mathsf{E}) \longrightarrow 0. \tag{2.11}$$

Proof. We use the notation in the proof of Proposition 2.3.

Recall from the proof of Proposition 2.3 that $\mathsf{S}_h^* = (\mathsf{E}^*\mathbb{I}_n)\mathsf{S}_0^*$. Since S_0^* is a normal subgroup of S_h^*, it follows by using the commutator identity (2.9) that $[\mathsf{S}_h^*, \mathsf{S}_h^*]/[\mathsf{S}_h^*, \mathsf{S}_0^*]$ is generated by the images of $[c\mathbb{I}_n, c'\mathbb{I}_n] = [c,c']\mathbb{I}_n$ for $c, c' \in \mathsf{E}^*$. Note that

$$\det_{\mathsf{S}_0}([c\mathbb{I}_n, c'\mathbb{I}_n]) \;=\; [c,c']^n\,[\mathsf{E}_0^*, \mathsf{E}_0^*].$$

Furthermore note that the commutators $[c,c']$ generate $[\mathsf{E}^*, \mathsf{E}^*]$. Since the isomorphism ξ maps $\overline{[\mathsf{S}_h^*, \mathsf{S}_0^*]}$ to $\overline{[\mathsf{E}_h^*, \mathsf{E}_0^*]}$ by Proposition 2.3, it therefore maps maps $[\mathsf{S}_h^*, \mathsf{S}_h^*]/[\mathsf{S}_0^*, \mathsf{S}_0^*]$ onto $[\mathsf{E}^*, \mathsf{E}^*]^n[\mathsf{E}^*, \mathsf{E}_0^*]/[\mathsf{E}_0^*, \mathsf{E}_0^*]$. Hence,

$$\mathrm{SK}_1^h(\mathsf{S}) = \mathsf{S}^{(1)}/[\mathsf{S}_h^*, \mathsf{S}_h^*] \;\cong\; \mathsf{E}^{(1)}/[\mathsf{E}^*, \mathsf{E}^*]^n[\mathsf{E}^*, \mathsf{E}_0^*],$$

which yields the exact sequence (2.10).

For (i)–(iii), recall from [HW$_1$, Th. 3.4, Lemma 3.5] that there is a well-defined epimorphism $\psi\colon \Lambda = \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \wedge \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \to [\mathsf{E}^*, \mathsf{E}^*]/[\mathsf{E}_0^*, \mathsf{E}^*]$, given as follows: For $\gamma, \delta \in \Gamma_{\mathsf{E}}$, take any nonzero $x_\gamma \in \mathsf{E}_\gamma$ and $x_\delta \in \mathsf{E}_\delta$. Then,

$$\psi\big((\gamma + \Gamma_{\mathsf{T}}) \wedge (\delta + \Gamma_{\mathsf{T}})\big) \;=\; [x_\gamma, x_\delta] \bmod [\mathsf{E}_0^*, \mathsf{E}^*].$$

This ψ induces an epimorphism

$$\Lambda/n\Lambda \;\to\; \big([\mathsf{E}^*, \mathsf{E}^*]/[\mathsf{E}_0^*, \mathsf{E}^*]\big)\big/\big([\mathsf{E}^*, \mathsf{E}^*]/[\mathsf{E}_0^*, \mathsf{E}^*]\big)^n \cong [\mathsf{E}^*, \mathsf{E}^*]/[\mathsf{E}^*, \mathsf{E}^*]^n[\mathsf{E}^*, \mathsf{E}_0^*],$$

which yields (i). Assertion (ii) follows immediately from (i) since the epimorphism ψ shows that the exponent of $[\mathsf{E}^*, \mathsf{E}^*]/[\mathsf{E}_0^*, \mathsf{E}^*]$ divides that of Λ. Also, (iii) is immediate from (i) and the exact sequence (2.10), since $\Lambda/n\Lambda$ is trivial when $\gcd(n,e) = 1$.

For (iv), let E be an unramified graded division algebra with center T, i.e., suppose $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{T}}$. Then we have $\mathsf{E}^* = \mathsf{E}_0^*\mathsf{T}^*$, so $[\mathsf{E}^*, \mathsf{E}^*] = [\mathsf{E}^*, \mathsf{E}_0^*]$ and it follows immediately from (2.10) that $\mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E})) \cong \mathrm{SK}_1(\mathsf{E})$, for any $n \in \mathbb{N}$. (Compare this with Corollary 1.4). The isomorphism $\mathrm{SK}_1(\mathsf{E}) \cong \mathrm{SK}_1(\mathsf{E}_0)$ for E unramified is given in [HW$_1$, Cor. 3.6(i)].

For (v), let E be a totally ramified graded division algebra with center T, i.e., $\mathsf{E}_0 = \mathsf{T}_0$. Then $[\mathsf{E}^*, \mathsf{E}_0^*] = [\mathsf{E}^*, \mathsf{T}_0^*] = 1$. Also, by [HwW, Prop. 2.1], $[\mathsf{E}^*, \mathsf{E}^*] \cong \mu_{e'}(\mathsf{T}_0) \cong \mathbb{Z}/e'\mathbb{Z}$, where e' is the exponent of the torsion abelian group $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$. But since E is totally ramified, there is a nondegenerate symplectic pairing on $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ induced by commutators in E (see [HwW, Prop. 2.1, Remark 2.2(ii)]). Hence, $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \cong H \times H$ for some finite abelian group H, which implies that the exponent e' of $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ coincides with the exponent e of Λ. With this information, exact sequence (2.11) follows from (2.10). The formula for $\mathrm{SK}_1(\mathsf{E})$ was given in [HW$_1$, Cor. 3.6(ii)] $\square$

Example 2.5. For any positive integers $e > 1$ and s with $e \mid s$ and s having the same prime factors as e, it is easy to construct examples of graded division algebras E with center T such that E is totally ramified over T with $\exp(\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}) = e$ and $\operatorname{ind}(\mathsf{E}) = s$, and $\mathrm{SK}_1(\mathsf{E}) \cong \mu_s/\mu_e$. For example, T could be an iterated Laurent polynomial ring over the complex numbers, $\mathsf{T} = \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_k, X_k^{-1}]$ graded by multidegree in $X_1, \ldots, X_k$ (so $\Gamma_{\mathsf{T}} = \mathbb{Z}^k$). For k sufficiently large, one can take E to be a tensor product of suitable graded symbol algebras over T, cf. [HW_2, Ex. 5.3]. By choosing e arbitrarily and choosing n not relatively prime to e, one obtains explicit examples where $\mathrm{SK}_1(\mathbb{M}_n(\mathsf{E})) \ncong \mathrm{SK}_1(\mathsf{E})$ by Theorem 2.4(v).

The exact sequence (2.10), along with part (i) of Theorem 2.4 shows that $\mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E}))$ is a finite abelian group with exponent dividing $n \operatorname{ind}(\mathsf{E})$ (since $\mathrm{SK}_1(\mathsf{E})$ is finite abelian with exponent dividing $\operatorname{ind}(\mathsf{E})$ by [D, §23, Lemma 2]). However if we permit shifting in the grading on matrices, we can construct more complicated reduced Whitehead groups. In the example below we construct a simple graded algebra such that its homogenous SK_1 is not even a torsion group when T_0^* is not torsion.

Example 2.6. Let E be a graded division algebra totally ramified over its center T, with grade group $\Gamma_{\mathsf{E}} \subseteq \Gamma$. Consider $\mathsf{S} = \mathbb{M}_n(\mathsf{E})(\overline{\delta})$, where $n > 1$ and $\overline{\delta} = (\delta_1, \ldots, \delta_n) = \big(0, \delta, \ldots, (n-1)\delta\big)$, with $\delta \in \Gamma$ chosen so that the order m of $\delta + \Gamma_{\mathsf{E}}$ in $\Gamma/\Gamma_{\mathsf{E}}$ exceeds $3n$. Let $s = \operatorname{ind}(\mathsf{E})$. We will show that

$$\mathrm{SK}_1^h(\mathbb{M}_n(\mathsf{E})(\overline{\delta})) \cong \left(\left(\prod_{i=1}^{n-1} \mathsf{T}_0^* \right) \times \mu_s(\mathsf{T}_0) \right) \Big/ H, \tag{2.12}$$
$$\text{where } H = \{(\omega, \ldots, \omega, \omega^{2-n}) \mid \omega \in \mu_e(\mathsf{T}_0)\} \cong \mu_e.$$

Note that since the δ_i are distinct modulo Γ_{E}, the grading on matrices (1.2) shows that S_0 consists of all diagonal matrices with entries from E_0. We show further that $\Gamma_{\mathsf{S}}^* = \Gamma_{\mathsf{E}}$. For, recall that Γ_{S}^* is a subgroup of Γ with $\Gamma_{\mathsf{E}} \subseteq \Gamma_{\mathsf{S}}^* \subseteq \Gamma_{\mathsf{S}}$. From (1.3), we have

$$\Gamma_{\mathsf{S}} \;=\; \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (\delta_i - \delta_j) + \Gamma_{\mathsf{E}} \;=\; \bigcup_{k=-(n-1)}^{n-1} k\delta + \Gamma_{\mathsf{E}}.$$

If $\Gamma_{\mathsf{S}}^* \supsetneqq \Gamma_{\mathsf{E}}$, then $\ell\delta \in \Gamma_{\mathsf{S}}^*$ for some integer ℓ with $1 \le |\ell| \le n-1$. Take the integer q with $n \le q\ell < n + \ell$. For any integer k with $|k| \le n-1$, we have

$$1 \;\le\; q\ell - k \;<\; 2n + \ell - 1 \;<\; 3n \;\le\; m.$$

Hence, $(q\ell - k)\delta \notin \Gamma_{\mathsf{E}}$; so, $(q\ell)\delta + \Gamma_{\mathsf{E}} \neq k\delta + \Gamma_{\mathsf{E}}$ for any k with $|k| \le n-1$. Hence, $q\ell\delta \notin \Gamma_{\mathsf{S}}$, But, $q\ell\delta$ lies in the group Γ_{S}^*, a contradiction. Thus, $\Gamma_{\mathsf{S}}^* = \Gamma_{\mathsf{E}}$.

The formula for Γ_{S}^* implies that $\mathsf{S}_h^* = \mathsf{S}_0^*(\mathsf{E}^*\mathbb{I}_n)$. Since $\mathsf{S}_0^* = \prod_{i=1}^n \mathsf{E}_0^* = \prod_{i=1}^n \mathsf{T}_0^*$, which is abelian and centralized by $\mathsf{E}^*\mathbb{I}_n$, it follows that $[\mathsf{S}_h^*, \mathsf{S}_h^*] = [\mathsf{E}^*, \mathsf{E}^*]\mathbb{I}_n$. By [HwW, Prop. 2.1], $[\mathsf{E}^*, \mathsf{E}^*] = \mu_e(\mathsf{T}_0) = \mu_e$, where e is the exponent of the torsion abelian group $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$. Hence, $[\mathsf{S}_h^*, \mathsf{S}_h^*] = \mu_e\mathbb{I}_n$.

By Lemma 2.1, $\mathsf{S}_h^{(1)} \subseteq \mathsf{S}_0^* \subseteq \mathbb{M}_n(\mathsf{T}_0)$. Now, for any matrix

$$U = \operatorname{diag}(u_1, \ldots, u_n)\mathsf{S}_0^*,$$

we have

$$\mathrm{Nrd}_{\mathsf{S}_0}(U) = u_1 \dots u_n,$$

so by Lemma 2.2,

$$\mathrm{Nrd}_{\mathsf{S}}(U) = (u_1 \dots u_n)^s,$$

where $s = \mathrm{ind}(\mathsf{E})$. It follows that

$$\begin{aligned} \mathsf{S}_h^{(1)} &= \big\{\mathrm{diag}(u_1, \dots, u_n) \mid \text{each } u_i \in \mathsf{T}_0^* \text{ and } u_1 \dots u_n \in \mu_s(\mathsf{T}_0)\big\} \\ &\cong \big\{(u_1, \dots, u_{n-1}, \omega) \mid \text{each } u_i \in \mathsf{T}_0^* \text{ and } \omega \in \mu_s(\mathsf{T}_0\big\} \\ &\cong \left(\prod_{i=1}^{n-1} \mathsf{T}_0^*\right) \times \mu_s(\mathsf{T}_0). \end{aligned}$$

In the isomorphism

$$\mathsf{S}_h^{(1)} \cong \left(\prod_{i=1}^{n-1} \mathsf{T}_0^*\right) \times \mu_s(\mathsf{T}_0),$$

for any $\omega \in \mu_s(\mathsf{T}_0)$, the matrix $\omega \mathbb{I}_n$ maps to $(\omega, \dots, \omega, \omega^{2-n})$. This yields formula (2.12) for $\mathrm{SK}_1^h(\mathsf{S}) = \mathsf{S}_h^{(1)}/[\mathsf{S}_h^*, \mathsf{S}_h^*]$.

One natural question still unanswered is whether *inhomogeneous* SK_1 is Morita invariant in the graded setting, i.e., whether for a graded division algebra E, we have a natural isomorphism $\mathrm{SK}_1(\mathbb{M}_n(\mathsf{E})) \cong \mathrm{SK}_1(\mathsf{E})$, for $n \in \mathbb{N}$. This seems to be a difficult question, in particular as there does not seem to be a notion of (inhomogeneous) Dieudonné determinant, which is what furnishes the Morita isomorphism for division algebras. A key fact which one uses frequently for invertible matrices over fields and division rings is that they are diagonizable modulo their elementary subgroups. However, the work of Bass, Heller and Swan ([R, Lemma 3.2.21]) shows that the decomposition of an invertible matrix over the graded field $F[X, X^{-1}]$ modulo its elementary subgroup is not necessarily diagonal.

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